9/27/21 6.302

A Brief Intro to Z-Transforms

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Big Picture

1) Transform variables
   $x[n] \Rightarrow X(z)$ $\quad$ $v[n] \Rightarrow V(z)$, etc

2) Use scaling & shifting identity to convert difference equations into polynomials in z or z^{-1}

3) Recognize that the abstract 'K' and 'H' operators can be represented as rational functions of z. (Roots of the numerator poly are known as zeros, roots of the den polynomial
4) Use pattern matching to convert back to time

   a) To generate system difference equations
   b) To determine natural freqs
   c) To compute specific outputs (partial fraction expansion)

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Identity we want

If $y[n] = x[n-1]$

Then $\mathcal{Z}(y) = z^{-1}\,\mathcal{Z}(x)$

Difference Eqn

$$y[n] = a_1 y[n-1] + \dots a_L y[n-L] + b_0 x[n] + b_1 x[n-1] + \dots b_{L'} x[n-L']$$

$$Y(z) = \left(a_1 z^{-1} + a_2 z^{-2} + \dots a_L z^{-L}\right) Y(z) + \left(b_0 + b_1 z^{-1} + \dots b_{L'} z^{-L'}\right) X(z)$$

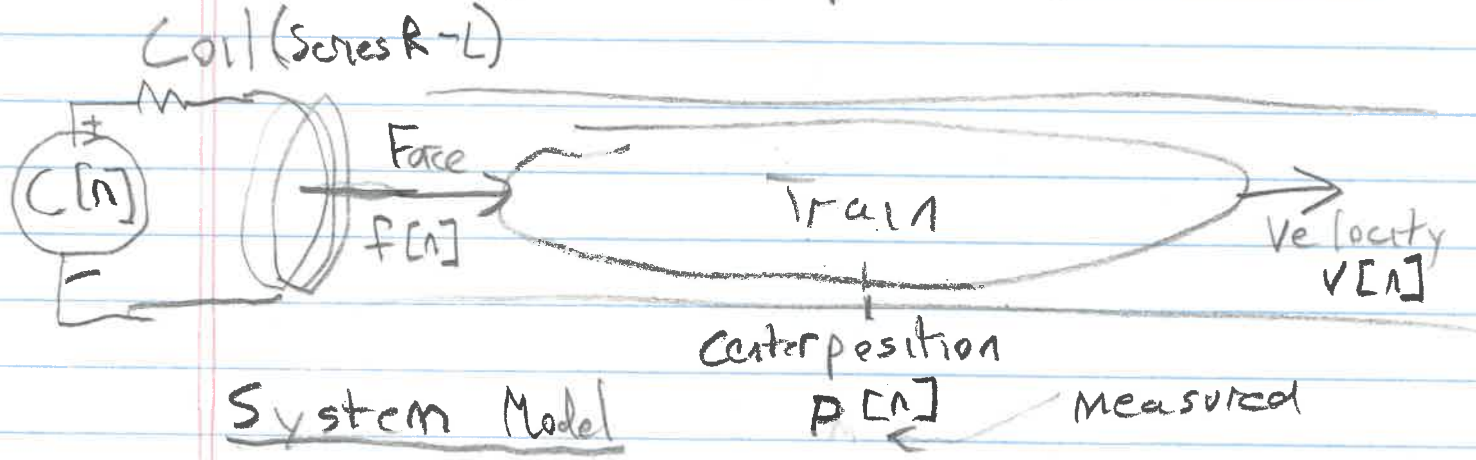
$$Y(z) = \left(\frac{\sum_{k=0}^{L'} b_k z^{-k}}{1 - \sum_{\ell=1}^{L} a_\ell z^{-\ell}} \right) X(z)$$

transfer function

So?

