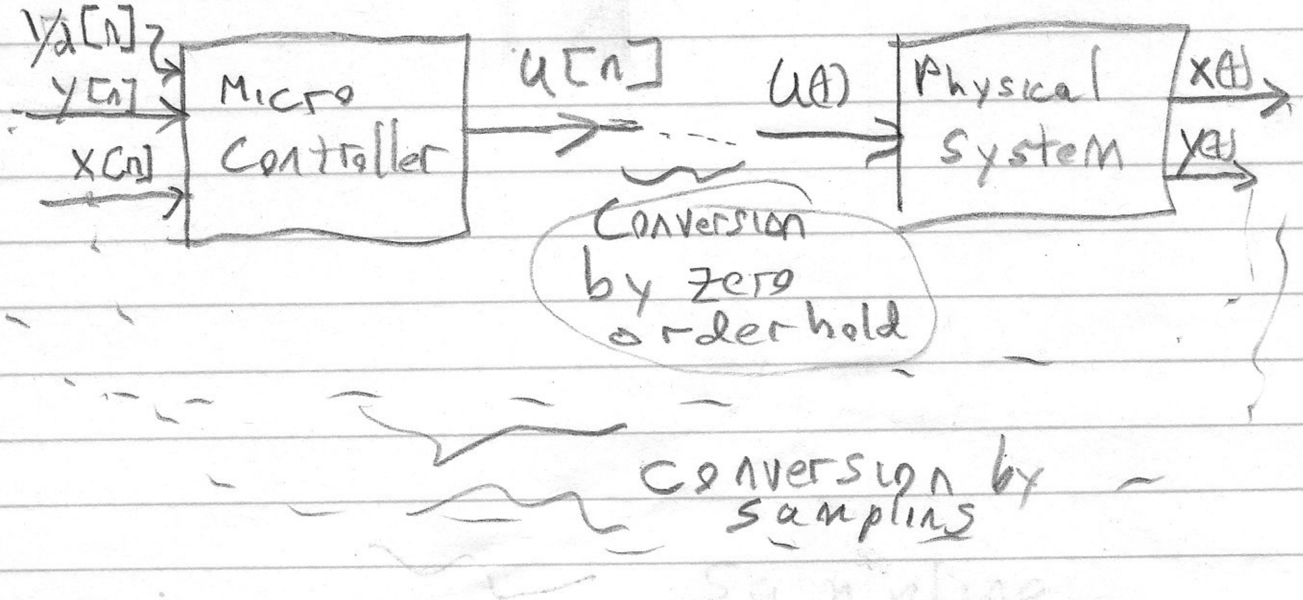
6.302 11/29/21-> 12/1/21

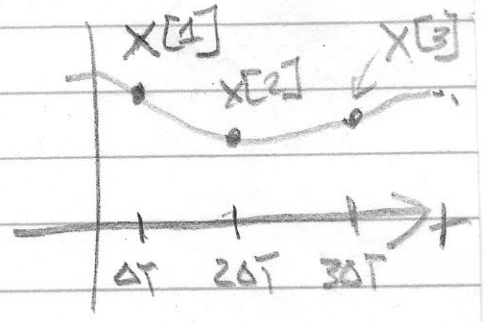
Mixed Discrete - Continuous System



ignoring finite # bits effect!

Sampling Analog → Digital Conversion every ΔT seconds

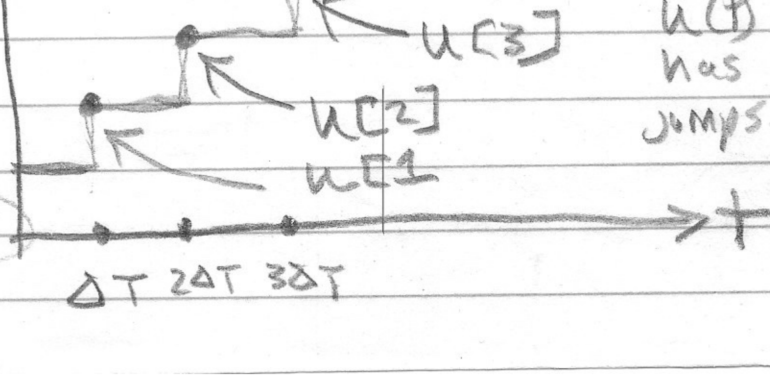
$$x[n] = x(n\Delta T)$$
$$y[n] = y(n\Delta T)$$
$$y_d[n] = y_d(n\Delta T)$$



Zero Order Hold

$$u(t) = u\left[\left\lfloor \frac{t}{\Delta T} \right\rfloor\right]$$

$\lfloor x \rfloor \equiv$ Greatest integer $\leq x$



6.302

Quick Reminder about C2b

Solution to constant input case:

