1

DYNAMIC EQUATIONS AND THEIR SOLUTIONS FOR SIMPLE CIRCUITS

1.0 Introduction

The goal of this first chapter is twofold: to remind the reader of the basic principles of electrical circuit analysis, and to formulate these principles in appropriate ways so that we can develop them further in the chapters to come. Circuits (or networks) are, of course, arrangements of *interconnected elements*. But the word "circuit" can refer either to a *real* reticulated structure that we build in the laboratory out of elements such as resistors, capacitors, and transistors, interconnected by wires or printed-circuit busses, or it can refer to a *model* that we develop abstractly. For the most part in this book, we shall be discussing circuits in this latter sense (although we should never forget for long that, as engineers, we are interested in circuit models primarily as aids to the design and understanding of real systems). Our first task is therefore to define what we choose the words "interconnected" and "elements" to imply as abstractions. The circuit model then becomes a graphic way of specifying a set of *dynamic equations* that describe the behavior of the circuit. But such a description is usually only implicit; in the latter part of this chapter, we shall explore how simple dynamic equations can be solved to yield an explicit specification of the circuit response to simple stimuli. A goal of later chapters will be to extend and refine the ideas of this chapter into a collection of powerful tools for the analysis and design of the complex systems that characterize modern engineering practice.

1.1 Constitutive Relations for Elements

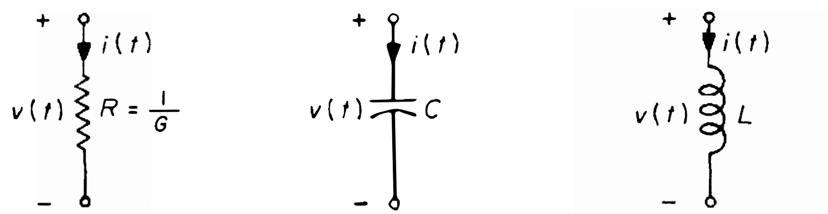
In models of electrical circuits, the elements or branches are characterized by equations (called *constitutive relations*) relating branch voltages and currents.* The simplest abstract electrical elements are the linear resistors, capacitors,

*It is perhaps useful to point out that most of the ideas to be studied in this book also apply to a variety of other situations in which the important dynamic variables are *efforts* and *flows* (e.g., mechanical forces and velocities, temperatures and heat flows, chemical potentials and reaction rates). In addition, many models proposed in the social and biological sciences are described by equations similar to those we shall be investigating. Some texts go to elaborate lengths to formalize these *analogies*. We are not convinced such efforts are worthwhile, since most students make the necessary translations easily. Examples from non-electrical applications are scattered throughout the problems in this book.

inductors, and ideal sources described in Figure 1.1–1. Note that the reference directions for current, $i(t)$, and voltage, $v(t)$, in the constitutive relations are always *associated* as shown; that is, the positive direction for $i(t)$ is selected to be through the element from the positive reference terminal for $v(t)$ towards the negative terminal. The units of $i(t)$ and $v(t)$ are *amperes* and *volts* respectively.

Circuit elements may have more than two terminals. Perhaps the most important abstract multiterminal element is the *ideal controlled* (or *dependent*) *source*, of which there are four basic types as shown in Figure 1.1–2. Ideal controlled sources arise most commonly as idealizations for such active elements as transistors and op-amps in their linear regions. The *ideal op-amp*, for example, is an important special case of an ideal voltage-controlled voltage source obtained

Resistor | **Capacitor** | **Inductor**



$v(t) = Ri(t)$
or
$i(t) = Gv(t)$

Units of R are ohms (Ω)

$i(t) = C\dfrac{dv(t)}{dt}$
or
$v(t) - v(0) = \dfrac{1}{C}\displaystyle\int_0^t i(\tau)\,d\tau$

Units of C are farads (F)

$v(t) = L\dfrac{di(t)}{dt}$
or
$i(t) - i(0) = \dfrac{1}{L}\displaystyle\int_0^t v(\tau)\,d\tau$

Units of L are henries (H)

Ideal Independent Voltage Source | **Ideal Independent Current Source**

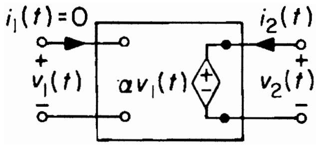
$v(t) = v_0(t)$ independent of $i(t)$

$i(t) = i_0(t)$ independent of $v(t)$

Figure 1.1–1. Simple linear 2-terminal lumped electrical elements and their constitutive relations. Note how current sources are distinguished from voltage sources; the orientation of the arrow or of the + and − signs inside the source symbol identifies the positive reference direction for the source quantity.

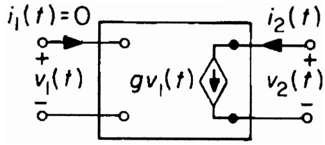
in the limit as the gain, α, becomes very large. It has its own special symbol as shown in Figure 1.1–3. The ideal op-amp is always used in a feedback circuit that achieves a finite output voltage by driving the input voltage difference, $\Delta v(t)$, nearly to zero. Other examples of multiterminal elements, such as coupled coils and transformers, transducers, and gyrators, are discussed in the problems at the end of Chapter 3.

Voltage-Controlled Voltage Source



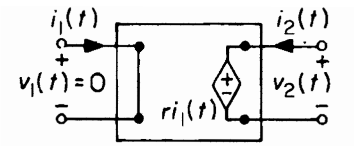
$v_2(t) = \alpha v_1(t)$ independent of $i_2(t)$

Voltage-Controlled Current Source



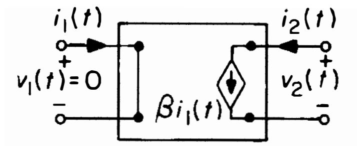
$i_2(t) = gv_1(t)$ independent of $v_2(t)$

Current-Controlled Voltage Source



$v_2(t) = ri_1(t)$ independent of $i_2(t)$

Current-Controlled Current Source



$i_2(t) = \beta i_1(t)$ independent of $v_2(t)$

Figure 1.1–2. Ideal controlled (dependent) sources and their constitutive relations. Note that diamonds are used to identify dependent sources and circles to identify independent sources.

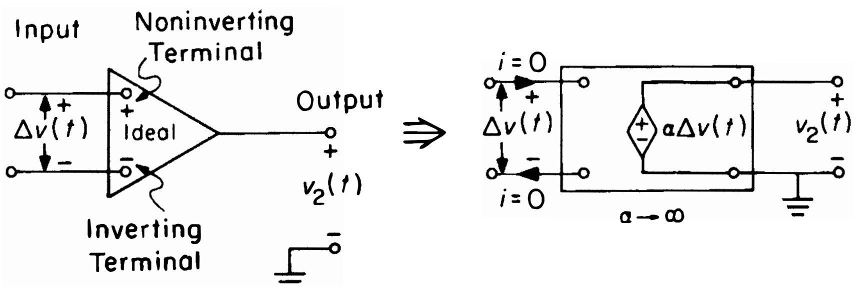


Figure 1.1–3. Ideal op-amp.

The ideal 2-terminal elements (excluding the independent sources) shown in Figures 1.1–1, 2, 3 are *linear*; that is, their dynamic variables satisfy the

*SUPERPOSITION (LINEARITY) PRINCIPLE:**

If $i'(t)$ and $v'(t)$ are *any* pair of functions that satisfy the constitutive relation of an element, and if $i''(t)$ and $v''(t)$ are any other pair satisfying the same constitutive relation, then the element is said to obey the *superposition principle* (or equivalently to be *linear*) if the pair of functions $i(t) = ai'(t) + bi''(t)$ and $v(t) = av'(t) + bv''(t)$ also satisfy the constitutive relation for any choices of the constants a and b.

The 2-terminal elements described in Figures 1.1–1, 2, 3 (again excluding independent sources) also satisfy the

*TIME-INVARIANCE PRINCIPLE:**

If $i(t)$ and $v(t)$ are *any* pair of functions that satisfy the constitutive relation of the element, then the element is *time-invariant* if $i(t-T)$ and $v(t-T)$ also satisfy the constitutive relation for *any* value of T.

Circuits composed entirely (except for independent sources) of linear time-invariant elements are examples of *linear time-invariant (LTI) systems*. The concept of an LTI system is more general, however, as we shall see in later chapters.

The most common *non*-linear element is probably the *diode*, whose idealized constitutive relation is shown in Figure 1.1–4. Also shown in this figure is the constitutive relation for what is surely the most important time-varying element—the *switch*. Circuits containing non-linear or time-varying elements are extremely useful. (See Problems 1.13–1.15 for some examples.) But the analysis of such circuits is often difficult. There are relatively few general principles or techniques for studying the behavior of non-linear circuits; each new circuit is likely to present a new analytical problem. In contrast, the theory of LTI systems consists of a rich collection of theorems, concepts, and methods providing powerful tools for understanding and design. As a result, the necessary non-linearities in practical electronic circuits are often restricted to isolated locations interconnected by LTI systems. Such an arrangement may vastly simplify the analysis while providing enough design freedom to achieve the desired dynamic effects. When such isolation and localization are impossible, as for example in some high-speed integrated circuits, the design process may reduce to employing numerical methods to study the performance of the device as various parameters are systematically varied. Computerized circuit simulation programs intended for this purpose have been developed, but the wide availability of such simulation

*The extension of these definitions to multiterminal elements is straightforward. For other examples of 2-terminal elements that are or are not linear and/or time-invariant, see Exercise 1.1.

programs has not eliminated the need to understand the mathematics of LTI systems, which remains a powerful *language* in terms of which complex system behavior can be discussed.

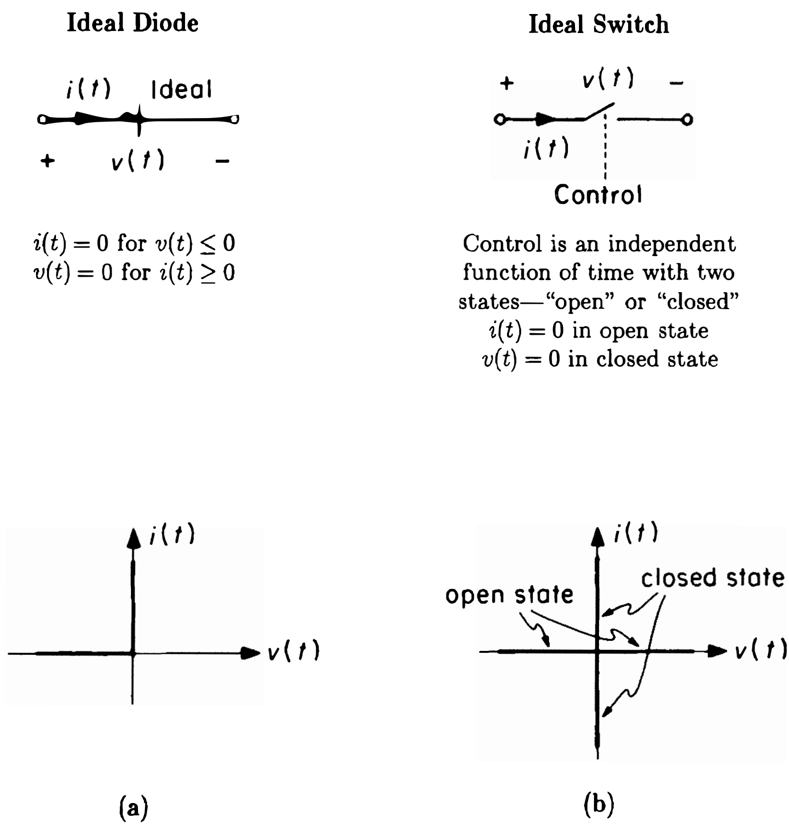


Figure 1.1–4. The ideal diode (a) and the ideal switch (b).

