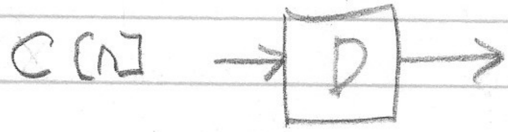
# 6.302 Poles, Zeros, Root-Locus, and the Partial Fraction Expansion



## Block Diagram

**Delay**

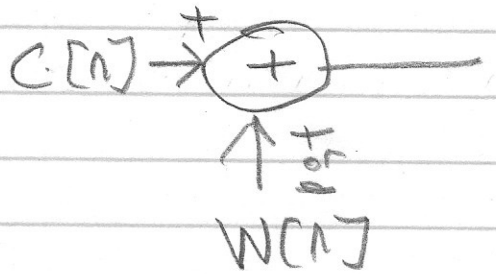



$x[n] = c[n-1]$


$X(z) = z^{-1}C(z)$

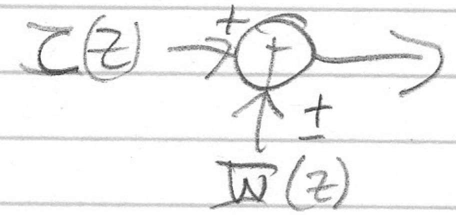
$C(z) = \sum_{n=-\infty}^{\infty} c[n] z^{-n}$ $\qquad$ $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

**Adder**



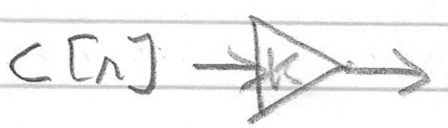

$x[n] = c[n] \pm w[n]$

linearity of transform

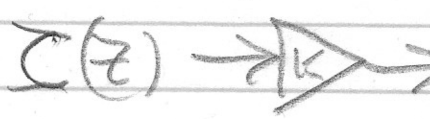



$X(z) = C(z) \pm W(z)$

**Gain**

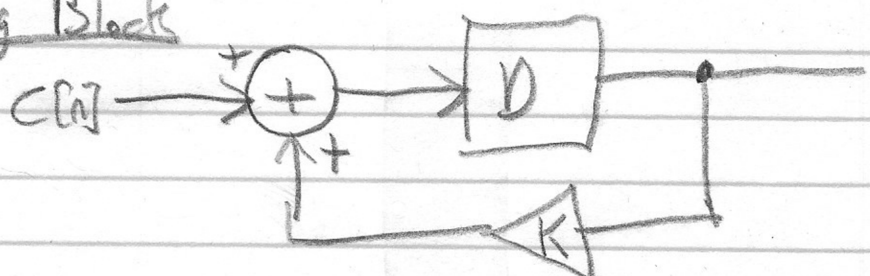



$x[n] = K\,c[n]$

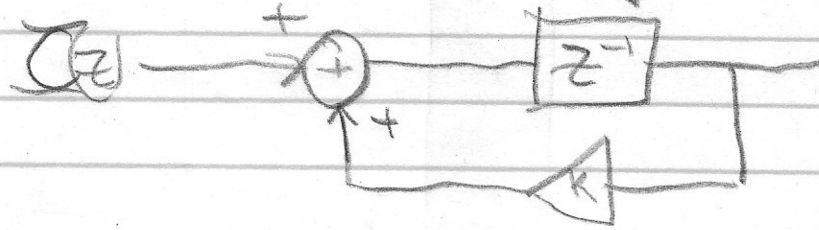



$X(z) = K\,C(z)$

**Building Block**




$x[n] = K x[n-1] + c[n-1]$




$X(z) = z^{-1}\left(K X(z) + C(z)\right)$

$X(z) = \dfrac{z^{-1}}{1-Kz^{-1}}\,C(z)$

# Systems of Difference Eqns

**Accel Control Ex** $v[n] = v[n-1] + \Delta T \, \gamma \, c[n-1]$

$$d[n] = d[n-1] + \Delta T \, v[n-1]$$

$$V(z) = \frac{\Delta T \, \gamma z^{-1}}{1 - z^{-1}} C(z)$$

$$D(z) = \frac{\Delta T z^{-2}}{1 - z^{-1}} V(z) = \underbrace{\frac{\Delta T^{2} \gamma z^{-2}}{(1 - z^{-1})^{2}}}_{H(z)} C(z)$$