Scalar Case

Eqn $$\frac{d}{dt}x(t) = a x(t) + b \underset{\uparrow \text{ constant}}{u_0}$$

$$x(t)\Big|_{t=0} = x(0)$$

Soln $$x(t) = e^{at}x(0) + (e^{at}-1)\frac{b}{a}u_0$$

Proof $$x(t)\Big|_{t=0} = \overset{=1}{e^{a0}}\, x(0) + \cancel{\overset{=0}{(1-1)}\frac{b}{a}u_0}$$

$$\overset{\checkmark}{=} x(0)$$ Matches I.C.

$$\frac{d}{dt}x(t) = ae^{at}x(0) + ae^{at}\frac{b}{a}u_0$$

$$= a\underbrace{\left(e^{at}x(0) + (e^{at}-1)\frac{b}{a}u_0\right)}_{x(t)} + bu_0$$

$$\overset{\checkmark}{=} a\,x(t) + bu_0$$

Matches Diff. Eqn.

Vector Case

$$\frac{d}{dt}x = Ax + Bu_0$$

$$x(t)\Big|_{t=0} = x(0)$$

$$x(t) = e^{At}x(0) + (e^{At}-I)A^{-1}Bu_0$$

Since $e^{At}\Big|_{t=0} = I$

$$x(t)\Big|_{t=0} \overset{\checkmark}{=} x(0)$$

Since $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$

$$\frac{d}{dt}x(t) = Ae^{At}x(0) + Ae^{At}A^{-1}Bu_0$$

$$\overset{\checkmark}{=} A\left(e^{At}x(0) + (e^{At}-I)A^{-1}Bu_0\right) + Bu_0$$

$e^{at}x = \left(1 + at + \frac{1}{2}(at)^2 + \frac{1}{6}(at)^3 + \dots\right)x$

$e^{At}\vec{x} = \left(I + At + \frac{1}{2}(At)^2 + \dots\right)\vec{x}$

1B

Repeating From Previous Page

Does $x(t) = e^{at}x(0) + (e^{at}-1)\frac{b}{a}u_0$

Solve $\frac{d}{dt}x(t) = ax(t) + bu_0$?

If $a < 0$ look at steady state

$\lim_{t\to\infty} x(t) = x(\infty) = x_{ss}$

↑ steady state

$$\overset{=0}{\frac{d}{dt}x_{ss}} = ax_{ss} + bu_0$$

$$\Rightarrow x_{ss} = \frac{-b}{a}u_0$$

$$x(t) = e^{at}x(0) + (e^{at}-1)\frac{b}{a}u_0$$

$$\xRightarrow[t\to\infty]{\lim} 0\cdot x(0) + (0-1)\frac{b}{a}u_0$$

$$\Rightarrow x_{ss} \overset{✓}{=} \frac{-b}{a}u_0$$

✓

Does $\vec{x}(t) = e^{At}\vec{x}(0) + (e^{At}-I)A^{-1}Bu_0$

Solve $\dot{\vec{x}}(t) = Ax(t) + Bu_0$?

If real(eig(A)) < 0 look at steady state

$\lim_{t\to\infty} \vec{x}(t) = \vec{x}(\infty) = \vec{x}_{ss}$

$$\overset{=0}{\cancel{\frac{d}{dt}\vec{x}_{ss}}} = A\vec{x}_s + Bu_0$$

$$\Rightarrow A\vec{x}_{ss} = -Bu_0$$

$$\vec{x}_{ss} = -A^{-1}Bu_0$$

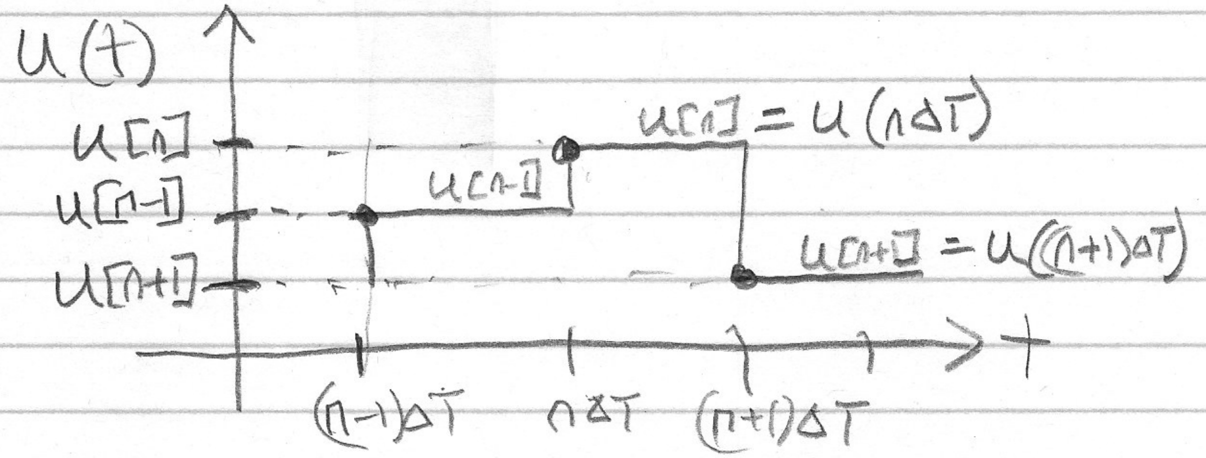
$$\vec{x}(t) = \underset{=0}{e^{At}}\vec{x}(0) + (\underset{=0}{e^{At}} - I)A^{-1}Bu_0$$