1.2 Interconnection Constraints: Kirchhoff's Laws

In addition to the constraints imposed by the constitutive relations of the branch elements, the branch voltages and currents in electric circuits are further constrained by the two fundamental laws portrayed in Figure 1.2–1.

KIRCHHOFF'S CURRENT LAW (KCL):
The algebraic sum of the currents entering any circuit node is zero. (More generally, the sum of the currents passing inward through any network cut set must equal zero. A *cut set* is any set of branches which, if cut, would divide the circuit into two parts.)

KIRCHHOFF'S VOLTAGE LAW (KVL):
The algebraic sum of the directed voltage drops around any circuit mesh is zero. (More generally, the sum of the voltage drops around any closed path in the circuit must equal zero.)

Both laws follow from Maxwell's equations* provided that the circuit is so designed and the variables are changed sufficiently slowly that all significant electromagnetic energy is stored inside the "elements" rather than in the spaces between the elements; the energy storage can then be described as *lumped*. For circuits of bench-top size composed of typical lumped elements, Kirchhoff's Laws are a good approximation for signal frequencies less than a few tens of megahertz.

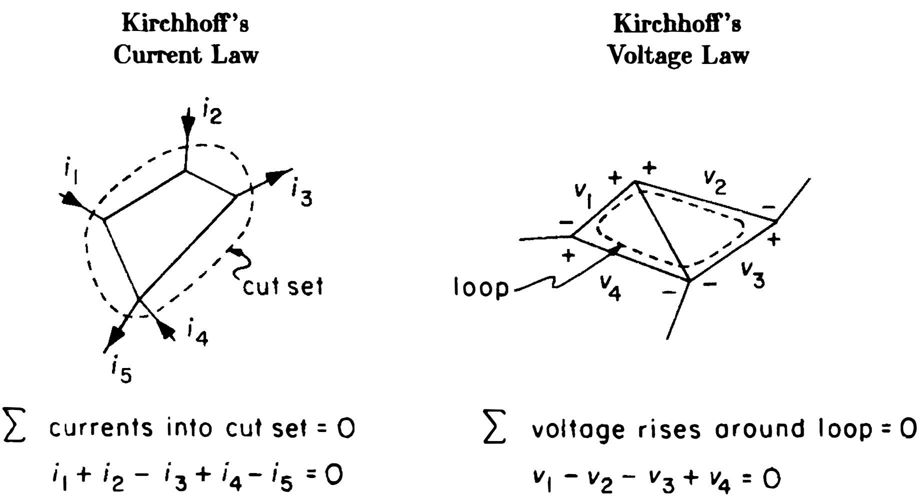


Figure 1.2–1. Kirchhoff's Laws.

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*For a careful investigation of the conditions under which Kirchhoff's Laws are valid, see, for example, R. M. Fano, L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces* (New York, NY: John Wiley, 1960).

1.3 Dynamic Equations in Node and State Form

Together, Kirchhoff's Laws and the constitutive relations provide a set of $2N$ independent equations for the N voltages and N currents associated with the N branches of a circuit. The formulation and solution of these dynamic equations* for a circuit is, however, usually much simplified by employing one or another of several special procedures that substantially reduce the number of unknowns. Two such special procedures—leading to what are called *node equations* or *state equations*—are particularly important in applications. The first special procedure is the

NODE EQUATIONS PROCEDURE:

1. Pick a *reference node*. The resulting equations will usually be simplest if the chosen node is the one that is common to the largest number of voltage sources and/or the largest number of branches.

2. Assign a *node voltage variable* to every other node, except that only one of two nodes connected by an ideal voltage source (whether independent or dependent) need be assigned a node voltage variable. (In particular, we do not need to assign a node voltage variable to any node connected to the reference node by a chain of one or more ideal voltage sources.) The number of assigned node voltage variables is thus one less than the number of nodes minus the number of ideal voltage sources.† Each node voltage variable measures the voltage of the corresponding node with respect to the reference node.

3. Write a KCL equation in terms of node voltage variables at each node to which such a variable is assigned. (If one or more ideal voltage sources are connected to the node, write the KCL equation for a cut set enclosing the desired node and the voltage sources, as shown in Example 1.3–1.)

The node equations procedure thus leads to as many equations and unknowns as there are assigned node voltage variables. In general, this number is very much less than twice the number of branches. Once the node voltages are known, any desired branch voltages or currents can usually be found quite easily.

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*In much of the electrical engineering literature these are called *equilibrium equations*, but this seems a rather inappropriate label since a circuit is rarely in equilibrium in a mechanical or thermodynamical sense. In physics the analogous equations are called *dynamic equations* or *equations of motion*, and we shall adopt the former term.

†If the network contains a loop of voltage sources (or a cut set of current sources), Kirchhoff's Laws imply that the values of these sources are not independent; one source can thus be deleted without modifying the behavior of the circuit. This rule for the number of independent node variables assumes that such deletion has been carried out.

Example 1.3–1

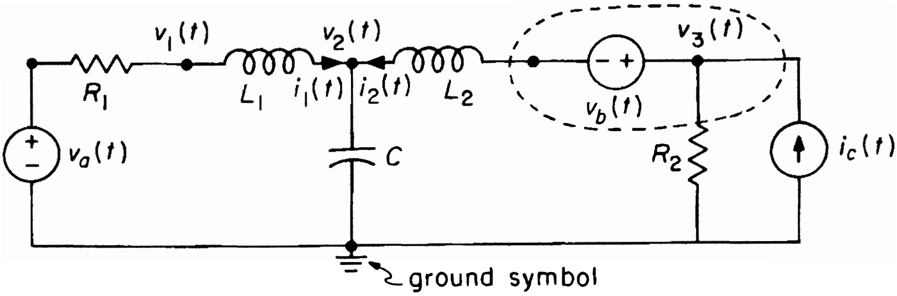

Figure 1.3–1. Circuit to illustrate the node equations procedure.

Following the procedure above for the circuit of Figure 1.3–1, we write the node equations as follows:

1. As our reference node we pick the one marked by the ground symbol. This node is chosen because four branches including one voltage source join there.
2. We assign three node voltage variables as shown. The circuit has six nodes; note that 6 nodes − 2 voltage sources − 1 = 3 independent node voltages.
3. We then write KCL equations at these three labelled nodes:

   i) For the currents leaving the node labelled $v_1(t)$:

$$\frac{v_1(t)-v_a(t)}{R_1}+\frac{1}{L_1}\int_0^t [v_1(\tau)-v_2(\tau)]\,d\tau+i_1(0)=0\,. \tag{1.3–1}$$

   ii) For the currents leaving the node labelled $v_2(t)$:

$$\frac{1}{L_1}\int_0^t [v_2(\tau)-v_1(\tau)]\,d\tau-i_1(0)+C\frac{dv_2(t)}{dt}+\frac{1}{L_2}\int_0^t [v_2(\tau)-v_3(\tau)+v_b(\tau)]\,d\tau-i_2(0)=0\,. \tag{1.3–2}$$

   iii) For the currents leaving the cut set defined by the dotted loop in Figure 1.3–1, enclosing the node labelled $v_3(t)$ and the voltage source $v_b(t)$:

$$\frac{1}{L_2}\int_0^t [v_3(\tau)-v_b(\tau)-v_2(\tau)]\,d\tau+i_2(0)+\frac{v_3(t)}{R_2}=i_c(t)\,. \tag{1.3–3}$$

The result is three simultaneous integro-differential equations in the three unknown node voltages, $v_1(t)$, $v_2(t)$, and $v_3(t)$.

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Example 1.3–2

As a second example of the node equations procedure, consider the circuit of Figure 1.3–2.

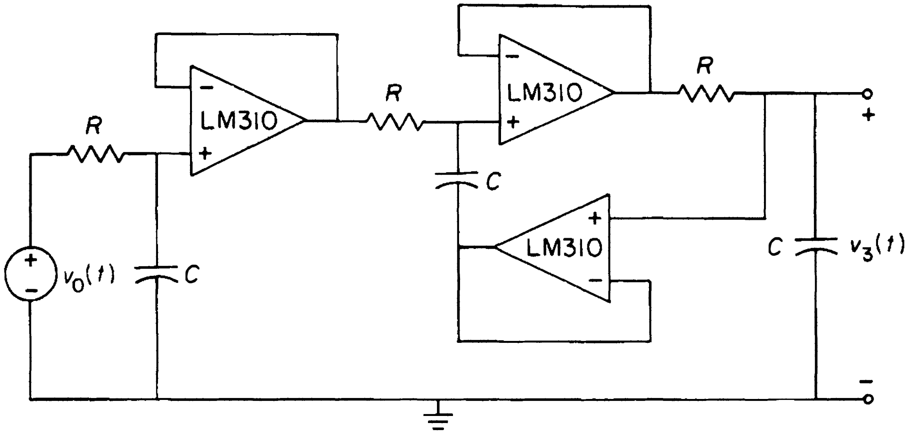

Figure 1.3–2. Another circuit to illustrate the node equations procedure.

If the op-amps are ideal, each voltage follower in Figure 1.3–2 can be replaced by the unit-gain controlled source shown in Figure 1.3–3.

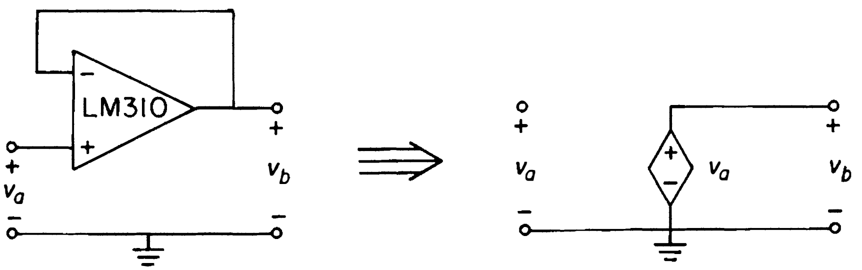

Figure 1.3–3. Controlled source equivalent of a voltage follower.

The result is the equivalent circuit of Figure 1.3–4, which contains three dependent voltage sources. (Note that the input current of an ideal voltage follower is zero. A voltage follower thus acts as a *buffer* or *isolator*. Despite the fact that its output voltage is equal to its input voltage, a voltage follower cannot in general be replaced by a wire between input and output without altering circuit behavior.)