**In General** $X(z) = H(z) C(z)$

# Blacks Formula

**P.D. Example** $c[n] = K_p\left(d_d[n] - d[n]\right)$

desired

$$+ K_d \left( (d_d[n] - d[n]) - (d_d[n-1] - d[n-1]) \right)$$

$$C(z) = \underbrace{\left(K_p + K_d(1 - z^{-1})\right)}_{K(z)} \left(D_d(z) - D(z)\right)$$

**In General** $X(z) = G(z) X_d(z)$

$$G(z) = \frac{K(z) H(z)}{1 + K(z) H(z)}$$

## P.D. ~~A~~ccel Control Ex

$$K(z)\,H(z) = \frac{z^3\left(K_p + K_d(1-z^{-1})\right)\Delta T^2 \gamma z^{-2}}{z^3\left(1-2z^{-1}+z^{-2}\right)}$$

Scale Factor

$$= K_0\,\frac{\left(\frac{K_p}{K_0}+\frac{K_d}{K_0}\right)\Delta T^2 \gamma z - \frac{K_d}{K_0}\Delta T^2 \gamma}{z^3 - 2z^2 + z}$$

## In General

$$K(z)\,H(z) = K_0\,\frac{n(z)}{d(z)}$$

poly in $z$ roots = zeros

gain scaling factor

poly in $z$ roots = poles

## Blacks Formula

$$K(z)\,H(z) = K_0\,\frac{n(z)}{d(z)}$$

$$G(z) = \frac{\frac{d(z)}{K_0}\;K_0\frac{n(z)}{d(z)}}{\frac{d(z)}{K_0}\left(1 + K_0\frac{n(z)}{d(z)}\right)} = \frac{n(z)}{\frac{1}{K_0}d(z) + n(z)}$$

## Root Locus Idea

Poles of $G(z)$ → $K_0 \to 0$ = poles of $K(z)H(z)$

Poles of $G(z)$ → $K_0 \to \infty$ = zeros of $K(z)H(z)$

In General (For any transfer function and input)

$$X(z) = H(z)\, C(z)$$

and

$$c[n] = A\lambda_{in}^{n} \quad n \geq 0$$
$$= 0 \quad n < 0$$

$$\Rightarrow \quad C(z) = A\sum_{n=0}^{\infty} \lambda_{in}^{n} z^{-n} = \sum_{n=0}^{\infty} (\lambda_{in} z^{-1})^{n} = \frac{A}{1-\lambda_{in}z^{-1}}$$

if $|z|$ big enough

If $\quad X(z) = H(z)\, C(z) = \dfrac{n(z)}{d(z)} \cdot \dfrac{A}{1-\lambda_{in}z^{-1}}$

$$= \underbrace{\frac{n(z)}{(1-\lambda_1 z^{-1})(1-\lambda_2 z^{-1})\cdots(1-\lambda_L z^{-1})}}_{} \cdot \frac{A}{(1-\lambda_{in} z^{-1})}$$

where $d(z)$ is an $L^{th}$ order poly

$$1 - \sum_{\ell=1}^{L} a_\ell z^{-\ell} = d(z)$$

no repeated poles!

And $\quad \lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_L \neq \lambda_{in}$

Then $\quad x[n] = \displaystyle\sum_{\ell=1}^{L} \alpha_\ell \lambda_\ell^{n} + H(z)\Big|_{z=\lambda_{in}} A\lambda_{in}^{n}$

Result holds in general $X(z) = G(z) X_d(z)$ 4A

Ex $$K(z)H(z) = \frac{\left(K_p(\Delta T)^2\gamma + K_d \Delta T \gamma (1-z^{-1})\right)z^{-1}}{1-2z^{-1}+z^{-2}}$$

$$D(z) = G(z)\, D_d(z)$$

$$G(z) = \frac{\left(K_p + \frac{K_d}{\Delta T}\right)(\Delta T)^2 \gamma z - \frac{K_d}{\Delta T}(\Delta T)^2\gamma}{\left(z^3 - 2z^2 + z + \left(K_p + \frac{K_d}{\Delta T}\right)(\Delta T)^2 \gamma z - \left(\frac{K_d}{\Delta T}\right)(\Delta T)^2 \gamma\right)}$$

