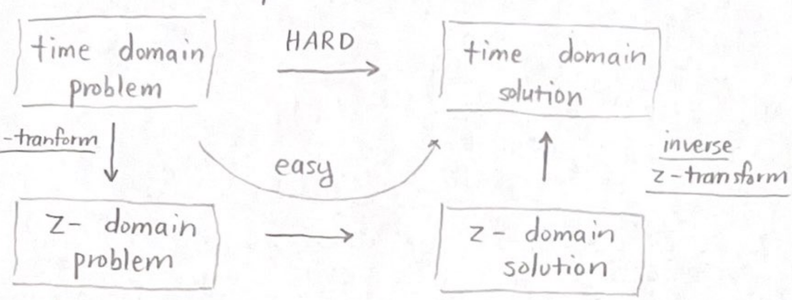
Today

1. Z- transform example
   - I closed-loop vs open-loop
   - II stability
   - III steady state error
2. PID controller

1. Z - transform example:



recall from lab 1

$$w[n] = w[n-1] + \Delta T\left(\beta w[n-1] + \gamma c[n-1]\right) \qquad c[n-1] = k_p\left(w_d[n-1] - w[n-1]\right)$$

I. Find the closed-loop transfer function:

method 1:

$$w[n] = w[n-1] + \Delta T\left(\beta w[n-1] + \gamma k_p\left(w_d[n-1] - w[n-1]\right)\right)$$

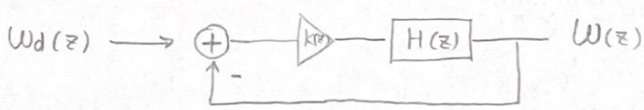
$$w[n] = w[n-1]\left(1 + \Delta T\beta - \Delta T\gamma k_p\right) + \Delta T\gamma k_p\, w_d[n-1]$$

$$W(z) = z^{-1}W(z)\left(1 + \Delta T\beta - \Delta T\gamma k_p\right) + \Delta T\gamma k_p\, W_d(z)\, z^{-1}$$

$$W(z)\left(1 - z^{-1}\left(1 + \Delta T\beta - \Delta T\gamma k_p\right)\right) = \Delta T\gamma k_p\, W_d(z)\, z^{-1}$$

$$W(z) = \underbrace{\frac{\Delta T\gamma k_p z^{-1}}{1 - z^{-1}\left(1 + \Delta T\beta - \Delta T\gamma k_p\right)}}_{\text{closed-loop tansfer function (TF)}} W_d(z)$$

method 2: Black formula:



where $K(z) = k_p$

$H(z)$ = open-loop TF

$$w[n] = w[n-1] + \Delta T\left(\beta w[n-1] + \gamma c[n-1]\right)$$

$$W(z)\left(1 - z^{-1} - \Delta T z^{-1}\beta\right) = \gamma z^{-1}\Delta T\, C(z) \Rightarrow W(z) = \overbrace{\frac{\gamma z^{-1}\Delta T}{1 - z^{-1} - \Delta T z^{-1}\beta}}^{H(z)} C(z)$$

$$H_{\text{closed-lopp}} = \frac{K(z)H(z)}{1 + K(z)H(z)} = \frac{k_p \frac{\gamma z^{-1}\Delta T}{1 - z^{-1} - \Delta T z^{-1}\beta}}{1 + k_p \frac{\gamma z^{-1}\Delta T}{1 - z^{-1} - \Delta T z^{-1}\beta}} \overset{\text{simplify}}{=} \underbrace{\frac{\Delta T\gamma k_p z^{-1}}{1 - z^{-1}\left(1 + \Delta T\beta - \Delta T\gamma k_p\right)}}_{\text{closed loop TF}}$$

II. Stability

The stability of a control system is determined by the maximum root of the closed-loop transfer function's denominator

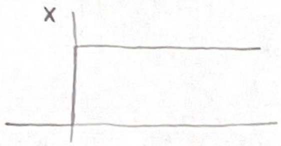
$$1 - z^{-1}(1 + \Delta T\beta - \Delta T g k_p) = 0$$

$$z - \underbrace{(1 + \Delta T\beta - \Delta T g k_p)}_{\text{root}} = 0$$

stable if $|1 + \Delta T\beta - \Delta T g k_p| < 1$

III. Steady state error

1. consider a step input :



$$x[n] = 1 \quad \text{if } n \geq 0$$
$$= 0 \quad \text{if } n < 0$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + \cdots = \frac{z}{z-1}$$

we let $\alpha = 1 + z^{-1} + z^{-2} + \cdots$

then $\alpha z = z + 1 + z^{-1} + \cdots$

$$\alpha z - \alpha = z + \cancel{1} + \cancel{z^{-1}} + \cdots \quad - \cancel{1} - \cancel{z^{-1}} - z^{-2} = z$$

$$\alpha(z-1) = z$$

$$\alpha = \frac{z}{z-1}$$

2. solve the full problem

$$W(z) = W_d(z)\, \frac{\Delta T g k_p}{z - (1 + \Delta T\beta - \Delta T g k_p)} = \frac{z}{z-1}\, \frac{\Delta T g k_p}{z - (1 + \Delta T\beta - \Delta T g k_p)}$$