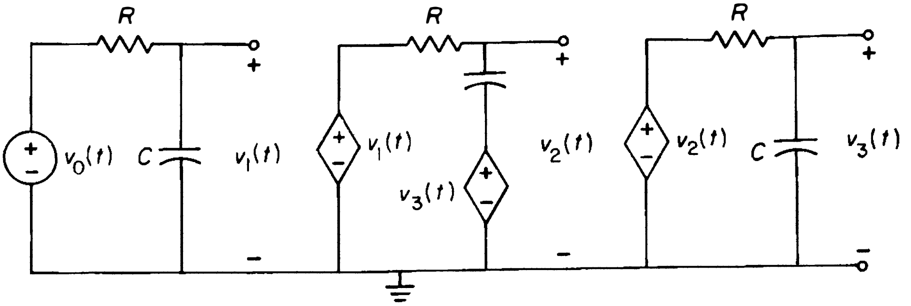

Figure 1.3–4. Equivalent circuit of Figure 1.3–2.

The circuit of Figure 1.3–4 has 8 nodes and 4 voltage sources; hence $8-4-1=3$ node voltage variables, $v_1(t)$, $v_2(t)$, and $v_3(t)$, are required. The most appropriate choice of reference node is, again, the real circuit ground, since it is a junction of 6 branches and is common to all the voltage sources. Writing KCL equations at each variable node yields three equations in three unknowns:

$$C\frac{dv_1(t)}{dt}+\frac{1}{R}[v_1(t)-v_0(t)]=0 \tag{1.3–4}$$

$$C\frac{d}{dt}[v_2(t)-v_3(t)]+\frac{1}{R}[v_2(t)-v_1(t)]=0 \tag{1.3–5}$$

$$C\frac{dv_3(t)}{dt}+\frac{1}{R}[v_3(t)-v_2(t)]=0\,. \tag{1.3–6}$$

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The second special procedure for writing dynamic equations is the

STATE EQUATIONS PROCEDURE:

1. Replace each inductor L_j temporarily by an ideal current source of value $i_j(t)$, and each capacitor C_k by an ideal voltage source of value $v_k(t)$.
2. Solve the resulting circuit, consisting of resistors and sources only, for the voltages $v_j(t)$ (across the current sources replacing the inductors) and the currents $i_k(t)$ (through the voltage sources replacing the capacitors). For an LTI circuit, this will yield a set of equations of the form $v_j(t)$ (or $i_k(t)$) equals a weighted sum of inductor currents, $i_j(t)$, capacitor voltages, $v_k(t)$, and independent source quantities.
3. Replace $v_j(t)=L_j\frac{di_j(t)}{dt}$ and $i_k(t)=C_k\frac{dv_k(t)}{dt}$ on the left in these equations to give a set of *first-order* differential equations in the set of *state variables*—the inductor currents and the capacitor voltages.

Example 1.3–3

To help make these formal steps more concrete, let's reconsider the circuit of Example 1.3–1. Assigning branch voltage and current variables to the inductors and capacitors leads to the situation shown in Figure 1.3–5.

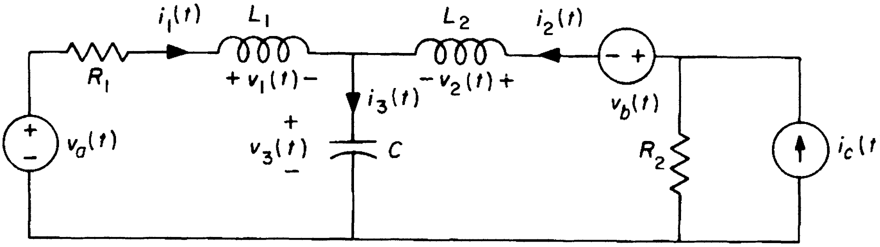

Figure 1.3–5. Circuit to illustrate the state equations procedure.

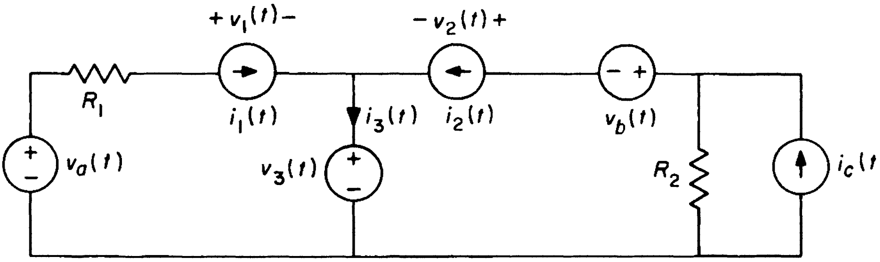

Figure 1.3–6. Figure 1.3–5 with energy-storage elements replaced by sources.

Replacing inductors and capacitors by ideal current and voltage sources respectively yields the circuit of Figure 1.3–6. Elementary resistive circuit analysis then gives

$$v_1(t) = -R_1 i_1(t) - v_3(t) + v_a(t) \tag{1.3–7}$$

$$v_2(t) = -R_2 i_2(t) - v_3(t) - v_b(t) + R_2 i_c(t) \tag{1.3–8}$$

$$i_3(t) = i_1(t) + i_2(t)\,. \tag{1.3–9}$$

Since $v_1(t) = L_1 \dfrac{di_1(t)}{dt}$, $v_2(t) = L_2 \dfrac{di_2(t)}{dt}$, and $i_3(t) = C\dfrac{dv_3(t)}{dt}$, we obtain the dynamic equations in state form:

$$\frac{di_1(t)}{dt} = -\frac{R_1}{L_1} i_1(t) - \frac{1}{L_1} v_3(t) + \frac{1}{L_1} v_a(t) \tag{1.3–10}$$

$$\frac{di_2(t)}{dt} = -\frac{R_2}{L_2} i_2(t) - \frac{1}{L_2} v_3(t) - \frac{1}{L_2} v_b(t) + \frac{R_2}{L_2} i_c(t) \tag{1.3–11}$$

$$\frac{dv_3(t)}{dt} = \frac{1}{C} i_1(t) + \frac{1}{C} i_2(t)\,. \tag{1.3–12}$$

Note that the left-hand side of each of these equations is the first derivative of a state variable ($i_1(t)$, $i_2(t)$, or $v_3(t)$) and the right-hand side is a function of state variables and independent sources ($v_a(t)$, $v_b(t)$, and $i_c(t)$) only.

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The inductor currents and capacitor voltages are called *state* variables because their present values summarize the accumulated effects of past experiences insofar as these may influence future behavior. This follows because the inductor currents and capacitor voltages determine the present distribution of stored energy in the circuit.* The choice of inductor currents and capacitor voltages as state variables is not, however, unique; for example, any equal number of independent linear combinations of the inductor currents and capacitor voltages could also serve as a set of state variables because the capacitor voltages and inductor currents can be uniquely derived from them. (See Problem 1.1.)

The number of independent state variables is called the *order* (or *degree*) of the circuit. The procedure described above suggests that the order is equal to the number of capacitors and inductors in the circuit. If, however, the circuit has loops of capacitors and voltage sources, or cut sets of inductors and current sources, the number of independent state variables (and the number of state equations) is reduced by one for each such loop or cut set. This happens because KVL or KCL equations constrain the values of the ideal sources in the first step of the state equations procedure. (See Problem 1.2.)

The state equations describe the local evolution of the state. Their form is important: The rate of change of the state is a function of the present state and the present inputs. As a result of this orderly structure, the state form of the dynamic equations has certain advantages over the node form, particularly for proving theorems or for describing general properties of circuits. However, different procedures (such as node or state) for writing dynamic equations will in general require different numbers of variables and equations, and will yield equations of different complexity. Selecting the "best" procedure is an art, not a science, and depends on the particular circuit and the objectives of the analysis. Both the node and state procedures can be extended to networks of arbitrary complexity, including time-varying and non-linear elements. Formal circuit analysis algorithms exist for either procedure that will automatically produce the dynamic equations once the network topology and constitutive relations for the elements are specified. Moreover, for LTI circuits at least, the effort required to obtain explicit analytical solutions of the dynamic equations is roughly independent of the procedure used to formulate the equations—it is primarily dependent on the *order* of the system, as we shall see.

*Recall that the product of a branch voltage and the associated current is the instantaneous power input to that branch element. For an LTI capacitor C, the integral of the instantaneous power, which is the stored energy at time t, is

$$\int_{-\infty}^{t} v_C(\tau) i_C(\tau)\, d\tau = \int_{-\infty}^{t} v_C(\tau) C \frac{dv_C(\tau)}{d\tau}\, d\tau = \tfrac{1}{2} C v_C^2(t).$$

Similiarly, for an LTI inductor L, the stored energy at time t is $\frac{1}{2} L i_L^2(t)$. In both of these formulas we have tacitly assumed that at $t = -\infty$ the element is in the *zero state*, $v_C(-\infty) = i_L(-\infty) = 0$.

1.4 Block Diagrams

Once the dynamic equations have been written in state form for any system—electrical circuit or not—it is easy to devise a *block diagram*, and from this an electronic circuit, that behaves analogously. By "analogously" we mean that if the inputs to the electronic circuit have the same waveform or time shape as the inputs to the actual system, then the waveforms of the state variables (or any combinations of state variables and inputs that may be selected as outputs) will also be the same in the actual system and in the electronic analog. Of course, in the process of translation from the actual system to the electronic analog, we are free to choose the units of the electronic circuit variables corresponding to the units of the actual system variables in any convenient way. We can also choose the time scale of the analog to speed up the representation of some slow action (such as a geological process) or slow down a fast action (such as the effects of an explosion). For many years, electronic *analog computers* of this kind have been effective design tools for complex, expensive, hard-to-modify systems such as aircraft or missile control systems. Today, to be sure, the dynamic analysis of such systems is usually done digitally, but block diagrams remain useful conceptually, and electronic circuits designed on the basis of such diagrams still have many applications in real-time situations such as audio signal processing.

Some of the more common elements appearing in simple block diagrams are shown in Figure 1.4–1. Each block defines a relationship between one or more *outputs* (the labels on arrows leaving the block) and one or more *inputs* (the labels on arrows entering the block). When blocks are connected together, the output of one becomes an input to another. The following example illustrates how interconnections of such blocks can describe a given set of state equations.