$$= 0\cdot\vec{x}(0) + (-I)A^{-1}Bu_0$$

$$\overset{✓}{=} -A^{-1}Bu_0$$

✓

Zero-Order Hold



Let $x(0) = x((n-1)\Delta T) = x[n-1]$, $u_0 = u((n-1)\Delta T) = u[n-1]$

in constant input solution

Scalar Case

$$x(n\Delta T) = e^{a\Delta T}x((n-1)\Delta T) + (e^{a\Delta T}-1)\frac{b}{a}u((n-1)\Delta T)$$

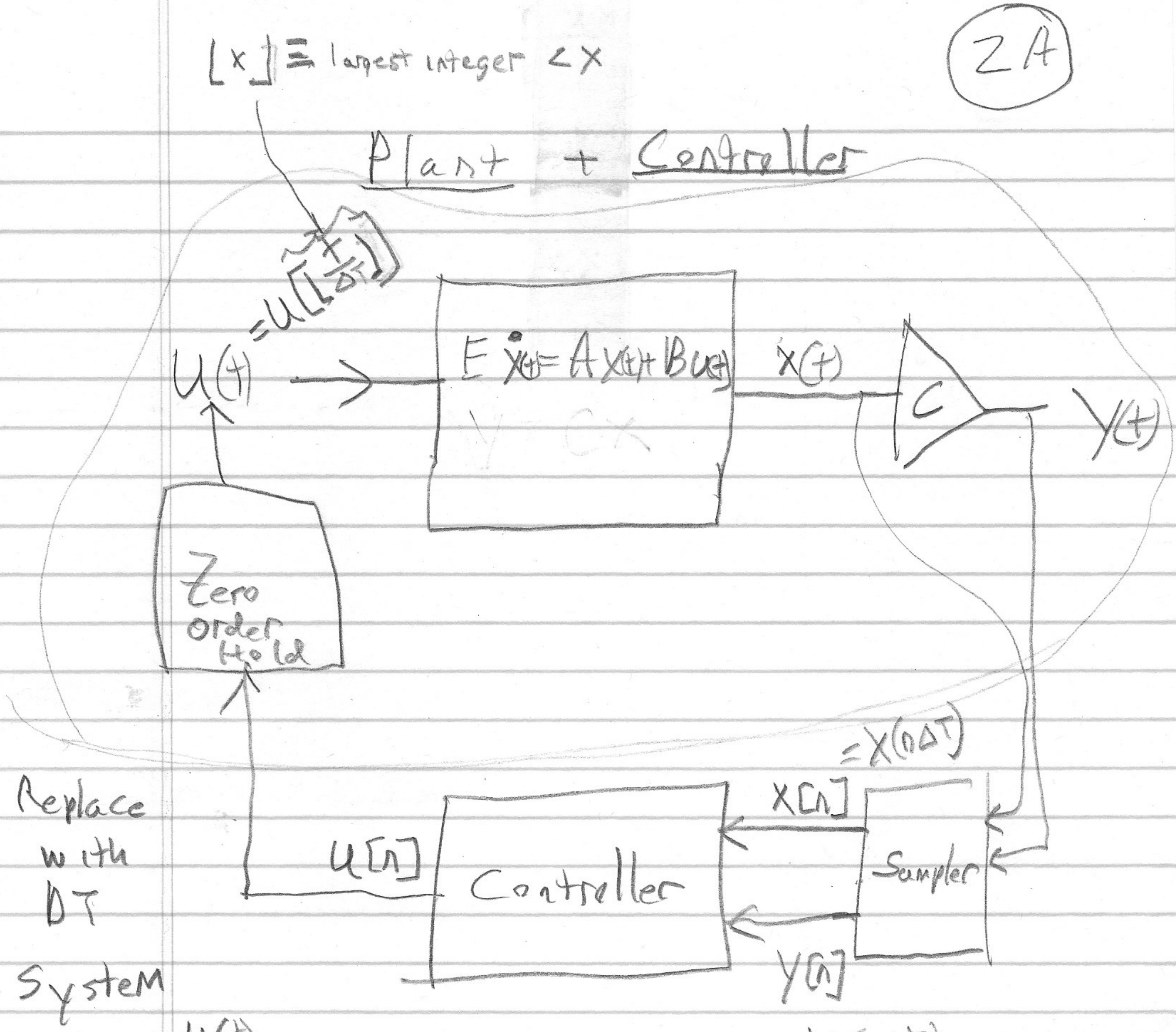
$$x[n] = e^{a\Delta T}x[n-1] + (e^{a\Delta T}-1)\frac{b}{a}u[n-1]$$