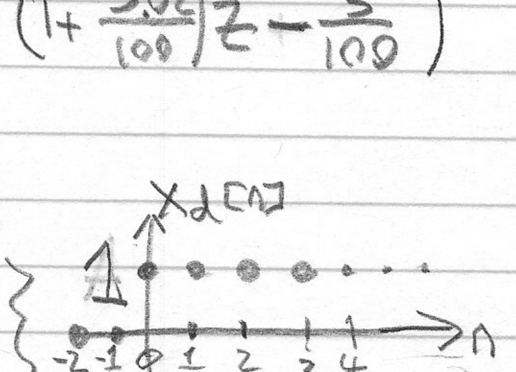
Let $K_p = 20$ $K_d = 5$ $\Delta T = 1e^{-3}$ $\gamma = 10$

$$G(z) = \frac{\overbrace{(20+5000)(1e^{-6})10}^{5.02e^{-2}}\, z - 5e^{-2}}{\left(z^3 - 2z^2 + 1.0502 z - 0.05\right)}$$

$$= \frac{\frac{5.02}{100} z - \frac{5}{100}}{\left(z^3 - 2z^2 + \left(1 + \frac{5.02}{100}\right) z - \frac{5}{100}\right)}$$

If $X_d$ is a unit step

$$x_d[n] = 1 \quad n \geq 0$$
$$= 0 \quad n < 0$$




$$X_d(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

4B

$$G(z) = \frac{\overbrace{\frac{5.02}{100}}^{K_0}\left(1 - \overbrace{\frac{5}{5.02}}^{B} z^{-1}\right)}{(1-\lambda_1 z^{-1})(1-\lambda_2 z^{-1})(1-\lambda_3 z^{-1})}$$

$$\lambda_1 \simeq 0.996 \quad \lambda_2 \simeq 0.952 \quad \lambda_3 \simeq 0.053$$

$$X(z) = G(z)\, X_d(z) = \frac{K_0(1-Bz^{-1})}{(1-\lambda_1 z^{-1})(1-\lambda_2 z^{-1})(1-\lambda_3 z^{-1})} \cdot \frac{A}{(1-z^{-1})}$$

$$X_d[n] = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \alpha_3 \lambda_3^n + \alpha_{in} 1^n$$

Use PFE

Example $\alpha_1$

$$\alpha_1 = G(z)(1-\lambda_1 z^{-1})\Big|_{z=\lambda_1} = \left.\cancel{(1-\lambda_1 z^{-1})} \frac{K_0(1-Bz^{-1})A}{\cancel{(1-\lambda_1 z^{-1})}(1-\lambda_2 z^{-1})(1-\lambda_3 z^{-1})(1-z^{-1})}\right|_{z=\lambda_1}$$

$$= \frac{K_0(1-B/\lambda_1)A}{\left(1-\frac{\lambda_2}{\lambda_1}\right)\left(1-\frac{\lambda_3}{\lambda_1}\right)\left(1-\frac{1}{\lambda_1}\right)}$$

$$\boxed{X_d[n] = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \alpha_3 \lambda_3^n + \alpha_{in} 1^n}$$

4C

$$\alpha_{1n} = \cancel{(1-z^{-1})}\, G(z) \cdot \frac{1}{\cancel{1-z^{-1}}} \Bigg|_{z=1}$$

$$= G(z) \Big|_{z=1}$$

$$= \left. \frac{\frac{5.02}{100} z - \frac{5}{100}}{\left(z^3 - 2z^2 + \left(1 + \frac{5.02}{100}\right) z - \frac{5}{100}\right)} \right|_{z=1}$$