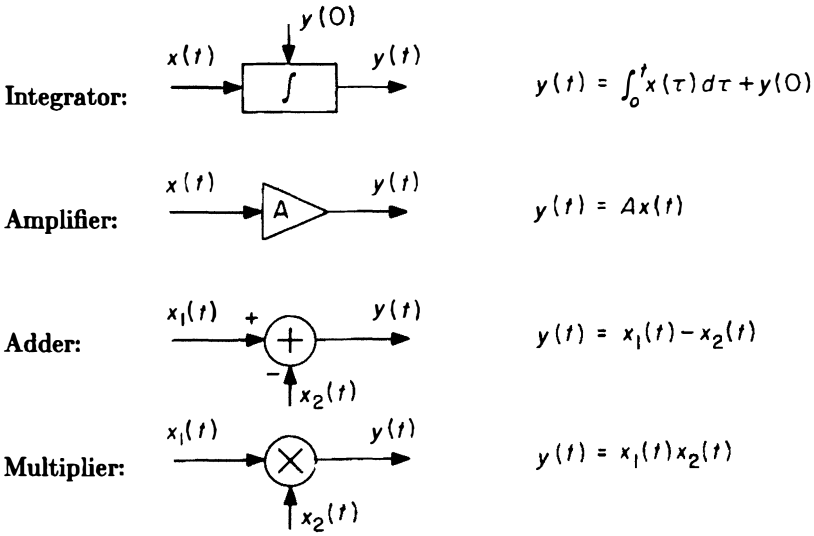

Figure 1.4–1. Simple block diagram elements.

Example 1.4–1

The circuit of Figure 1.3–5 led to three state equations:

$$\frac{di_1(t)}{dt} = -\frac{R_1}{L_1}i_1(t) - \frac{1}{L_1}v_3(t) + \frac{1}{L_1}v_a(t) \qquad (1.4\text{—}1)$$

$$\frac{di_2(t)}{dt} = -\frac{R_2}{L_2}i_2(t) - \frac{1}{L_2}v_3(t) - \frac{1}{L_2}v_b(t) + \frac{R_2}{L_2}i_c(t) \qquad (1.4\text{—}2)$$

$$\frac{dv_3(t)}{dt} = \frac{1}{C}i_1(t) + \frac{1}{C}i_2(t)\,. \qquad (1.4\text{—}3)$$

These equations can be simulated by the block diagram of Figure 1.4–2, composed of integrators, adders, and amplifiers. Because the system is 3$^{\text{rd}}$-order, three integrators are needed whose outputs represent the state variables, $i_1(t)$, $i_2(t)$, and $v_3(t)$. The key to devising or analyzing such diagrams is to focus on the inputs to the integrators, that is, on the derivatives of the state variables. Each derivative is composed of a sum of weighted inputs and state variables in accordance with the corresponding state equation. The way in which the diagram is drawn and the dashed boxes in Figure 1.4–2 should make this structure readily apparent. If we identify, say, the voltage $v_c(t)$ across R_2 in the circuit of Figure 1.3–5 as the output of interest, then we may add several blocks to the diagram to realize $v_c(t) = R_2(i_c(t) - i_2(t))$.

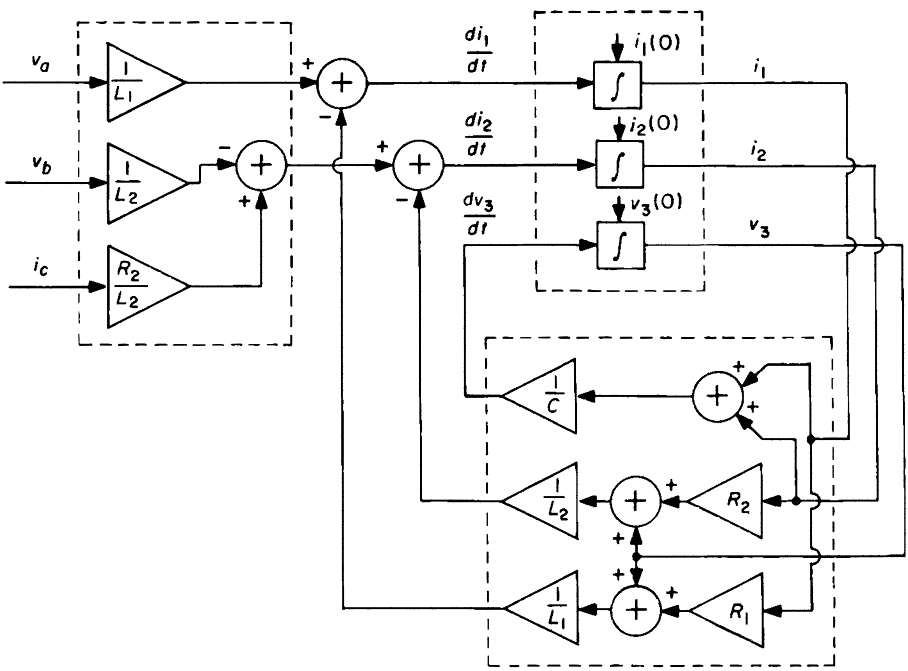

Figure 1.4–2. Block diagram simulation of equations (1.4—1, 2, 3).

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Example 1.4–2

To synthesize an electronic circuit analogous to the block diagram of Figure 1.4–2, we can interconnect the op-amp circuits of Figure 1.4–3 with appropriate element values.* Each of the circuits in Figure 1.4–3 actually combines several of the functions of the basic blocks of Figure 1.4–1. An appropriate interconnection is shown in Figure 1.4–4. Note that all the variables, independent of symbol, are in fact voltages. The labels next to the resistors are resistances but, since only ratios of resistances influence behavior, they can all be scaled by a common factor to any convenient range of values. The value C_0 of the integrator capacitors determines the time scale of the analog and may be chosen as desired.

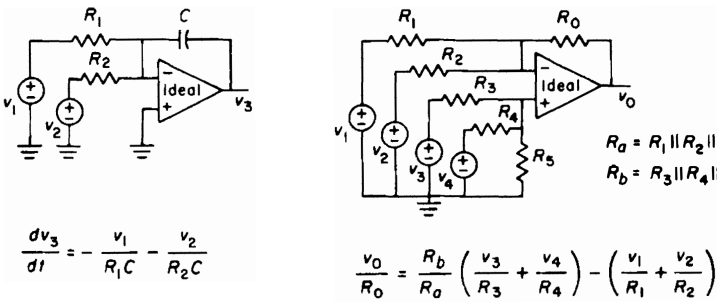

Figure 1.4–3. Op-amp realizations of integrators and adders.

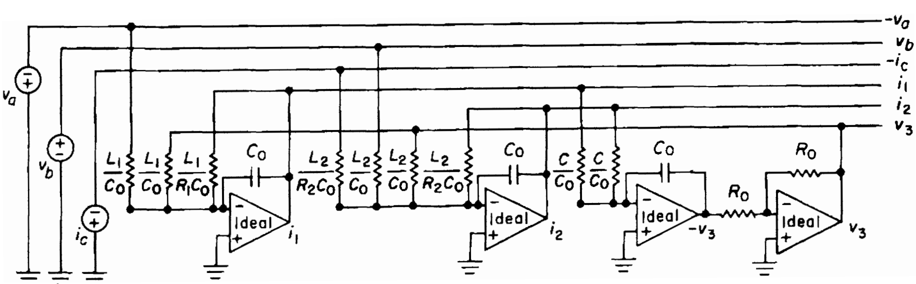

Figure 1.4–4. Electronic circuit implementation of (1.4–1, 2, 3).

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*The name *operational amplifier* was originally assigned because these units were first used to realize such operations as integrator and adder in analog computers.

1.5 Solutions of the Dynamic Equations

However derived—via the node or the state procedure described in Section 1.3 or some other procedure—the dynamic equations for a lumped circuit typically have the form of a set of differential equations in several variables.* Such equations describe the unknown responses to known stimuli only implicitly; they have the form

$$\text{(operations on responses)} = \text{(operations on stimuli)}.$$

On the other hand, what is frequently sought is an explicit (or operational) description in the form

$$\text{response} = \text{(operations on stimuli)}.$$

To achieve an operational description, the dynamic equations must be solved (integrated) rather than simply evaluated. Moreover, the solution may not be unique (because the results of operating on two different responses may be the same, so that both satisfy the dynamic equations). To obtain a unique solution, we must have additional auxiliary information, such as initial conditions or initial state.

If the circuit is LTI, an explicit closed-form solution of the dynamic equations can nearly always be achieved (at least in principle and for a wide class of input functions). The simplest such solutions are composed by combining solutions to two special situations:

a) The drives or inputs are zero.

b) The drives or inputs are exponentials in time.

The remainder of this chapter will review these two special solution situations for LTI circuits. And one of the purposes of the chapters that follow is to show that these situations are not in fact as "special" as they seem to be.

1.6 Solutions of the Dynamic Equations When the Inputs Are Zero

If all the independent sources are zero in some finite ($t_0 < t < t_1$) or semi-infinite ($t > t_0$) time interval, then the branch voltages and currents may be zero in that interval, but they do not have to be zero—energy stored in the circuit by inputs during $t < t_0$ may act as an effective drive for what is variously called the *natural* or *homogeneous* or *zero-input response* (ZIR) of the circuit. It is easy to show that the voltages and currents in a lumped LTI circuit during a zero-input interval may have nonzero values if and only if their waveforms are sums of particular appropriate exponential time functions (called *normal modes*) with specific (in general, complex) *time constants* whose reciprocals are the *natural frequencies* of the circuit. The basic issues are most readily explained through an example.

*Node equations for circuits containing inductors will also include integrals of the node voltages (see Example 1.3–1). But such integrals can be "cleared" by differentiating the equations.

Example 1.6–1

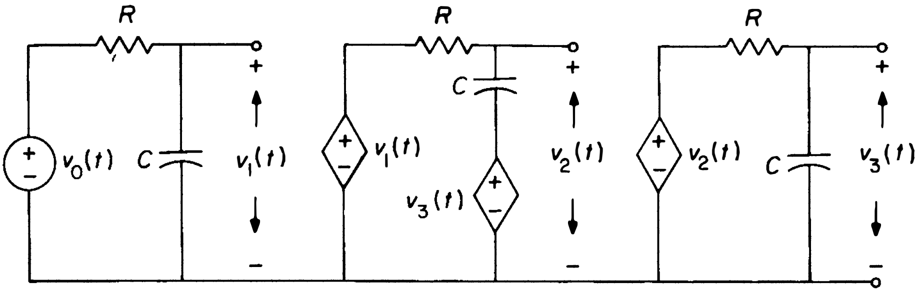

Figure 1.6–1. Equivalent circuit of Figure 1.3–4.