Composing Systems and the Z Transform

Maglev / Hyperloop example



System Model

acceleration proportional to force $\longrightarrow$ $$v[n] \cong v[n-1] + \Delta t\, f[n-1]$$

$$p[n] \cong p[n-1] + \Delta t\, v[n-1]$$

$$f[n] \cong f[n-1] + \Delta t\, \gamma\, c[n-1]$$

Controller

$$e[n] = p_d[n] - p[n]$$

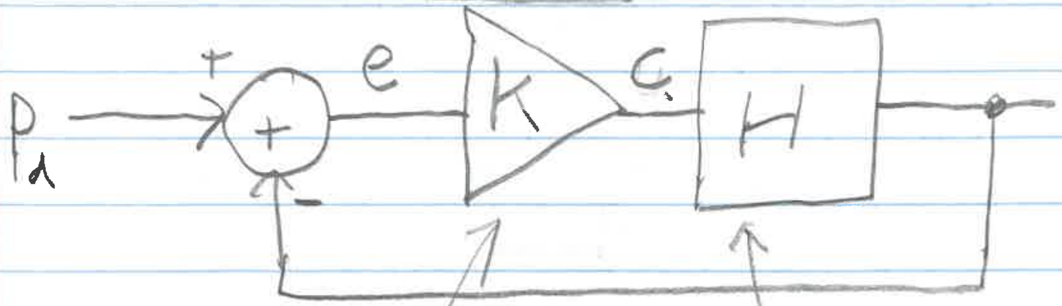
error

desired position

$$c[n] = K_p e[n] + \frac{K_d}{\Delta t}\left(e[n] - e[n-1]\right)$$

Scaling to make it "like" a derivative

Abstract Model



$v[n] = v[n-1] + \Delta t\, f[n-1]$

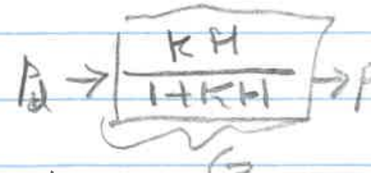
$p[n] = p[n-1] + \Delta t\, v[n-1]$

$f[n] = f[n-1] + \Delta t\, c[n-1]$

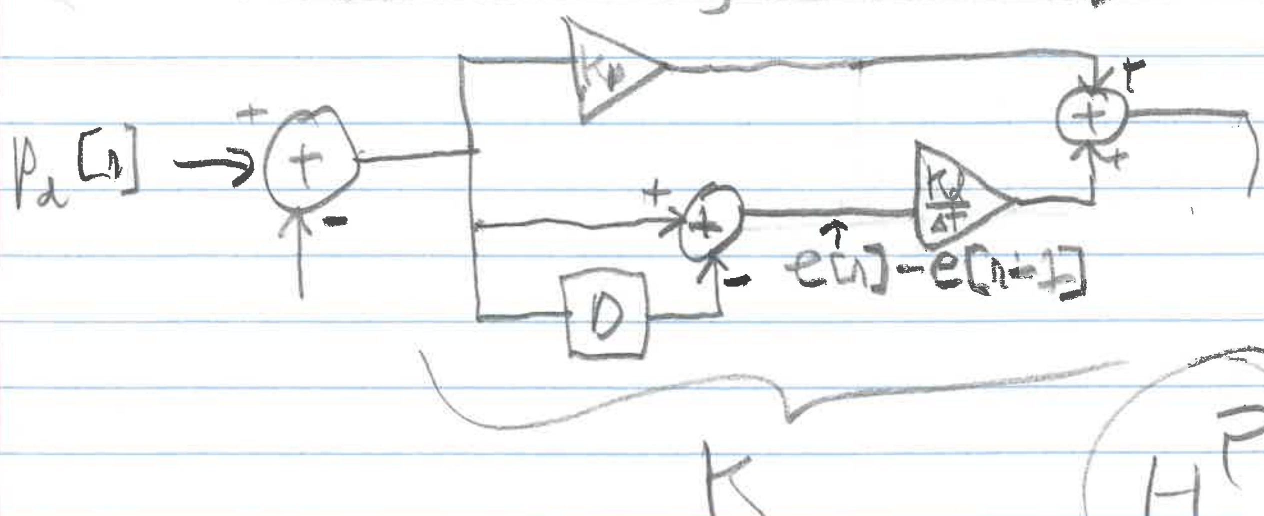
$e[n] = p_d[n] - p[n]$

$c[n] = K_p\, e[n] + \frac{K_d}{\Delta t}\left(e[n] - e[n-1]\right)$

Blacks Formula



Could Start Drawing Block Diagrams



H?

Too Complicated!!