Vector Case

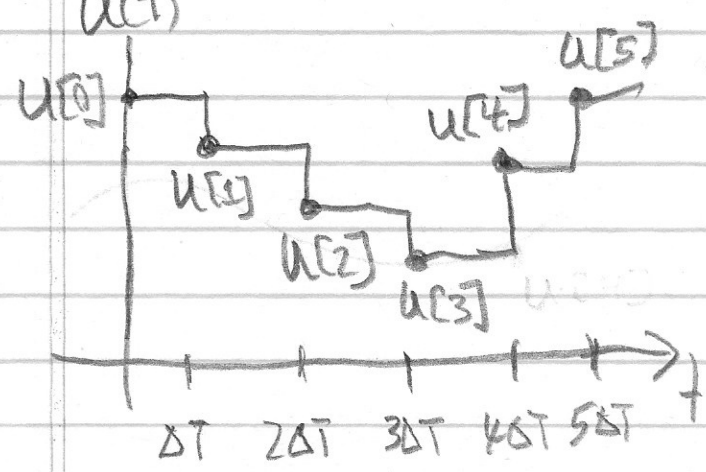
$$x(n\Delta T) = e^{A\Delta T}x((n-1)\Delta T) + (e^{A\Delta T}-1)A^{-1}Bu((n-1)\Delta T)$$

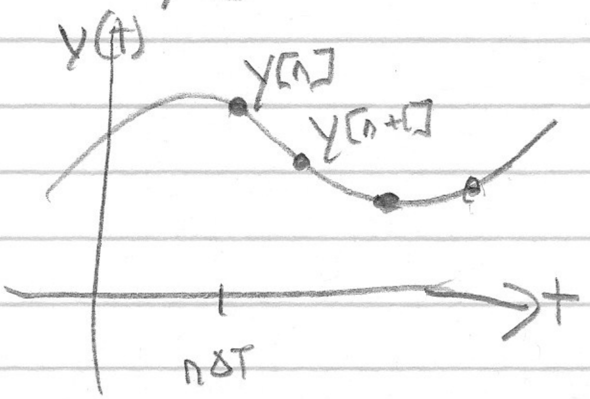
$$x[n] = \underbrace{e^{A\Delta T}}_{A_d}x[n-1] + \underbrace{(e^{A\Delta T}-1)A^{-1}B}_{B_d}u[n-1]$$

Plant + Controller



Replace with DT System





Discrete-Time State-Space Model

$$x[n] = A_d\, x[n-1] + B_d\, u[n-1]$$

$$e^{A\Delta T}$$

$$y[n] = C\, x[n]$$

CT & DT C's are equal!

$$\left(e^{A\Delta T} - I\right) A^{-1} B$$

State Feedback Control

$$u[n] = K_{r_d}\, y_d[n] - K_d\, x[n]$$

Closed-Loop System

$$x[n] = \left(A_d - B_d K_d\right) x[n-1] + B_d K_{r_d}\, y_d[n-1]$$

If $y_d = 0$

$$x[n] = \left(A_d - B_d K_d\right)^n x[0]$$

Want $x[n] \rightarrow 0$ as fast as possible?