The voltage-follower circuit of Example 1.3–2 led to the equivalent circuit of Figure 1.6–1. In that example, we derived the following dynamic equations for this circuit by the node method:

$$C\frac{dv_1(t)}{dt}+\frac{1}{R}v_1(t)=\frac{1}{R}v_0(t) \tag{1.6–1}$$

$$C\frac{d}{dt}[v_2(t)-v_3(t)]+\frac{1}{R}[v_2(t)-v_1(t)]=0 \tag{1.6–2}$$

$$C\frac{dv_3(t)}{dt}+\frac{1}{R}[v_3(t)-v_2(t)]=0\,. \tag{1.6–3}$$

Suppose that $v_0(t)=0$ in some interval $t_0<t<t_1$. Try solutions during that interval of the form

$$v_1(t)=V_1e^{st},\quad v_2(t)=V_2e^{st},\quad v_3(t)=V_3e^{st} \tag{1.6–4}$$

to obtain

$$\left(Cs+\frac{1}{R}\right)V_1e^{st} = 0 \tag{1.6–5}$$

$$-\frac{1}{R}V_1e^{st}+\left(Cs+\frac{1}{R}\right)V_2e^{st} - CsV_3e^{st}=0 \tag{1.6–6}$$

$$-\frac{1}{R}V_2e^{st}+\left(Cs+\frac{1}{R}\right)V_3e^{st}=0\,. \tag{1.6–7}$$

The common factor e^{st} can be cancelled since it is nonzero for any finite s and t. The resulting set of three linear algebraic equations in V_1, V_2, and V_3 is consistent with nonzero values of V_1, V_2, and V_3 if and only if the determinant of the coefficients vanishes, that is, if

$$\begin{vmatrix} Cs+\frac{1}{R} & 0 & 0 \\ -\frac{1}{R} & Cs+\frac{1}{R} & -Cs \\ 0 & -\frac{1}{R} & Cs+\frac{1}{R} \end{vmatrix} = \frac{(RCs)^3+2(RCs)^2+2RCs+1}{R^3} = 0 . \qquad (1.6\text{–}8)$$

The roots of this *characteristic equation* are the *characteristic* (or *natural*) *frequencies** of the circuit:

$$s = -\frac{1}{RC}, \quad \frac{-1 \pm j\sqrt{3}}{2RC} . \qquad (1.6\text{–}9)$$

Thus $v_1(t)$, $v_2(t)$, and $v_3(t)$ may have the form (1.6–4) with nonzero values of (at least some of) the amplitudes V_1, V_2, and V_3, provided that s has one of the values in (1.6–9). For each allowed value of s, however, constraints are imposed on V_1, V_2, and V_3 by (1.6–5, 6, 7). Thus, if $s = -1/RC$, it is immediately apparent from (1.6–7) that V_2 must be zero and from (1.6–6) that $V_1 = V_3$. Hence, one nonzero solution to our zero-input problem is

$$v_1(t) = v_3(t) = Ae^{-t/RC}, \quad v_2(t) = 0 \qquad (1.6\text{–}10)$$

where A is an arbitrary constant. Given these node voltages, one can readily compute all the branch voltages and currents. These, too, will be proportional to the arbitrary constant factor A.

A similar result holds separately for each of the natural frequencies. For each, the node voltages (and hence all the branch currents and voltages) have the form of constants times exponential factors in the corresponding frequency; the constants are determined by (1.6–5, 6, 7) up to a single common arbitrary (in general, complex) factor, independent for each natural frequency. Thus, for $s = \left(-1 \pm j\sqrt{3}\right)/2RC$, it is apparent from (1.6–5) that $V_1 = 0$, and from (1.6–7) that $V_2 = V_3 e^{\pm j\pi/3}$, so that two other solutions to our zero-input problem are

$$v_1(t) = 0 \qquad (1.6\text{–}11)$$

$$v_2(t) = B_1 e^{-t/2RC} e^{j\sqrt{3}t/2RC} \qquad (1.6\text{–}12)$$

$$v_3(t) = B_1 e^{-j\pi/3} e^{-t/2RC} e^{j\sqrt{3}t/2RC} \qquad (1.6\text{–}13)$$

and

$$v_1(t) = 0 \qquad (1.6\text{–}14)$$

$$v_2(t) = B_2 e^{-t/2RC} e^{-j\sqrt{3}t/2RC} \qquad (1.6\text{–}15)$$

$$v_3(t) = B_2 e^{j\pi/3} e^{-t/2RC} e^{-j\sqrt{3}t/2RC} \qquad (1.6\text{–}16)$$

If several sets of node voltages (and the associated branch voltages and currents) independently satisfy Kirchhoff's Laws and the branch constitutive relations for a circuit of linear elements under zero-input conditions, then it is easy to argue that node and branch variables composed by *superimposing* (adding) the corresponding variables from each set also satisfy Kirchhoff's Laws and the branch constitutive relations,

*Also called the *singularities* of the circuit because for these values of s the node equations are *singular* and have non-unique solutions.

that is, the combination obeys the dynamic equations for the circuit under zero-input conditions. Moreover, if the actual voltages and currents in the circuit are real, then the coefficients of the terms corresponding to complex conjugate natural frequencies must be complex conjugates. Hence, the most general form for the zero-input-response node voltages of the circuit of Figure 1.6–1 is

$$v_1(t) = Ae^{-t/RC} \tag{1.6–17}$$

$$\begin{aligned} v_2(t) &= Be^{-t/2RC}e^{j\sqrt{3}t/2RC} + B^* e^{-t/RC}e^{-j\sqrt{3}t/2RC} \\ &= 2|B|e^{-t/2RC}\cos\left(\frac{\sqrt{3}t}{2RC} + \angle B\right) \end{aligned} \tag{1.6–18}$$

$$\begin{aligned} v_3(t) &= Ae^{-t/RC} + Be^{-j\pi/3}e^{-t/2RC}e^{j\sqrt{3}t/2RC} \\ &\qquad + B^* e^{j\pi/3}e^{-t/2RC}e^{-j\sqrt{3}t/2RC} \\ &= Ae^{-t/RC} + 2|B|e^{-t/2RC}\cos\left(\frac{\sqrt{3}t}{2RC} + \angle B - \frac{\pi}{3}\right) \end{aligned} \tag{1.6–19}$$

where the asterisk (*) indicates complex conjugation. To find values for the real constant A and the complex constant $B = |B|e^{j\angle B}$, we must be given separately information equivalent to knowing the state of the system at some moment, typically the beginning of the zero-input interval (that is, initial conditions).

Note that the number of arbitrary (real) constants that must be determined from auxiliary information is equal to the *degree* of the characteristic equation, which is also the same as the number of normal modes, the number of natural frequencies, the number of state variables, or in general the *order* of the system. If the roots of the characteristic equation are not distinct or simple (that is, if the characteristic equation contains a repeated factor $(s - s_k)^n$), then the normal modes include not only the term $e^{s_k t}$ but also $te^{s_k t}$, $t^2e^{s_k t}, \ldots, t^{n-1}e^{s_k t}$; s_k is said to be a natural frequency of *order* or *multiplicity* n. The number of arbitrary constants must be expanded correspondingly. ▶▶▶

1.7 Solutions of the Dynamic Equations for Exponential Inputs

If all of the independent inputs to an LTI circuit during some finite or semi-infinite time interval are proportional to the same exponential time function e^{st} (where s may be complex), then all of the voltages and currents in the circuit during that interval may have the same form—proportional to the same time function e^{st}— and the proportionality factors may readily be found. When we say "may have the same form" we mean simply that such functions satisfy the dynamic differential equations. But we have already pointed out that such equations specify the circuit behavior only implicitly. The solutions of exponential form are thus not the only solutions when the inputs are exponentials. The *complete solution* during an interval in which the inputs are exponentials consists of a *particular solution* of the form e^{st} plus the natural response to zero input with the constants chosen so that the complete solution matches a given initial state or equivalent information—as we shall illustrate in the following examples.

Example 1.7–1

Constant inputs can be considered a special case of exponential inputs with $s = 0$. A particular solution for a circuit with constant inputs, then, corresponds to all of the branch variables having appropriate constant values; this is the "d-c" (for "direct-current") or steady-state solution that the complete response will approach asymptotically if the constant inputs are maintained indefinitely.

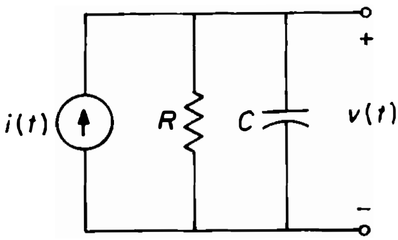

Figure 1.7–1. Example 1.7–1 circuit.

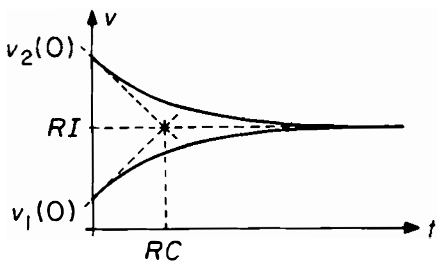

Figure 1.7–2. Constant input response.

For first-order systems the complete solution to a constant input can be written down by inspection. Thus the node equation (or state equation—they are essentially the same) for the circuit of Figure 1.7–1 is

$$i(t) = C\frac{dv(t)}{dt} + \frac{1}{R}v(t)\,. \tag{1.7–1}$$

If $i(t) = I =$ constant, $t > 0$, then a particular solution is $v(t) = V =$ constant. The $dv(t)/dt$ term in (1.7–1) is then zero so that

$$I = \frac{V}{R}\,. \tag{1.7–2}$$

The zero-input response is readily shown (by substitution into (1.7–1)) to have the form $Ae^{-t/RC}$ so that the complete solution is

$$v(t) = RI + Ae^{-t/RC}\,. \tag{1.7–3}$$

If the value of $v(t)$ at $t = 0$ is known to be $v(0)$, then it follows at once from (1.7–3) that

$$A = v(0) - RI \tag{1.7–4}$$

so that

$$v(t) = RI + (v(0) - RI)e^{-t/RC}, \quad t > 0\,. \tag{1.7–5}$$

The form of $v(t)$ is shown in Figure 1.7-2 for several values of $v(0)$. Note that the time $t = RC$ corresponds to the moment at which the response has progressed approximately 2/3 of the way from the initial to the final value. (The exact amount is $1 - \frac{1}{e} \approx 0.632$). Note also that a straight line with the initial slope of the response intercepts the final value at $t = RC$.

An ability to write down the complete response (1.7–5) for a first-order system with constant input, and to sketch it accurately as in Figure 1.7–2—directly from the problem statement without any intermediate steps—should be part of the skills of every engineer or scientist. Some examples of how this skill is useful in analyzing practical electronic circuits are given in Problems 1.13 and 1.14.

▶ ▶ ▶

Example 1.7–2

To illustrate the effect of exponential drives in a more complicated case, let's continue the analysis of the circuit of Examples 1.3–2 and 1.6–1. Let the drive be $v_0(t) = V_0e^{st}$, and assume that the node voltages have the form $v_i(t) = V_ie^{st}$. The dynamic node equations become

$$CsV_1e^{st} + \frac{1}{R}V_1e^{st} = \frac{1}{R}V_0e^{st} \tag{1.7–6}$$

$$Cs\left(V_2e^{st} - V_3e^{st}\right) + \frac{1}{R}\left(V_2e^{st} - V_1e^{st}\right) = 0 \tag{1.7–7}$$

$$CsV_3e^{st} + \frac{1}{R}\left(V_3e^{st} - V_2e^{st}\right) = 0\,. \tag{1.7–8}$$

Cancelling the common factor e^{st} and solving the simultaneous equations, we obtain

$$V_1 = \frac{V_0}{RCs+1} \tag{1.7–9}$$

$$V_2 = \frac{V_0(RCs+1)}{(RCs)^3 + 2(RCs)^2 + 2(RCs) + 1} \tag{1.7–10}$$

$$V_3 = \frac{V_0}{(RCs)^3 + 2(RCs)^2 + 2(RCs) + 1}\,. \tag{1.7–11}$$

We consider two special cases:

a) The *unit step response* of the circuit is the response when the input is the *unit step function* $u(t)$:

$$v_0(t) = u(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0\,. \end{cases} \tag{1.7–12}$$

By convention,* we take the fact that we are asking for the unit step response to imply that the circuit has been at rest for $t < 0$, that is,

$$v_1(t) = v_2(t) = v_3(t) = 0, \quad t < 0\,. \tag{1.7–13}$$

Since the voltages on the capacitors cannot change instantaneously, the state at $t = 0+$ is described by

$$v_1(0+) = v_2(0+) = v_3(0+) = 0\,. \tag{1.7–14}$$

*Unfortunately, this convention is not used by all workers in the field. If the circuit is not at rest at $t = 0$, we shall talk about the response to a *constant*, such as $v_0(t) = 1$, $t > 0$, rather than the unit step response.