$w[n]$ = inverse transform of $W(z)$

$$= \underbrace{\frac{1}{2\pi j} \oint_C W(z) z^{n-1}\, dz}_{\text{difficult to calculate}}$$

$\Rightarrow$ we can solve this through the method of partial fraction

$$W(z) = \frac{z}{z-1}\, \frac{\Delta T g k_p}{z - (1 + \Delta T\beta - \Delta T g k_p)} = \frac{z A}{z-1} + \frac{z B}{z - (1 + \Delta T\beta - \Delta T g k_p)},$$

where the coefficients A and B are given by :

$$A = W(z)\,(z-1)\Big|_{z=1} = \frac{g k_p}{g k_p - \beta}$$

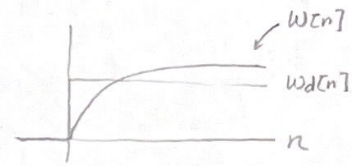
$$B = W(z)\,(z - (1 + \Delta T\beta - \Delta T g k_p))\Big|_{z = 1 + \Delta T\beta - \Delta T g k_p} = \frac{g k_p}{\beta - g k_p}$$

Next, we use the "linearity" property to solve each term independently:

$$w[n] = \mathcal{Z}^{-1}\left(\frac{zA}{z-1}\right) + \mathcal{Z}^{-1}\left(\frac{z\,B}{z-(1+\Delta T\beta-\Delta T g k_p)}\right)$$

$$= A\,(1) + B\,(\underbrace{1+\Delta T\beta-\Delta T g k_p}_{\lambda})^n$$

$$= \frac{g k_p}{g k_p-\beta} - \frac{g k_p}{g k_p-\beta}\,(1+\Delta T\beta-\Delta T g k_p)^n$$



Steady-state error:

$$\left|\,1-\frac{g k_p}{g k_p-\beta}\right| = \left|\frac{\beta}{g k_p-\beta}\right|$$

Is there a simpler method for solving the steady-state solution?

Final value theorem:

$$\lim_{n\to\infty} w[n] = \lim_{z\to 1}(z-1)\,W(z)$$

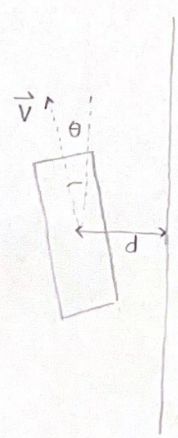
$$= \lim_{z\to 1}(z-1)\,\frac{z}{z-1}\,\frac{\Delta T\, g k_p}{z-(1+\Delta T\beta-\Delta T g k_p)} = \frac{g k_p}{g k_p-\beta}$$

Summary: The z-transform approach tells us:

- stability
- full solution
- an easy method to calculate the s.s value.

2. PID controller

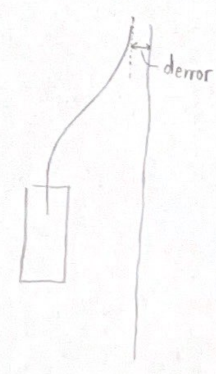
consider the line following example



P- controller → unstable ×

PP- controller → stable ✓

→ how about the steady-state error?



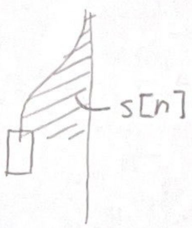
How can we remove the steady-state error ?

↳ accomulate (sum / integrate) the error !

let $e[n] = dd[n] - d[n]$

$$c[n] = k_p\, e[n] + k_d \left(\frac{e[n]-e[n-1]}{\Delta T} \right) + k_i \Delta T \underbrace{\sum_{m=0}^{m=n} e[m]}_{s[n] \Rightarrow \text{"sum" term}}$$

Intuition :



If there is a steady-state error, it will accumulate over time, the k_i term will remove the steady-state error for this problem.

Z- transform of "PID" controller

input $e[n] \rightarrow E(z)$

output $c[n] \rightarrow C(z)$

$$c[n] = k_p\, e[n] + k_i \sum_{m=0}^{n} e[m] \Delta T + \frac{k_d}{\Delta T} \left(e[n] - e[n-1] \right)$$

$$C(z) = k_p E(z) + k_i \left(E(z) + z^{-1} E(z) + \cdots + z^{-n} E(z) \right) \Delta T + \frac{k_d}{\Delta T} \left(E(z) - z^{-1} E(z) \right)$$

note : $1 + z^{-1} + \cdots + z^{-n} = \dfrac{1 - z^{-n-1}}{1 - z^{-1}} \approx \dfrac{1}{1 - z^{-1}}$

$$= \left[k_p + k_i \Delta T \frac{1}{1 - z^{-1}} + \frac{k_d}{\Delta T} (1 - z^{-1}) \right] E(z)$$