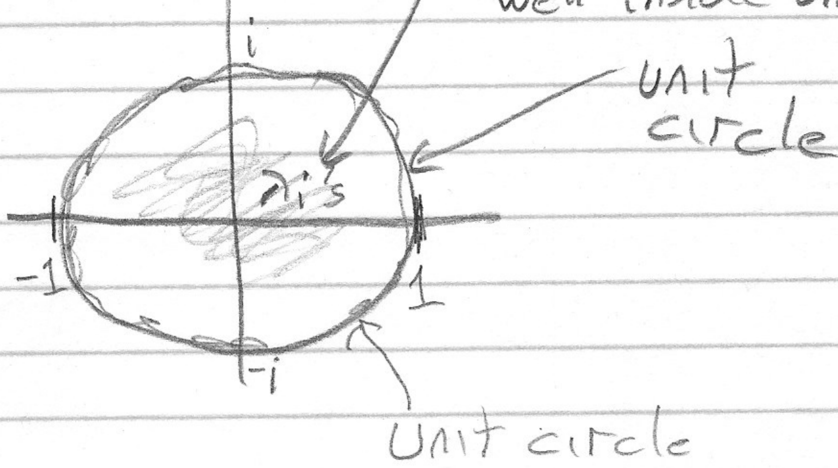
Picking Kd

Pole placement

$$\min_{K_d}\left(\max_{i}\left|\lambda_i(A_d - B_d K_d)\right|\right)$$

can be ineffective!

place λ_i's well inside unit circle



LQR for Discrete-Time

Pick Kd to minimize

$$\sum_{m=0}^{\infty} x^T[m]\, Q\, x[m] + u^T[m]\, R\, u[m]$$

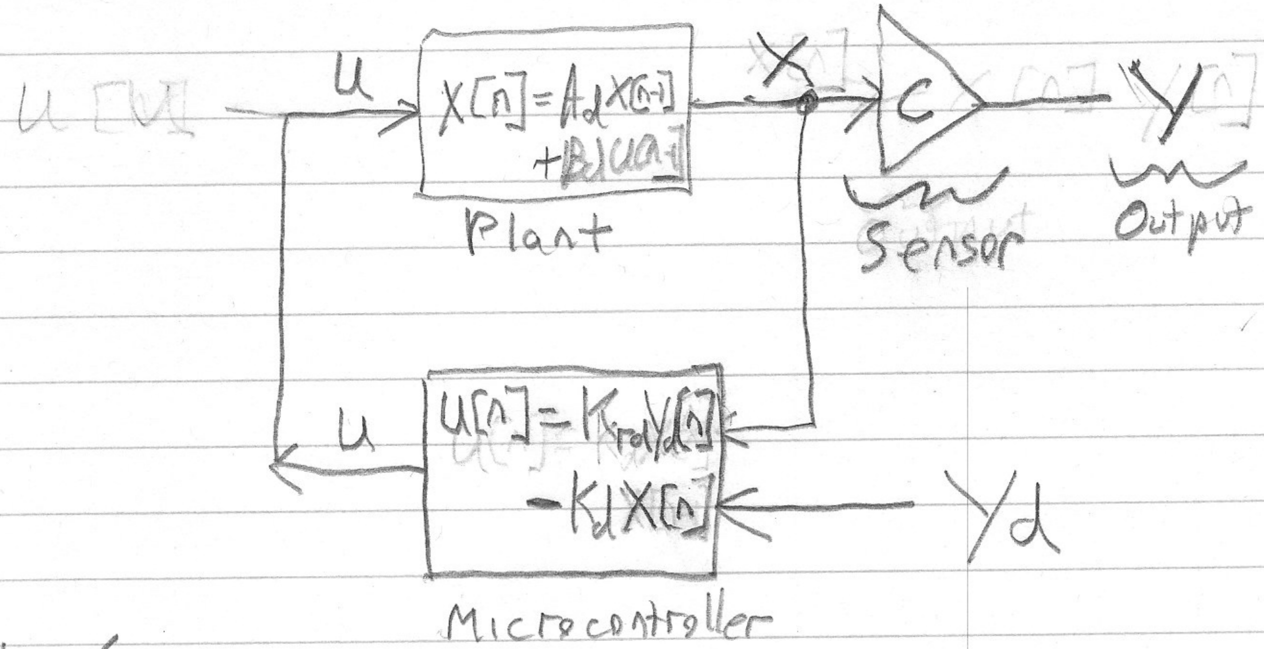
or if Q & R diagonal

$$\sum_{m=0}^{\infty}\left(\sum_{i=1}^{\#\text{ states}} Q_{ii}\, x_i^2[m]\right) + \sum_{j=1}^{\#\text{ inputs}} R_{jj}\, u_j^2[m]\Bigg)$$

4A

What about K_{rd}?