For $t \geq 0$, $v_0(t)$ has the form

$$v_0(t) = V_0 e^{st} \tag{1.7–15}$$

with $V_0 = 1$, $s = 0$. Thus particular solutions are, from (1.7–9, 10, 11),

$$v_1(t) = V_1 e^{st}\Big|_{V_0=1,s=0} = 1 \tag{1.7–16}$$

$$v_2(t) = V_2 e^{st}\Big|_{V_0=1,s=0} = 1 \tag{1.7–17}$$

$$v_3(t) = V_3 e^{st}\Big|_{V_0=1,s=0} = 1\,. \tag{1.7–18}$$

Using the results of Example 1.6–1, we have as complete solutions

$$v_1(t) = 1 + Ae^{-t/RC} \tag{1.7–19}$$

$$v_2(t) = 1 + Be^{-t/2RC}e^{j\sqrt{3}t/2RC} + B^* e^{-t/2RC}e^{-j\sqrt{3}t/2RC} \tag{1.7–20}$$

$$v_3(t) = 1 + Ae^{-t/RC} + Be^{-j\pi/3}e^{-t/2RC}e^{j\sqrt{3}t/2RC} + B^* e^{j\pi/3}e^{-t/2RC}e^{-j\sqrt{3}t/2RC}\,. \tag{1.7–21}$$

Matching the conditions at $t = 0+$ gives $A = -1$, $B = \dfrac{-e^{-j\pi/6}}{\sqrt{3}}$, $B^* = \dfrac{-e^{j\pi/6}}{\sqrt{3}}$, and

$$v_1(t) = \left[1 - e^{-t/RC}\right]u(t) \tag{1.7–22}$$

$$v_2(t) = \left[1 - \frac{2}{\sqrt{3}}e^{-t/2RC}\cos\left(\frac{\sqrt{3}t}{2RC} - \frac{\pi}{6}\right)\right]u(t) \tag{1.7–23}$$

$$v_3(t) = \left[1 - e^{-t/RC} - \frac{2}{\sqrt{3}}e^{-t/2RC}\sin\left(\frac{\sqrt{3}t}{2RC}\right)\right]u(t)\,. \tag{1.7–24}$$

These waveforms are sketched in Figure 1.7–3.

b) The *sinusoidal steady-state response* of the circuit is the response to the input

$$v_0(t) = V_0 \cos\omega t, \quad -\infty < t < \infty\,. \tag{1.7–25}$$

This input is not of the form $V_0 e^{st}$, but it is closely related to such a form since the cosine function can be written in either of two ways:

$$1.\quad \cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \tag{1.7–26}$$

$$2.\quad \cos\omega t = \Re e\left[e^{j\omega t}\right] \tag{1.7–27}$$

(where the symbol "$\Re e$" stands for "take real part of"). Because the circuit is LTI and composed of real elements, we can find the desired response by first finding the response to

$$v_0(t) = V_0 e^{j\omega t}, \quad -\infty < t < \infty \tag{1.7–28}$$

and then doing either of the following:

1. Adding the responses for ω positive and ω negative and dividing by 2,

or

2. Taking the real part of the response.

Usually the second procedure is simpler. Since we are interested in the sinusoidal steady state, the desired complete responses are just the components proportional to the drive exponential $e^{j\omega t}$. Any finite natural response terms that might have been initiated when the sinusoidal drive was applied "at" $t = -\infty$ have presumably long since died out. Taking $v_3(t)$ as typical, the steady-state response to $v_0(t) = V_0 e^{j\omega t}$ is

$$v_3(t) = \frac{V_0 e^{j\omega t}}{(j\omega RC)^3 + 2(j\omega RC)^2 + 2(j\omega RC) + 1} = H(j\omega)V_0 e^{j\omega t} \tag{1.7–29}$$

where

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)} = \frac{1}{(j\omega RC)^3 + 2(j\omega RC)^2 + 2(j\omega RC) + 1} \tag{1.7–30}$$

is called the *frequency response* of the circuit. Hence the steady-state response to $v_o(t) = V_0 \cos\omega t = \Re e\left[V_0 e^{j\omega t}\right]$ is

$$v_3(t) = \Re e\left[H(j\omega)V_0 e^{j\omega t}\right] = V_0|H(j\omega)| \cos\left[\omega t + \angle H(j\omega)\right], \quad -\infty < t < \infty . \tag{1.7–31}$$

The steady-state response to a sinusoidal waveform of arbitrary frequency can thus be described by plotting the magnitude and phase of the frequency response vs. ω as shown in Figure 1.7–4. It is easy to show from (1.7–30) that

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega) = \frac{1}{1 + (\omega RC)^6} . \tag{1.7–32}$$

The frequency response has a magnitude ≈ 1 for $\omega \ll 1/RC$, and it is small compared to 1 for $\omega \gg 1/RC$. Thus, this circuit with $v_0(t)$ as its input and $v_3(t)$ as its output *passes* low frequencies and (to a degree) *stops* or *rejects* high frequencies. It is thus called a *lowpass filter*. Since the transition between the pass and rejection *bands* occurs near $\omega = 1/RC$, this is called the *band-edge* or *cutoff frequency* of the filter. As we shall later discuss, there are many different forms of lowpass filters; filters for which

$$|H(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_0)^{2n}} \tag{1.7–33}$$

are called *Butterworth filters* of order n and cutoff frequency ω_0.

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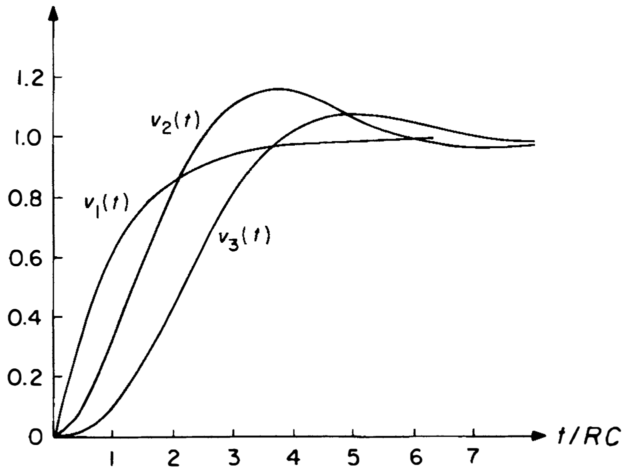


Figure 1.7–3. Step responses of third-order Butterworth lowpass filter.

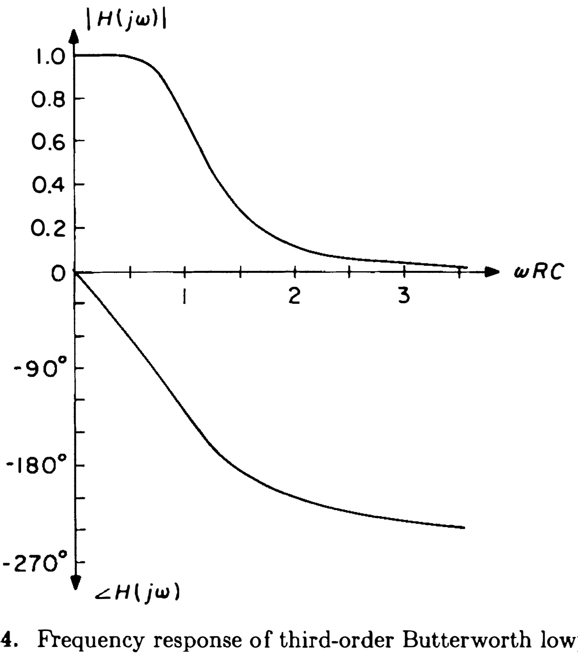


Figure 1.7–4. Frequency response of third-order Butterworth lowpass filter.

1.8 Summary

Using either the state or node methods, it is straightforward to derive a set of dynamic differential equations for any lumped circuit. If the circuit is LTI, the solution of these equations to find either the zero-input response or a particular response to an exponential drive begins by assuming exponential solutions of the form Ae^{st}. As a result, the differential equations are converted into a set of algebraic equations in the exponential amplitudes. In the case of the zero-input response, these equations are homogeneous and thus permit nonzero amplitudes only for certain characteristic frequencies. The general form of the zero-input response, then, is a sum of exponential terms at the characteristic frequencies with the amplitudes of the terms derived from knowledge of the circuit state at the beginning of the zero-input interval (or equivalent). In the case of an exponential drive proportional to e^{st}, the algebraic equations derived from assuming solutions of the form Ae^{st} are non-homogeneous and thus yield in general specific amplitudes for a particular solution to the problem. The complete solution is obtained by adding to this particular solution terms corresponding to a zero-input response, with appropriate amplitudes so that the complete solution matches the known state of the system at some moment (or equivalent information).

These "classical" methods for analyzing circuit behavior can be streamlined and extended in a variety of ways. For example, the algebraic equations that result from substituting trial solutions of the form Ae^{st} into the dynamic differential equations can be written directly from the circuit diagram (i.e., without first writing the differential equations) by using impedance methods. Probably you have already had some experience with impedance techniques in earlier courses. Developing impedance ideas through a new tool—the Laplace transform—will be one goal of the next chapter. The Laplace transform also provides a formal procedure for finding the response of an LTI circuit to an arbitrary input—not just the exponentials or sums of exponentials to which the "classical" methods of this chapter would seem to be limited. As we shall ultimately come to understand, however, the Laplace transformation has this power because virtually any waveform can be represented as a "sum" of exponentials; this is the essential idea behind what is called *Fourier analysis*. Thus the usefulness of frequency-domain methods for studying LTI system behavior, including impedance ideas and their generalization to the notion of a system function, is not restricted to exponential drives but is applicable in general.

The landscape of LTI analysis procedures is extraordinarily varied and rich. This wealth of perspectives, in turn, makes the language of LTI models an extraordinarily valuable tool for both the understanding and design of complex dynamical structures such as those employed in signal processing, communications and control, and measurements. Helping you learn to "speak" this language—not just to "solve" problems—is the real goal of this book.