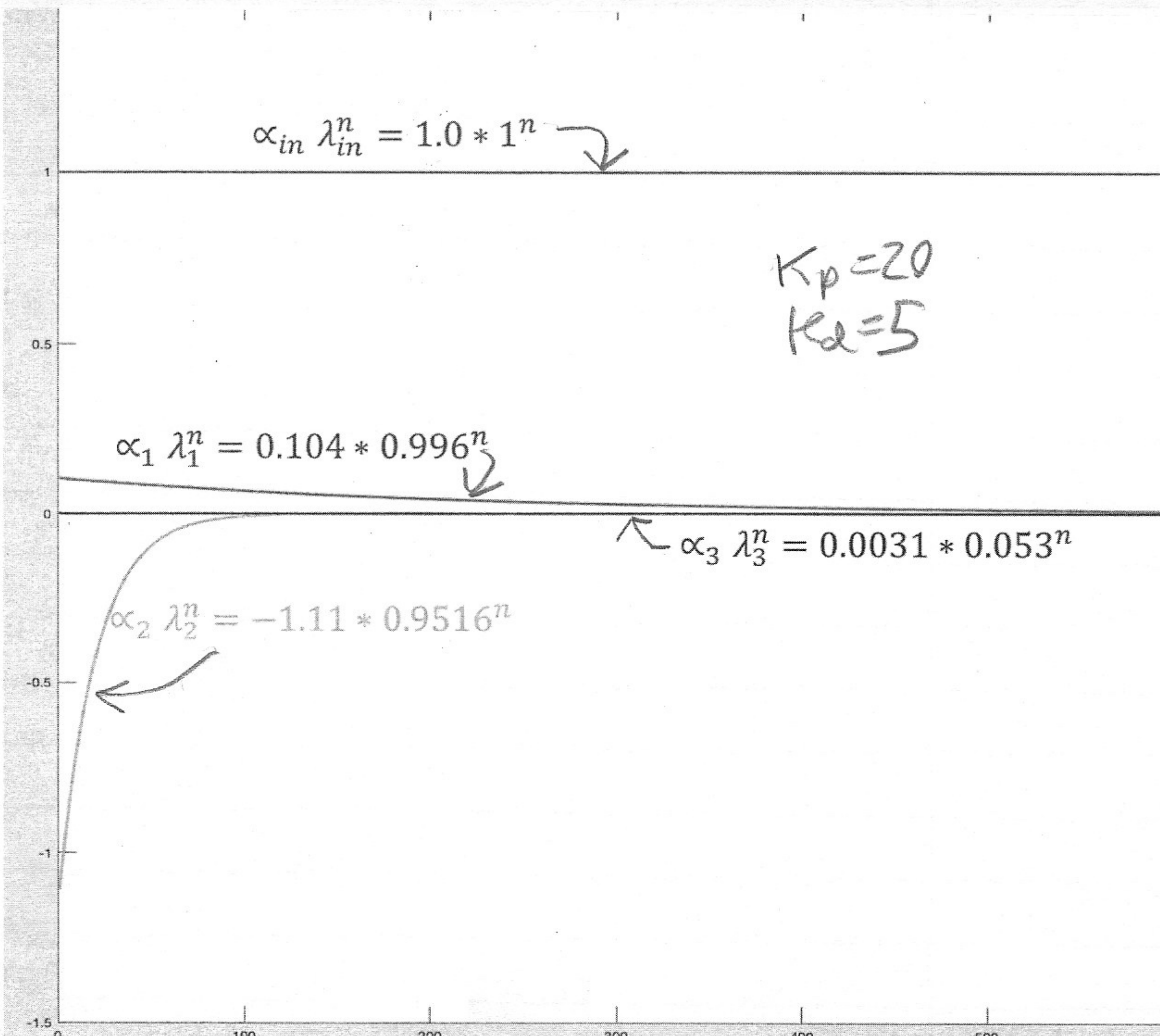
$$= \frac{\frac{0.02}{100}}{\frac{0.02}{100}} = 1$$

$$X_d[n] = 0.10 \cdot 0.996^n - 1.11 \cdot (0.952)^n + .0031 \cdot (0.053)^n + 1^n$$

$$X_d[0] \simeq 0$$

(4D)