Plant Model
$$\left\{\begin{aligned} x[n] &= A_d\, x[n-1] + B_d\, u[n-1] \\ y[n] &= C\, x[n] \end{aligned}\right.$$



Controller
$$\left\{ u[n] = \underset{?}{\underset{\uparrow}{K_{rd}}}\, y_d[n] - \underset{\text{minimize metric}}{\underset{\uparrow}{K_d}}\, x[n] \right.$$

Closed-Loop

$$x[n] = (A_d - B_d K_d)\, x[n-1] + B_d K_{rd}\, y_d[n-1]$$

In Steady State $x[n] \rightarrow x_\infty$ $y_d[n] \rightarrow y_\infty$

$$x_\infty = (A_d - B_d K_d)\, x_\infty + B_d K_{rd}\, y_\infty$$

What is Krd Continued

In Steady State (Continued)

$$\left(I-(A_d-B_dK_d)\right)X_\infty = B_dK_{rd}Y_{d_\infty}$$

$$\Rightarrow \underbrace{X_\infty}_{\substack{\text{Vector}\\ n\times n}} = \underbrace{\left(I-(A_d-B_dK_d)\right)^{-1}}_{\substack{\text{matrix}\\ n\times n}} \underbrace{B_dK_{rd}}_{\substack{\text{Vector}\\ n\times 1}} \underbrace{Y_{d_\infty}}_{\substack{\text{Scalar}\\ 1\times 1}}$$

$$\underbrace{Y_\infty}_{\text{scalar}} = CX_\infty = C\left(I-(A_d-B_dK_d)\right)^{-1}B_dK_{rd}\underbrace{Y_{d_\infty}}_{\text{Scalar}}$$

Should match in steady-state

$$\frac{Y_\infty}{Y_{d_\infty}} = 1 = \underbrace{C\left(I-(A_d-B_dK_d)\right)^{-1}B_d}_{\text{Scalar}}\underbrace{K_{rd}}_{\text{Scalar}}$$

$$\frac{1}{C\left(I-(A_d-B_dK_d)\right)^{-1}B_d} = K_{rd}\ !!$$

Discrete-Time Observers

u

$$\hat{x}[n] = A_d \hat{x}[n-1] + \underbrace{B_d u[n-1]}_{\substack{\text{No hat, we} \\ \text{know } \underline{u}}}$$

$$+ \underbrace{L_d(y[n-1] - \hat{y}[n-1])}_{\text{correction}}$$

$$u[n-1] = K_{r_d} y_d[n-1] - K_d \hat{x}[n-1]$$

↕ Compare Indexing

On a microcontroller

→ Every ΔT interval

1) Using predicted state, create control

$$u \Leftarrow K_{r_d} y_d - K_d \hat{x}$$

2) Measure observable and compare

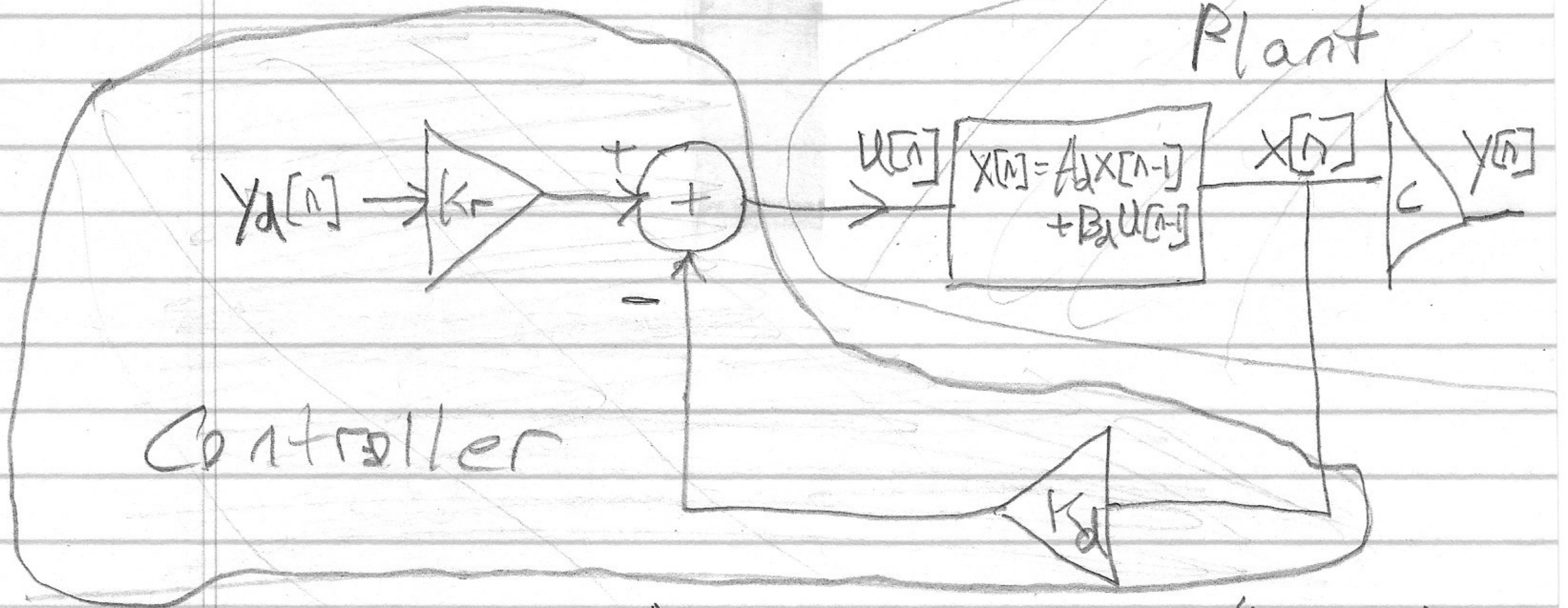
$$\text{correction} \Leftarrow L_d(y - \hat{y})$$