Z - Transform

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

If $x[n] = 0 \quad n < 0$

Then

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

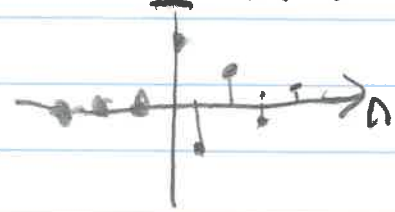
Examples $(x[n] = 0 \quad n < 0)$

$0 < a < 1$

$$x[n] = a^n \quad n \geq 0$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

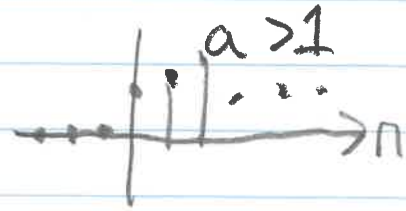
$-1 < a < 0$



$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} \quad \left(\text{for } |z| \text{ large enough}\right)$$

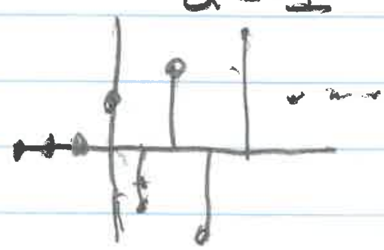
$a > 1$



$x[n] = 1 \quad n = 0$

$x[n] = 0 \quad n \neq 0$

$a < -1$



$$X(z) = 1$$

1st Key Identity (Scaling & Shifting)

If $w[n] = \alpha x[n-1]$ (Scaled & shifted)

$$W(z) = \alpha z^{-1} X(z)$$

$$\sum_{n=0}^{\infty} w[n] z^{-n} = \sum_{n=0}^{\infty} \alpha x[n-1] \underbrace{z^{-(n-1)} z^{-1}}_{= z^{-n}}$$

$$= \alpha z^{-1} \sum_{n=1}^{\infty} x[n-1] z^{-(n-1)}$$

$\tilde{n} = n-1$

$$= \alpha z^{-1} \sum_{\tilde{n}=0}^{\infty} x[\tilde{n}] z^{-\tilde{n}}$$

$$= \alpha z^{-1} X(z)$$

Consider

$$v[n] = v[n-1] + \Delta t\, f[n-1]$$

$$V(z) = z^{-1} V(z) + \Delta t z^{-1} F(z)$$

$$(1 - z^{-1}) V(z) = \Delta t z^{-1} F(z)$$

$$V(z) = \frac{\Delta t z^{-1}}{1 - z^{-1}} F(z)$$

And
$$P(z) = \frac{\Delta t z^{-1}}{1 - z^{-1}} V(z)$$

So
$$P(z) = \frac{\Delta t^2 z^{-2}}{(1 - z^{-1})(1 - z^{-1})} F(z)$$

$$F(z) = \frac{\Delta t \, \gamma z^{-1}}{1 - z^{-1}} I(z)$$

$$\Rightarrow \quad P(z) = \frac{(\Delta t)^3 \gamma z^{-3}}{(1-z^{-1})(1-z^{-1})(1-z^{-1})} I(z)$$

$$I(z) \longrightarrow \boxed{H(z)} \longrightarrow P(z)$$

<u>2nd Key Identity:</u> (General Linearity)

<u>If</u> $v[n] = c_1 x[n] + c_2 w[n]$

<u>Then</u> $V(z) = c_1 X(z) + c_2 W(z)$

$$\sum v[n] z^{-n} = \sum \left(c_1 x[n] + c_2 w[n] \right) z^{-n}$$

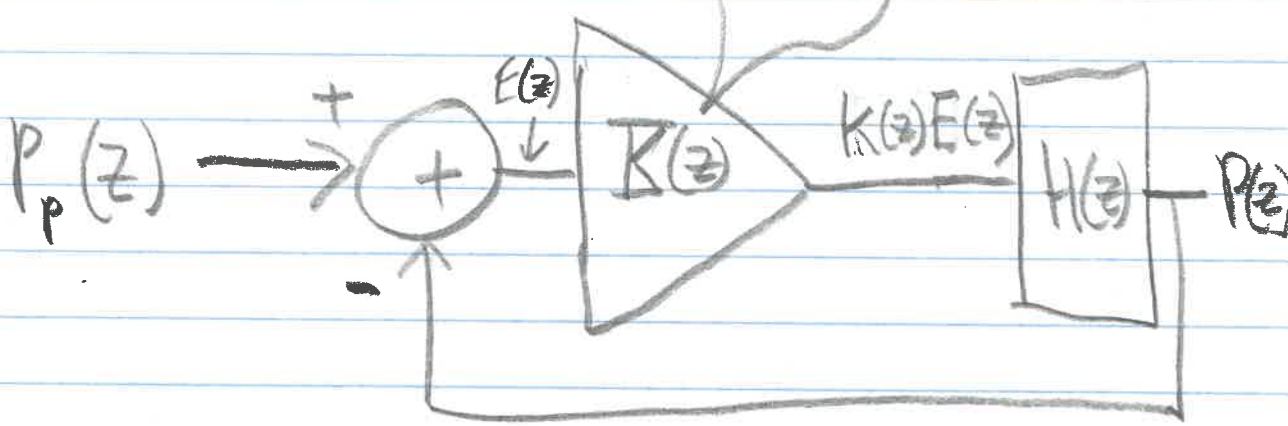
$$= c_1 \sum x[n] z^{-n} + c_2 \sum w[n] z^{-n}$$