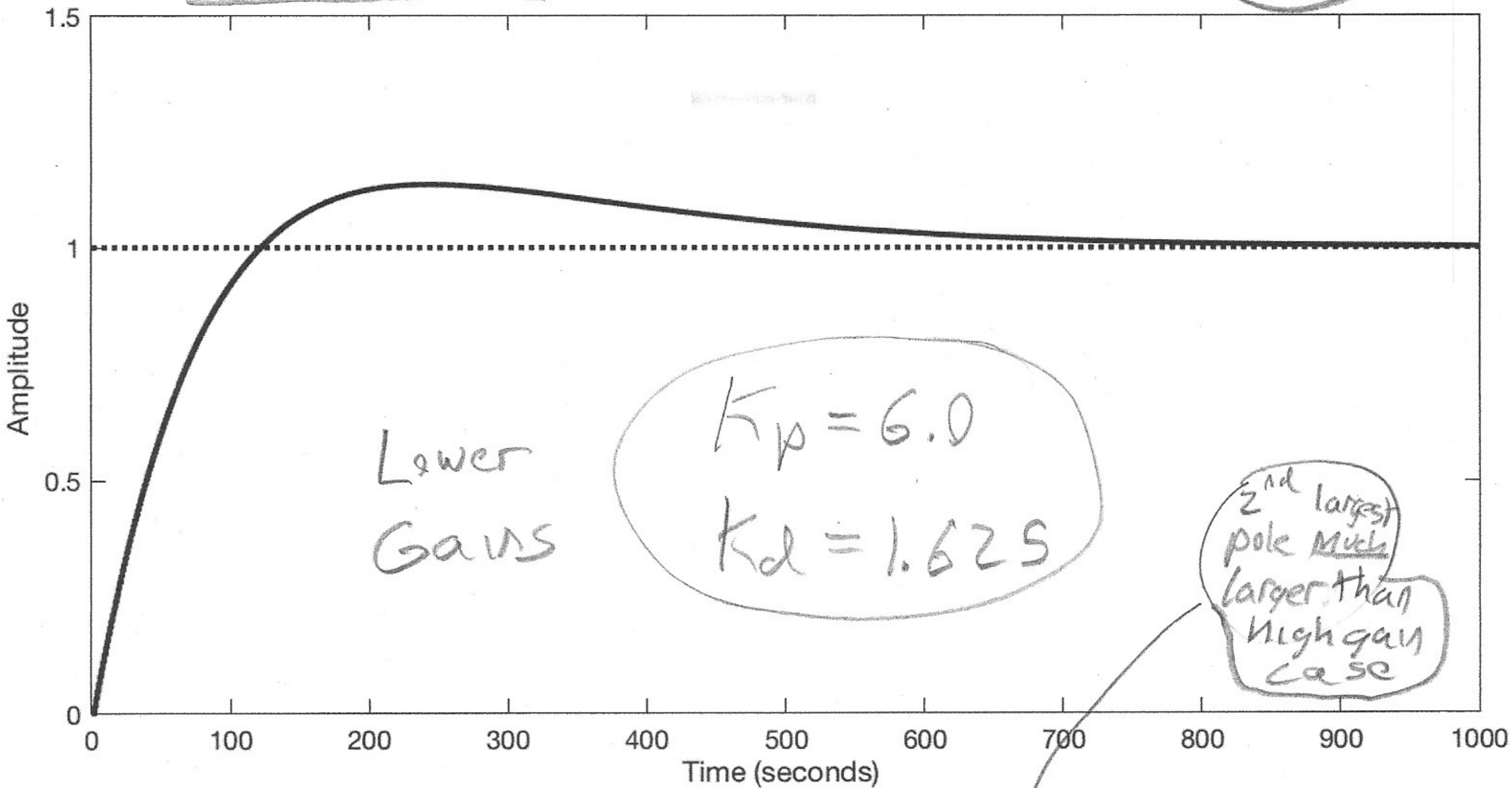
# Note Different Terms




Maximum Pole Does not Tell Whole Story!