EXERCISES FOR CHAPTER 1

Exercise 1.1

In the following assume that all waveforms are defined in the interval $-\infty < t < \infty$.

a) Demonstrate that each of the following constitutive relations describes a 2-terminal element that is both linear and time-invariant.

i) $\dfrac{dv(t)}{dt} + v(t) = i(t)$

ii) $v(t) = i(t-1)$

iii) $i(t) = \displaystyle\int_{-\infty}^{t} v(\tau)\cos(t-\tau)\,d\tau$

iv) $i(t) = \displaystyle\int_{0}^{t} v(\tau)\,d\tau + i(0)$

b) Demonstrate that each of the following constitutive relations describes an element that is not linear and/or is not time-invariant.

i) $v(t)i(t) = 1$

ii) $i(t) = \displaystyle\int_{0}^{t} v(\tau)\,d\tau$

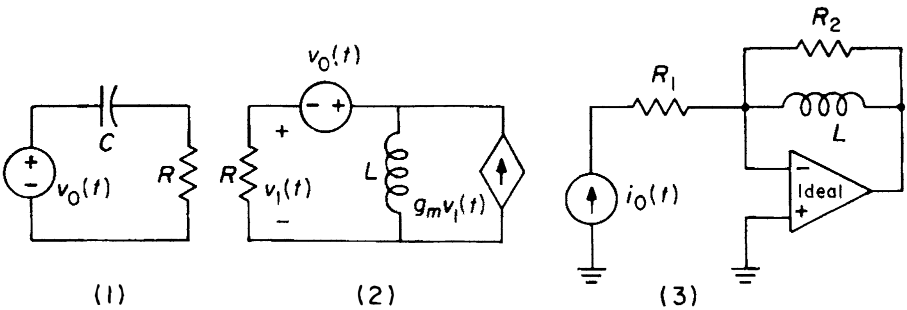
iii) $v(t) = i(t)\cos t$

iv) $v(t) = i(t) + 1$

Exercise 1.2

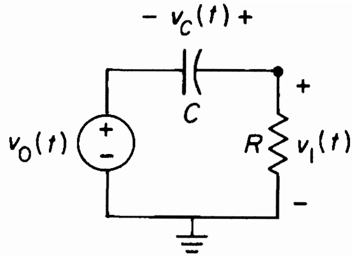
a) Pick a reference node and an appropriate (set of) node voltage(s) for each of the circuits below, and write the corresponding dynamic equation(s) in node form.

b) Pick an appropriate (set of) state variable(s) for each of the circuits below and write the corresponding dynamic equation(s) in state form.



Answers:

1a)

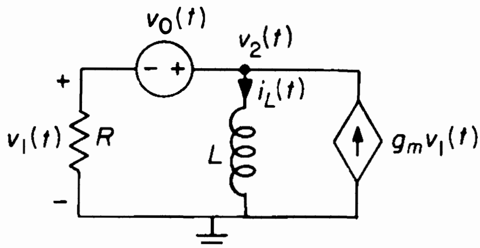


$$C\frac{d[v_1(t)-v_0(t)]}{dt}+\frac{v_1(t)}{R}=0$$

1b)

$$C\frac{dv_C(t)}{dt}=-\frac{v_0(t)+v_C(t)}{R}$$

2a)

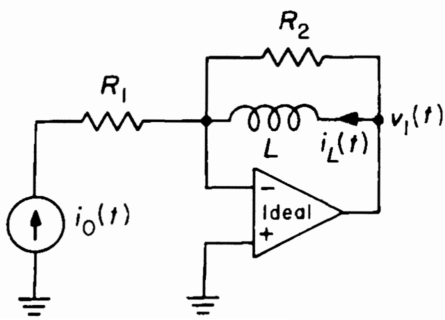


$$\frac{1}{L}\int_0^t v_2(\tau)\,d\tau+i_L(0)+\frac{v_2(t)-v_0(t)}{R}-g_m(v_2(t)-v_0(t))=0$$

2b)

$$L\frac{di_L(t)}{dt}=v_0(t)+\frac{i_L(t)}{g_m-\frac{1}{R}}$$

3a)

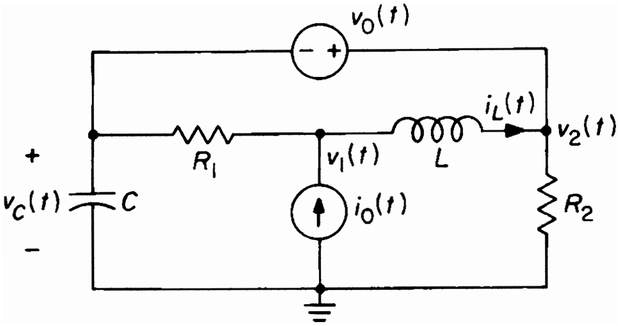


$$\frac{1}{L}\int_0^t v_1(\tau)\,d\tau+i_L(0)+\frac{v_1(t)}{R_2}=-i_0(t)$$

3b)

$$L\frac{di_L(t)}{dt}=-[i_0(t)+i_L(t)]R_2$$

Exercise 1.3



a) Pick the reference node as shown by the ground symbol (this is *not* necessarily the best reference node to pick in this circuit). Show that the node equations in the variables $v_1(t)$ and $v_2(t)$ are

$$\frac{1}{L}\int_0^t [v_1(\tau) - v_2(\tau)]\, d\tau + i_L(0) + \frac{1}{R_1}[v_1(t) - v_2(t) + v_0(t)] = i_0(t)$$

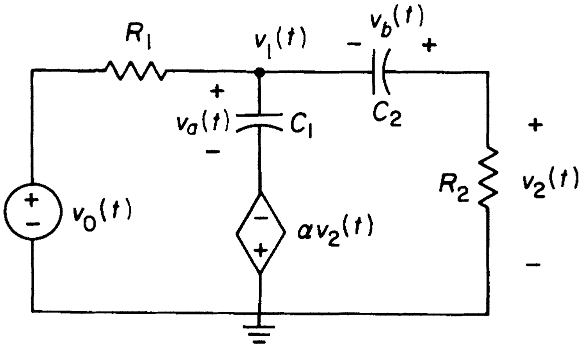
$$\frac{1}{R_2}v_2(t) + \frac{1}{R_1}[v_2(t) - v_0(t) - v_1(t)] + \frac{1}{L}\int_0^t [v_2(\tau) - v_1(\tau)]\, d\tau$$
$$- i_L(0) + C\frac{d}{dt}[v_2(t) - v_0(t)] = 0.$$

b) Show that state equations in the state variables $v_C(t)$ and $i_L(t)$ are

$$C\frac{dv_C(t)}{dt} = -\frac{1}{R_2}v_C(t) - \frac{1}{R_2}v_0(t) + i_0(t)$$
$$L\frac{di_L(t)}{dt} = -R_1 i_L(t) - v_0(t) + R_1 i_0(t).$$

c) Show that the equations derived in (a) and (b) are consistent.

Exercise 1.4



a) Show that the node equations in the variables $v_1(t)$ and $v_2(t)$ for the circuit above are

$$\frac{v_2(t)}{R_2} + C_2\frac{d}{dt}[v_2(t) - v_1(t)] = 0$$

$$C_2\frac{d}{dt}[v_1(t) - v_2(t)] + C_1\frac{d}{dt}[v_1(t) + \alpha v_2(t)] + \frac{1}{R_1}[v_1(t) - v_0(t)] = 0.$$

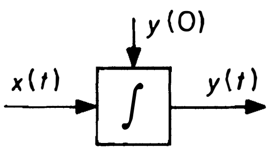
b) Show that state equations in the variables $v_a(t)$ and $v_b(t)$ are

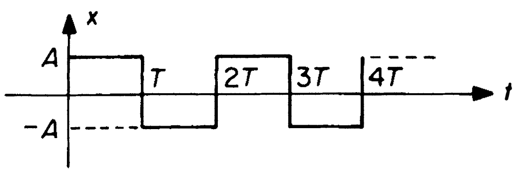
$$C_1\frac{dv_a(t)}{dt} = -\frac{R_1 + R_2}{R_1R_2(1+\alpha)}v_a(t) - \frac{R_1 - \alpha R_2}{R_1R_2(1+\alpha)}v_b(t) + \frac{v_0(t)}{R_1}$$

$$C_2\frac{dv_b(t)}{dt} = -\frac{1}{(1+\alpha)R_2}v_a(t) - \frac{1}{(1+\alpha)R_2}v_b(t).$$

c) Draw an actual circuit diagram employing an op-amp and appropriate other elements which would have the incremental equivalent circuit shown above.

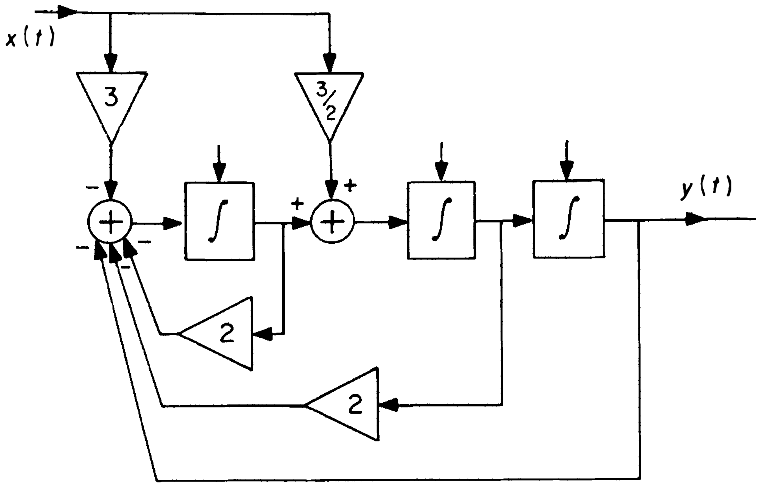
Exercise 1.5





Sketch the output of the ideal integrator above if the input is as shown. Illustrate the effect of various choices for $y(0)$.