$$c[n] = K_p e[n] + \frac{K_d}{\Delta t}\left(e[n] - c[n-1]\right)$$

$$\underline{C(z)} = K_p E(z) + \frac{K_d}{\Delta t}\left(E(z) - z^{-1}E(z)\right)$$

$$= \left(\left(K_p + \frac{K_d}{\Delta t}\right) - \frac{K_d}{\Delta t} z^{-1}\right) E(z)$$

$$= \left(K_p + \frac{K_d}{\Delta t}\left(1 - z^{-1}\right)\right) E(z)$$



<u>Blacks Formula</u> $$P(z) = \underbrace{\frac{K(z)H(z)}{1+K(z)H(z)}}_{G(z)} P_d(z)$$

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$$G(z) = \frac{\frac{\Delta t^3 \gamma z^{-3}}{(1-z^{-1})^3} \cdot \left(K_p + \frac{K_d}{\Delta t}(1-z^{-1})\right)}{1 + \left(K_p + \frac{K_d}{\Delta t}(1-z^{-1})\right)\frac{\Delta t^3 \gamma z^{-3}}{(1-z^{-1})^3}}$$

$$= \frac{\Delta t^3 \gamma z^{-3} \cdot \left(K_p + \frac{K_d}{\Delta t}(1-z^{-1})\right)}{(1-z^{-1})^3 + \underbrace{\Delta t^3 \gamma \left(K_p + \frac{K_d}{\Delta t} - \frac{K_d}{\Delta t} z^{-1}\right) z^{-3}}_{\alpha z^{-3} - \beta z^{-4}}}$$

$$= \frac{\alpha z^{-3} - \beta z^{-4}}{\left(1 - 3z^{-1} + 3z^{-2} - z^{-3}\right) + \left(\alpha z^{-3} - \beta z^{-4}\right)}$$

$$= \frac{\alpha z^{-3} - \beta z^{-4}}{1 - 3z^{-1} + 3z^{-2} + (1+\alpha) z^{-3} - \beta z^{-4}}$$

$$\alpha = \Delta t^3 \gamma \left(K_p + \frac{K_d}{\Delta t}\right)$$

$$\beta = \Delta t^3 \gamma \left(\frac{K_d}{\Delta t}\right)$$

Rational function in z to Difference Eqn

3rd Key Identity

If

$$P(z) = G(z)\,P_d(z)$$

$$G(z) = \frac{\sum_{\ell=0}^{L} b_\ell z^{-\ell}}{1-\sum_{\ell=1}^{L} a_\ell z^{-\ell}}$$

rational function in z

Then the Difference Eqn is

$$p[n] - \sum_{\ell=1}^{L} a_\ell\, p[n-\ell] = \sum_{\ell=0}^{L} b_\ell\, p_d[n-\ell]$$

OR $\Rightarrow$

$$p[n] = a_1 p[n-1] + \dots + a_L p[n-L] + b_0 p_d[n] + \dots + b_L p_d[n-L]$$

Often referred to as "update" form

And Natfreqs (substituting $z=\lambda$)

$$\lambda \text{ s.t. } \left(1-\sum_{\ell=1}^{L} a_\ell \lambda^{-\ell}\right) = 0$$

$\boxed{\text{or}}$

$$\lambda \in \text{roots}\left(\lambda^L - a_1\lambda^{L-1} \dots - a_{L-1}\lambda - a_L\right)$$