Maximum Pole Not Whole Story

4E




Lower Gains

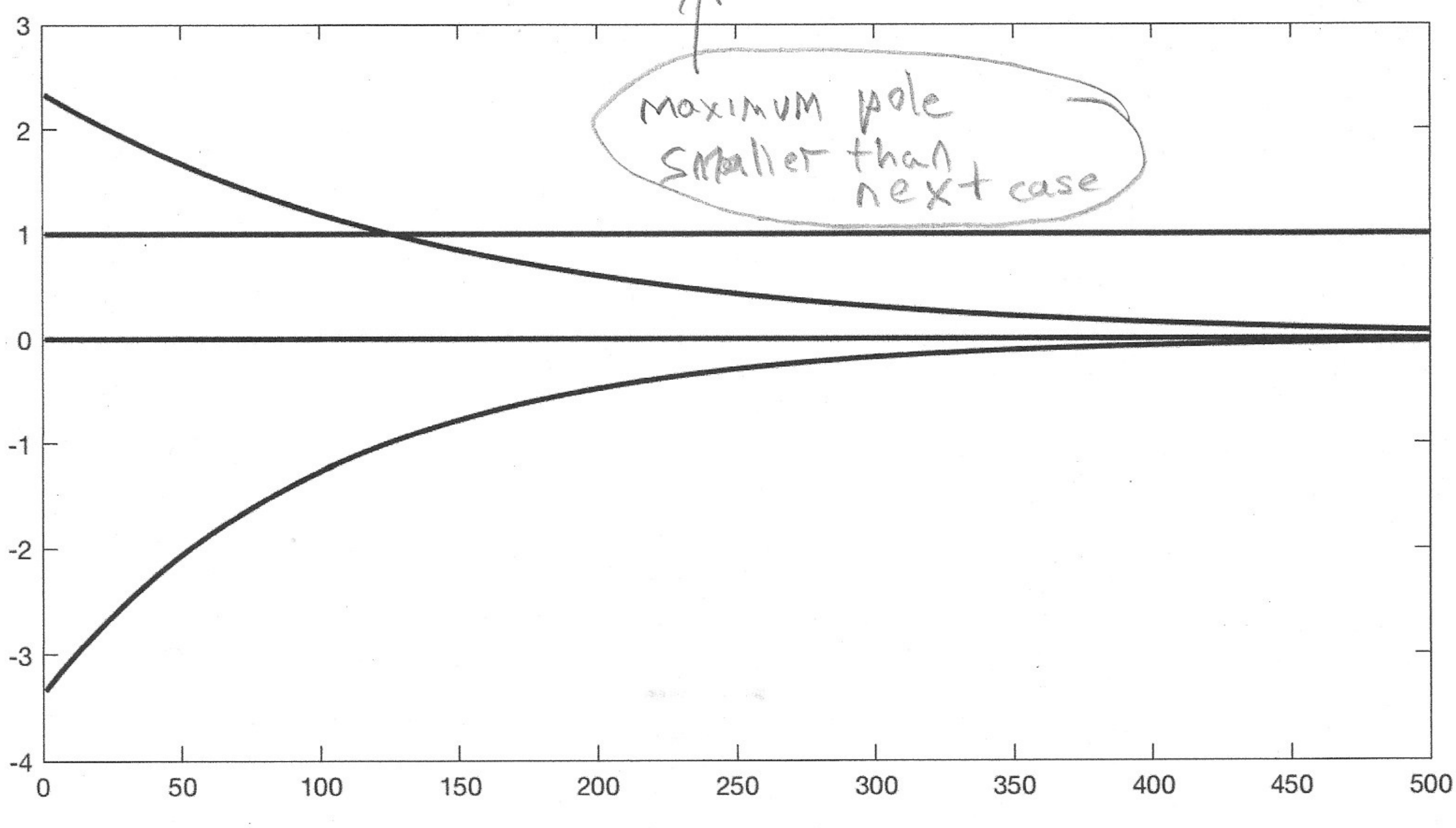
$K_p = 6.0$