3) Update estimates of state & observable

$$\hat{x} \Leftarrow A_d \hat{x} + B_d u + \text{correction}$$

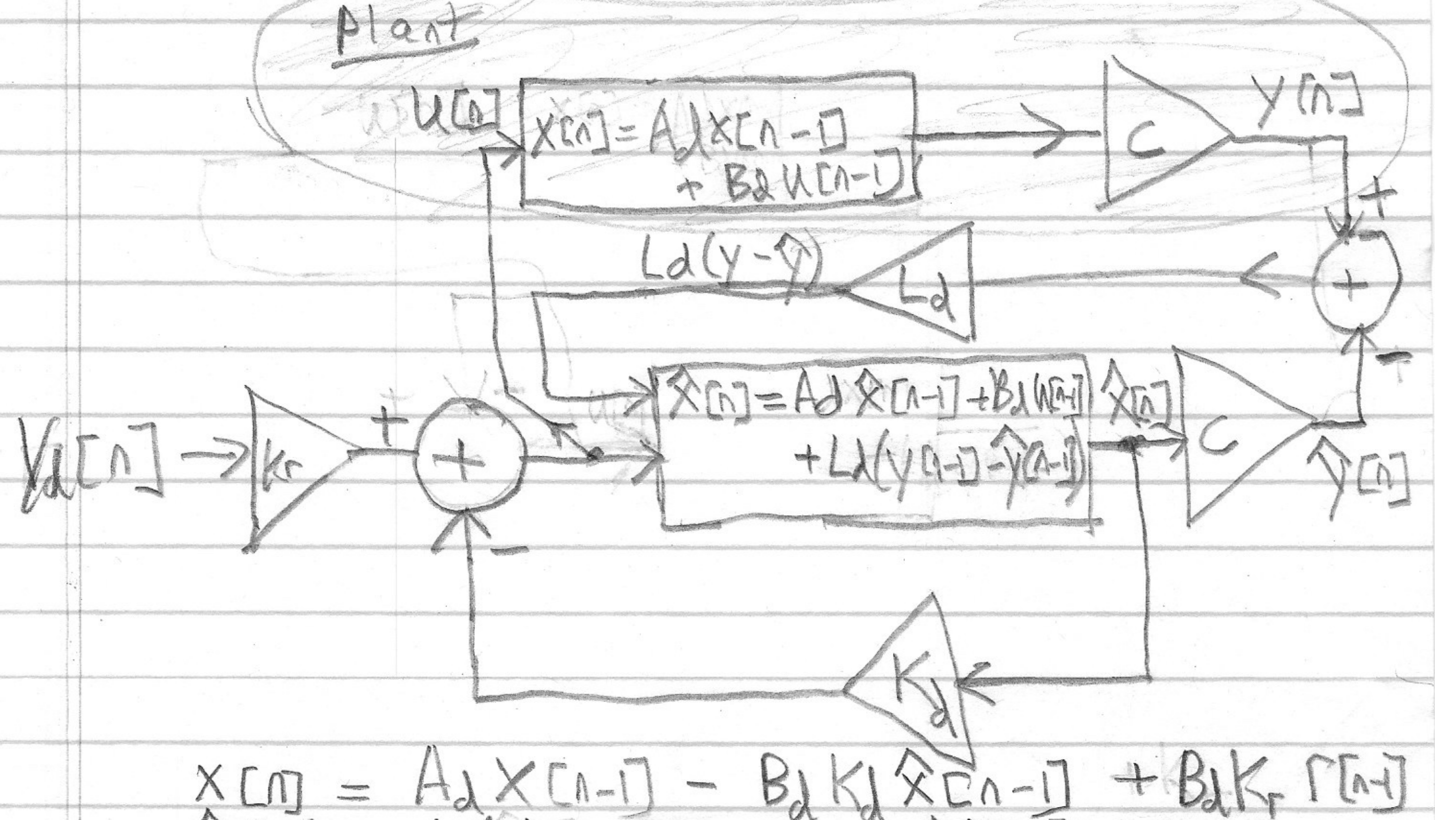
$$\hat{y} \Leftarrow C \hat{x}$$

Measured-State Feedback



$$x[n] = A_d x[n-1] + B_d u[n-1] = (A_d - B_d K_d) x[n-1] + B_d K_r y_d[n]$$

Observer-State Feedback



$$x[n] = A_d x[n-1] - B_d K_d \hat{x}[n-1] + B_d K_r r[n-1]$$

$$\hat{x}[n] = (A_d - B_d K_d - L_d C)\hat{x}[n-1] + L_d C x[n-1] + B_d K_r y_d[n-1]$$

Analyze Coupled System

Actual: $x[n] = A_d x[n-1] + B_d u[n-1]$

Estimate: $\hat{x}[n] = A_d \hat{x}[n-1] + B_d u[n-1] + L_d(y[n-1] - \hat{y}[n-1])$

$$e_{rr}[n] \equiv x[n] - \hat{x}[n]$$

$$\Rightarrow \hat{x}[n] = x[n] - e[n]$$

Subtracting

$$\underbrace{x[n] - \hat{x}[n]}_{e_{rr}[n]} = A_d \overbrace{\left(x[n-1] - \hat{x}[n-1]\right)}^{e_{rr}[n-1]} - L_d \underbrace{\left(y[n-1] - \hat{y}[n-1]\right)}_{C e_{rr}[n-1]}$$

$$e_{rr}[n] = A_d e_{rr}[n-1] - L_d C_d e_{rr}[n-1]$$

$$= (A_d - L_d C_d) e_{rr}[n-1]$$

$$\boxed{e_{rr} \text{ does } \underline{\text{not}} \text{ depend on } K_d!}$$

Coupled System Continued

Closed-Loop Control using Observer

$$x[n] = A_d x[n-1] + B_d\left(K_{r_d} y_d[n-1] - K_d \hat{x}[n-1]\right)$$

Since $\hat{x}[n-1] = x[n-1] - e[n-1]$

$$\Rightarrow x[n] = \left(A_d - B_d K_d\right) x[n-1] + B_d K_d e[n-1] + B_d K_{r_d} y_d[n-1]$$

$e[n] \equiv \hat{x}[n] - x[n]$

Compare to Closed loop measured-state

Assuming $u = K_{r_d} y_d - \underbrace{K_d x}_{\text{Measured State}}$