In matlab roots([1, -a_1, ... -a_L])

are the natural frequencies!

$$G(z) \text{ is stable} \equiv |\lambda| \le 1$$

$$\text{strictly stable} \equiv |\lambda| < 1$$

Example

$$v[n] = v[n-1] + \Delta t\, f[n]$$

$$p[n] = p[n-1] + \Delta t\, v[n]$$

$$V(z) = \frac{\Delta t\, z^{-1}}{1-z^{-1}} F(z)$$

$$P(z) = \frac{\Delta t\, z^{-1}}{1-z^{-1}} V(z)$$

$$P(z) = \underbrace{\frac{\Delta t^2\, z^{-2}}{1-2z^{-1}+z^{-2}}}_{H(z)} F(z)$$

$$= \frac{\Delta t^2}{z^2-2z+1} F(z)$$

update form
$p[n] =$
$2p[n-1] - p[n-2]$
$+ \Delta t^2 f[n-2]$

So

$$(1-2z^{-1}+z^{-2}) P(z) = (\Delta t)^2 z^{-2} F(z)$$

$$p[n] - 2p[n-1] + p[n-2] = \Delta t^2 f[n-2]$$

OR

$$(z^2-2z+1) P(z) = \Delta t^2 F(z)$$

$$p[n+2] - 2p[n+1] + p[n] = \Delta t^2 f[n]$$

Including feedback with proportional gain

$$f[n] = K_p(p_d[n] - p[n])$$

$$F(z) = K_p(P_d(z) - P(z))$$

$$K(z) = K_p$$

$$H(z) = \frac{\Delta t^2 z^{-2}}{1 - 2z^{-1} + z^{-2}}$$

$$G(z) = \frac{K_p \Delta t^2 z^{-2}}{1 - 2z^{-1} + z^{-2} + K_p \Delta t^2 z^{-2}}$$

$$= \frac{K_p \Delta t^2}{1 - 2z^{-1} + (1 + K_p \Delta t^2) z^{-2}}$$

$$P(z) = G(z) P_d(z)$$

$$p[n] - 2p[n-1] + (1 + K_p \Delta t^2) p[n-2]$$
$$= K_p \Delta t^2 p_d[n-2]$$

<u>Nat freqs</u> $\lambda \in \text{roots}(\lambda^2 - 2\lambda + (1 + K_p \Delta t^2)$

Can we convert back?

Ex $\quad P(z) = G(z)\, P_d(z)$

$$G(z) = \frac{K_p \Delta t^2 z^{-2}}{1 - 2z^{-1} + (1 + K_p \Delta t^2) z^{-2}}$$

Suppose $\quad p_d[n] = 1 \quad n \geq 0, \quad p_d[n] = 0, \ n < 0$

Always assuming this

$$P_d(z) = \frac{1}{1 - z^{-1}}$$

($p[n] = a^n$, $n \geq 0$ formula, $a = 1$)

$$\Rightarrow P(z) = \frac{K_p \Delta t^2 z^{-2}}{(1 - 2z^{-1} + (1 + K_p \Delta t^2) z^{-2})} \cdot \frac{1}{1 - z^{-1}}$$

$$p[n] = ?$$

If λ_1, λ_2 are nat freqs & $\lambda_1 \neq \lambda_2 \neq 1$

$$P(z) = \frac{K_p \Delta t^2 z^{-2}}{(1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1})(1 - z^{-1})}$$

$$= \frac{A}{1 - \lambda_1 z^{-1}} + \frac{B}{1 - \lambda_2 z^{-1}} + \frac{C}{1 - z^{-1}}$$

Follows from linearity & pattern matching

$$p[n] = A \lambda_1^n + B \lambda_2^n + C \quad \underline{n \geq 0}$$

$A = (1-\lambda_1 z^{-1}) P(z) \Big|_{z=\lambda_1}$ $\quad B = (1-\lambda_2 z^{-1}) P(z) \Big|_{z=\lambda_2}$ $\quad C = (1-z^{-1}) P(z) \Big|_{z=1}$

Why cross multiply and see!

(Only true if $\lambda_1 \neq \lambda_2 \neq 1$)!!

Notice

$$C = (1-z^{-1})\, P(z) \Big|_{z=1} = G(z) \Big|_{z=1}$$