$K_d = 1.625$

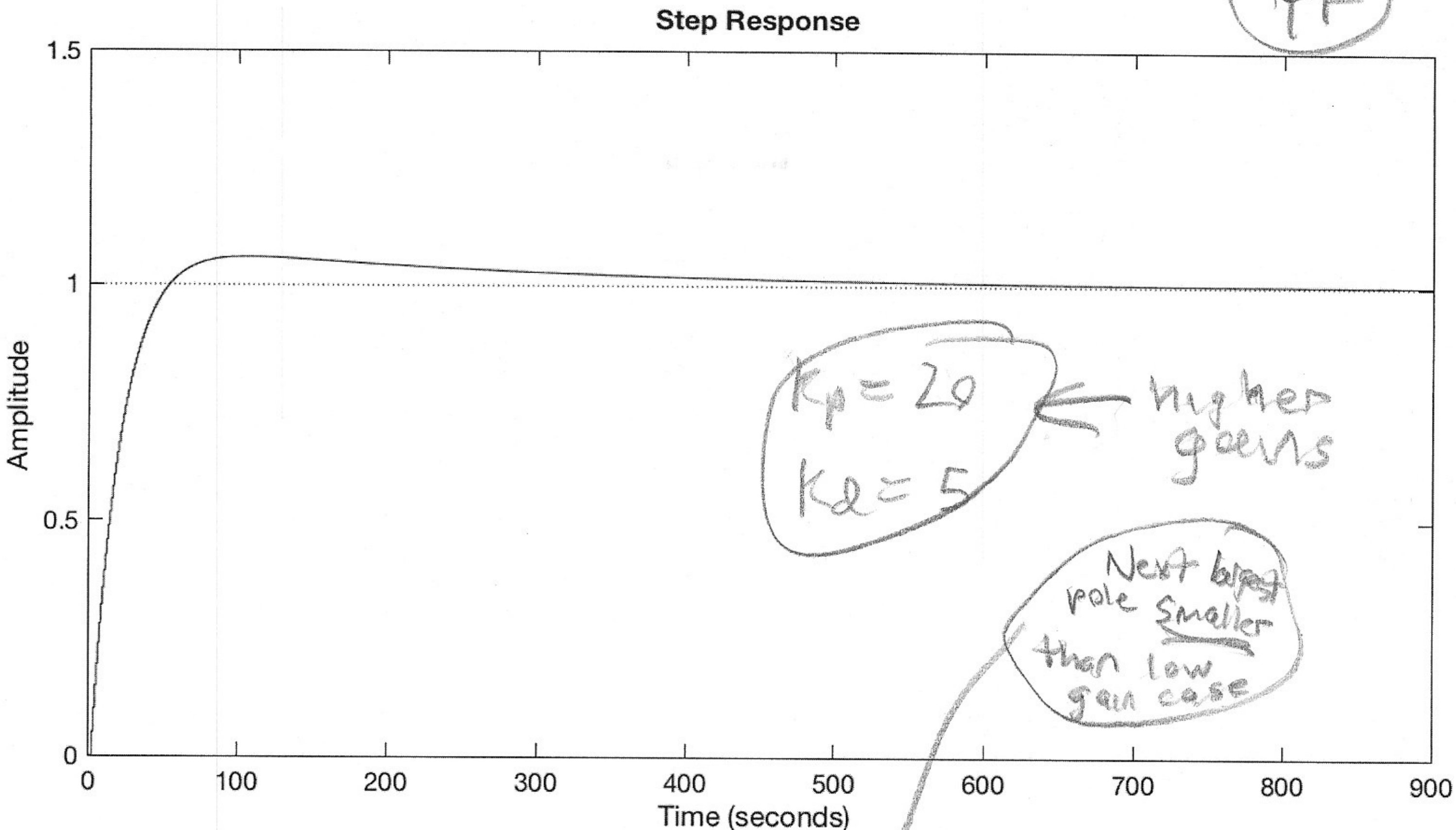
$2^{nd}$ largest pole much larger than high gain case