$$x_r[n] = \left(A_d - B_d K_d\right) x_r[n-1] + B_d K_{r_d} y_d[n-1]$$

x_r is state if **measured** state feedback is used

$$\left(x[n] - x_r[n]\right) = \left(A_d - B_d K_d\right)\left(x[n-1] - x_r[n-1]\right) + B_d K_d e[n-1]$$

x: State using observed state feedback

x_r: state using measured state feedback

Observer Feedback like "Real" State Feedback if $e \to 0$ fast

Relation between K_d & L_d

If $\mathrm{eig}(A_d - L_d C_d)$ are smaller than the $\mathrm{eig}(A_d - B_d K_d)$

Then State-Feedback with estimated state will perform like state-feedback with measured state.

Why not pick L_d such that

$$\max_i |\lambda_i(A_d - L_d C_d)| \simeq 0 ?$$

Because large values of L_d are bad!

Noise Example with (Measurement noise)

With noise

$$e_{rr}[n] = A_d\, e_{rr}[n-1] - L_d\left(y[n-1] - \hat{y}[n-1] + \underset{\overset{\uparrow}{\text{Noise}}}{W[n-1]}\right)$$

Suppose $W[n] = W_0$ (constant noise)

$$e_{rr}[n] = (A_d - L_d C_d)\, e_{rr}[n-1] + L_d W_0$$

Even in Best case Large L_d's mean high sensitivity to measurement Noise!

In steady-state $e_{rr}[n] = e_{rr}[n-1] = e_{rr}[\infty]$

$$\left(I - (A_d - L_d C_d)\right) e_{rr}[\infty] = L_d W_0$$

$$e_{rr}[\infty] = \left(I - (A_d - L_d C_d)\right)^{-1} L_d W_0$$

Best Case! If L_d's chosen for very fast decay

$$A_d - L_d C_d \simeq 0 \Rightarrow \left(I - (A_d - L_d C_d)\right)^{-1} \simeq I$$

and $err[\infty] \simeq L_d W_0$ Big L_d Big Errors