$G(z)$ poles $= 0.993, 0.990, 0.0165$

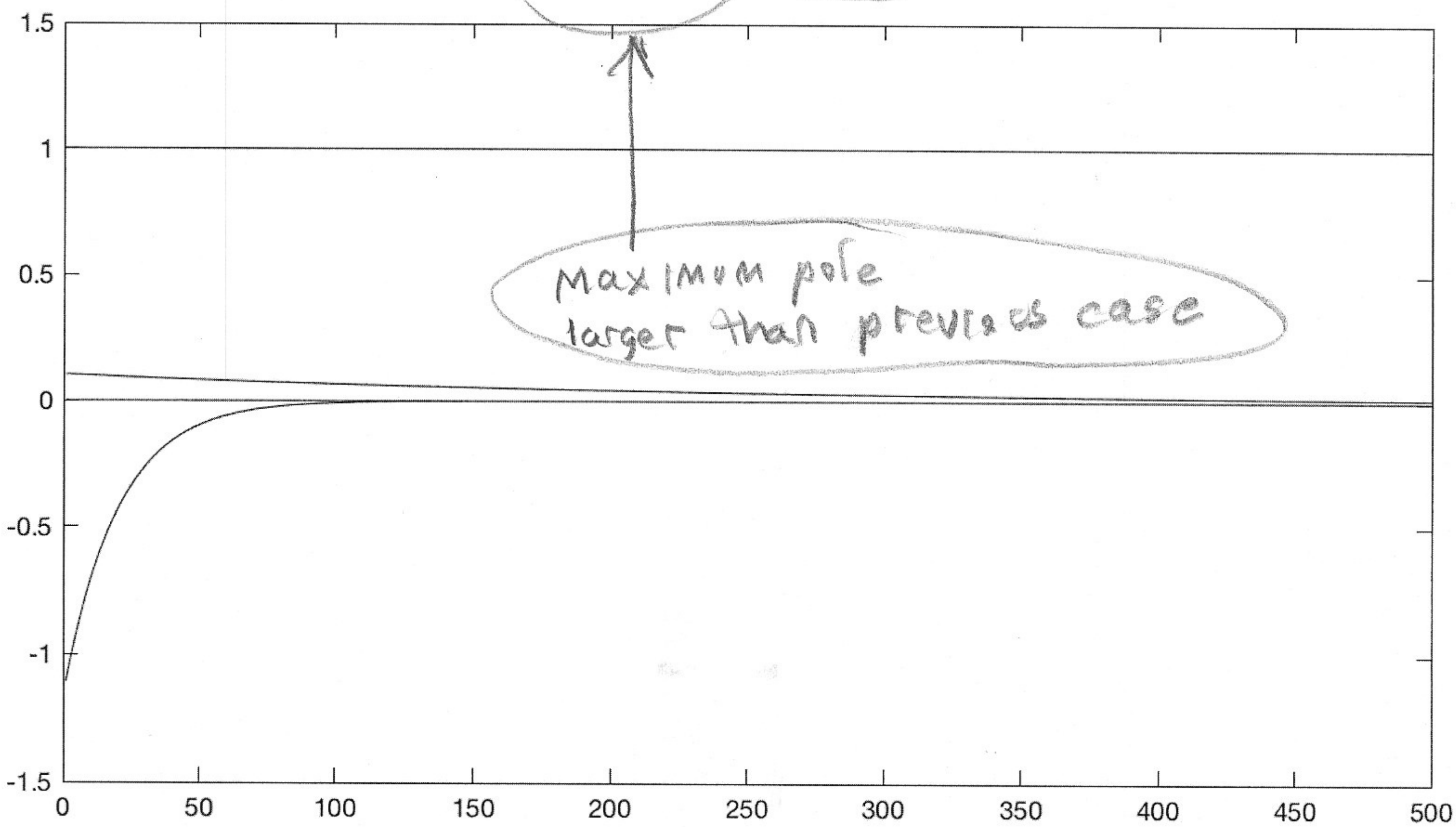
maximum pole smaller than next case



4F




G(z) poles 0.996, 0.951, 0.053



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## Why does PFE formula Hold

$$H(z)X(z) = \frac{n(z)}{d(z)} \cdot \frac{A}{1-\lambda_{in}z^{-1}} = \frac{\alpha_1}{1-\lambda_1 z^{-1}} + \cdots + \frac{\alpha_L}{1-\lambda_L z^{-1}} + \frac{\alpha_{in}A}{1-\lambda_{in}z^{-1}}$$

$$\alpha_i = (1-\lambda_i z^{-1})\, H(z) \left(\frac{A}{1-\lambda_{in}z^{-1}}\right)\Bigg|_{z=\lambda_i}$$

$$\alpha_{in} = \cancel{(1-\lambda_{in}z^{-1})}\, H(z) \frac{A}{\cancel{(1-\lambda_{in}z^{-1})}}\Bigg|_{z=\lambda_{in}}$$

<u>Cross-Multiply and See!</u>

<u>So</u> if $H(z)\Big|_{z=\lambda_{in}} \cong 0$ $\left(\text{i.e. } n(z)\Big|_{z=\lambda_{in}} \simeq 0\right)$

Small response to input! (Zero)

<u>But</u> if $\lambda_{in}$ is s.t. $d(z)\Big|_{z=\lambda_{in}} \simeq 0$

<u>Huge response</u> (Pole)