Exercise 1.6



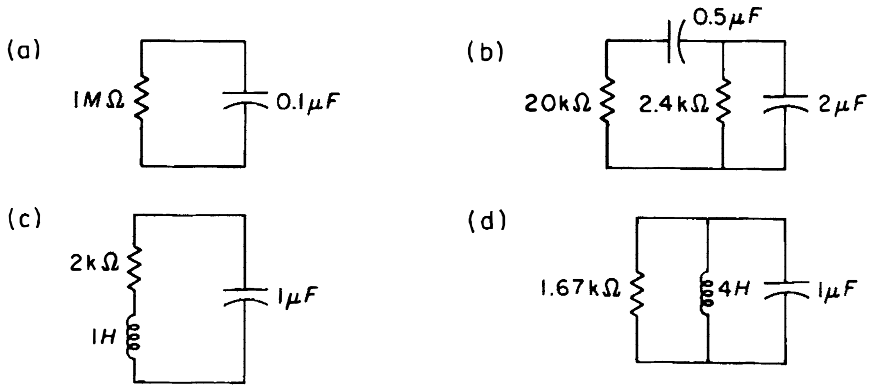
a) Taking the outputs of the integrators as state variables, write the dynamic equations for this system in state form.

b) Eliminate intermediate state variables to obtain the input-output differential equation

$$\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{3}{2}\frac{dx(t)}{dt}.$$

Exercise 1.7

Find the natural frequencies and the form of the ZIR for each of the following circuits-



Answers: (a) Ae^{-10t} (b) $Ae^{-250t} + Be^{-250t/3}$

(c) $Ae^{-10^3t} + Bte^{-10^3t}$ (d) $Ae^{-300t}\cos(400t + \theta)$

PROBLEMS FOR CHAPTER 1

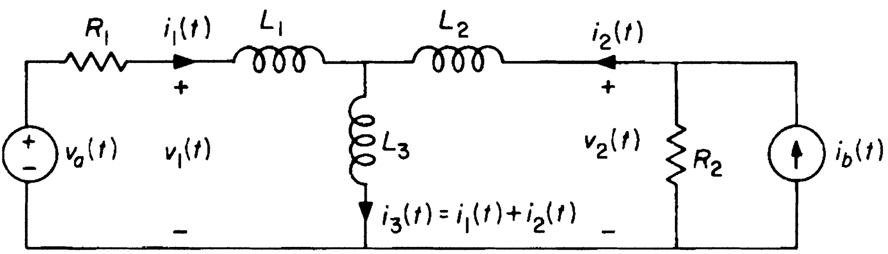
Problem 1.1

From the physical point of view, the voltages on capacitors and currents in inductors are obviously a satisfactory set of state variables since their present values describe the distribution of stored energies in the circuit and thus specify the entire effect that past stimuli can have on future behavior. But equally obviously they are not unique in this respect; any other set of quantities from which the capacitor voltages and inductor currents could be algebraically derived would also be an appropriate alternative set of state variables.

a) By manipulating the node equations of Example 1.3–2 into state form, prove that the node voltages for that circuit are an appropriate set of state variables.

b) Show by means of some simple examples that node voltages are not always an appropriate set of state variables.

Problem 1.2



The three inductors in the circuit above may represent three independent coils or an equivalent circuit for two coupled coils (see Problem 3.7). In either case, KCL at the central node allows just two of the inductor currents to be independent, so that the circuit has only two independent state variables and is hence of second rather than third order.

a) Choosing the independent state variables to be $i_1(t)$ and $i_2(t)$, write expressions for $v_1(t)$ and $v_2(t)$ entirely in terms of $i_1(t)$, $i_2(t)$, $v_a(t)$, and $i_b(t)$.

b) Considering the T of inductors as a 2-port (see appendix to Lecture 3), find equations characterizing it in the form

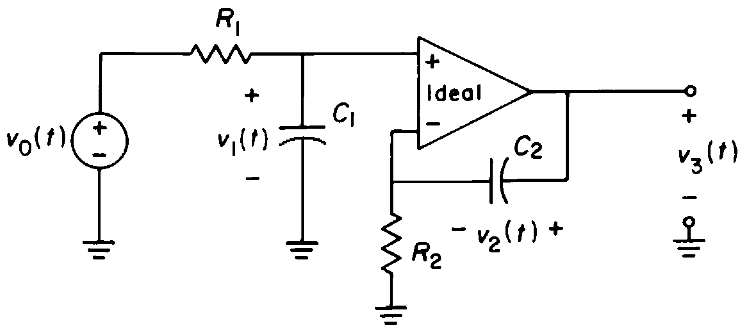
$$\frac{di_1(t)}{dt} = \Gamma_{11}v_1(t) + \Gamma_{12}v_2(t)$$
$$\frac{di_2(t)}{dt} = \Gamma_{21}v_1(t) + \Gamma_{22}v_2(t)$$

and determine $\{\Gamma_{ij}\}$ in terms of $\{L_k\}$.

c) Combine the results of (a) and (b) to obtain state equations in normal form for the circuit above.

Problem 1.3

With appropriate parameter values, the circuit shown below functions as a *non-inverting integrator.*



a) Write state equations for this circuit in terms of the capacitor voltages $v_1(t)$ and $v_2(t)$.

b) Show that the input-output differential equation relating $v_3(t)$ and $v_0(t)$ has the form

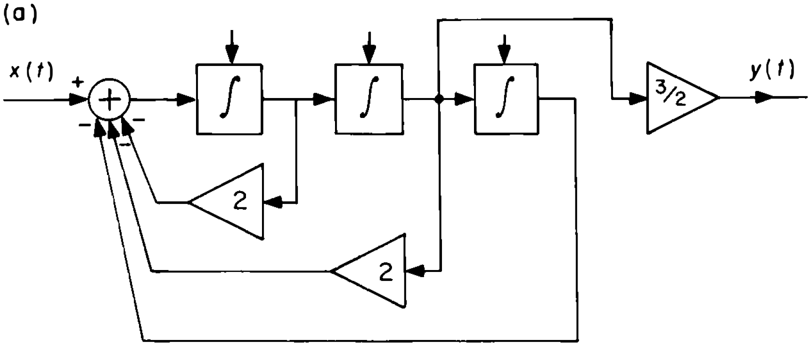
$$\frac{dv_3(t)}{dt} = Kv_0(t)$$

provided that $R_1C_1 = R_2C_2$. Find the magnitude and sign of K.

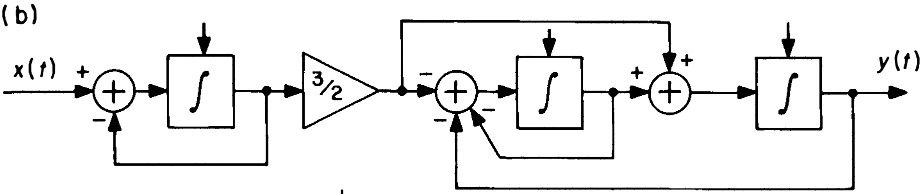
Problem 1.4

From the standpoint of input-output behavior, many different block diagrams may be *equivalent* in the sense of being described by the same input-output differential equation. Show that each of the following block diagrams is equivalent in this sense to the diagram of Exercise 1.6.

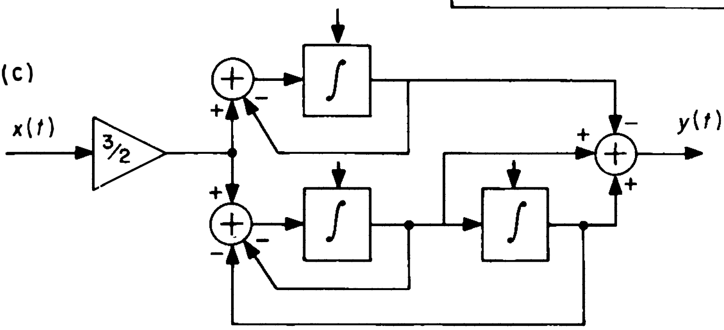
(a)



(b)

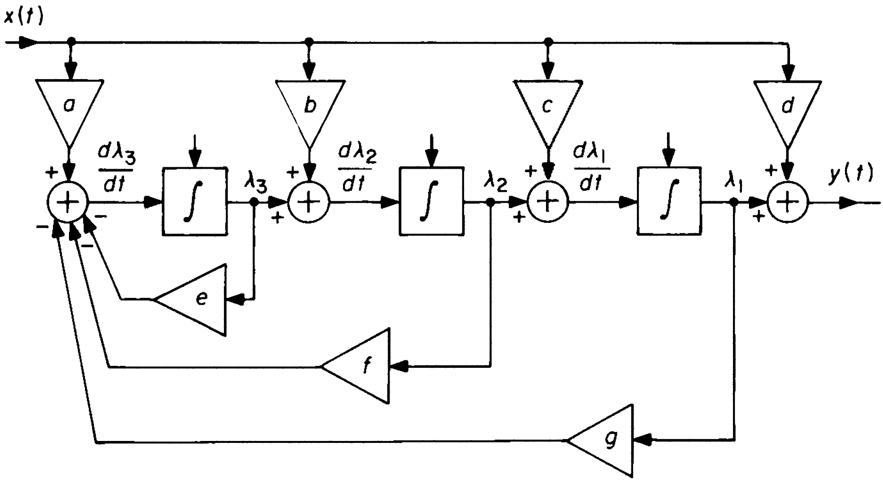


(c)



Problem 1.5

The input-output behavior of *any* 3[rd]-order system can be simulated by the following block diagram:



a) Write the dynamical equations for this system in state form. That is, express $d\lambda_1(t)/dt$, $d\lambda_2(t)/dt$, $d\lambda_3(t)/dt$, and the output $y(t)$ in terms of the state variables $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, and the input $x(t)$. (HINT: focus on the relationships among the variables imposed by the adder blocks.)

b) Eliminate the state variables and their derivatives to obtain a single differential equation of the form

$$\frac{d^3y(t)}{dt^3} + \alpha_2\frac{d^2y(t)}{dt^2} + \alpha_1\frac{dy(t)}{dt} + \alpha_0 y(t) = \beta_3\frac{d^3x(t)}{dt^3} + \beta_2\frac{d^2x(t)}{dt^2} + \beta_1\frac{dx(t)}{dt} + \beta_0 x(t)$$

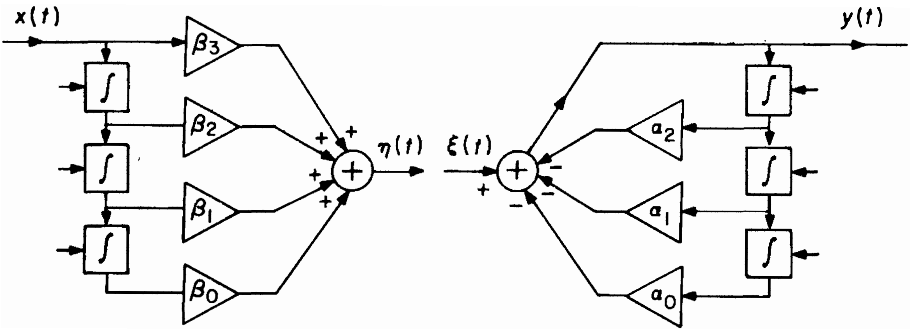
relating the input and the output.

c) Derive formulas for the gains a, b, $\ldots$, g in terms of the coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ and thus show that any input-output 3rd-order differential equation can be realized in this way.

The procedure illustrated by this problem generalizes to equations of arbitrary order.

Problem 1.6

This problem discusses another form of block diagram that can simulate the input-output behavior of any 3rd-order LTI system. Extension to LTI systems of arbitrary order should be obvious.



a) Show that the block diagram to the left above is described by the input-output differential equation

$$\frac{d^3\eta(t)}{dt^3} = \beta_3\frac{d^3x(t)}{dt^3} + \beta_2\frac{d^2x(t)}{dt^2} + \beta_1\frac{dx(t)}{dt} + \beta_0 x(t).$$

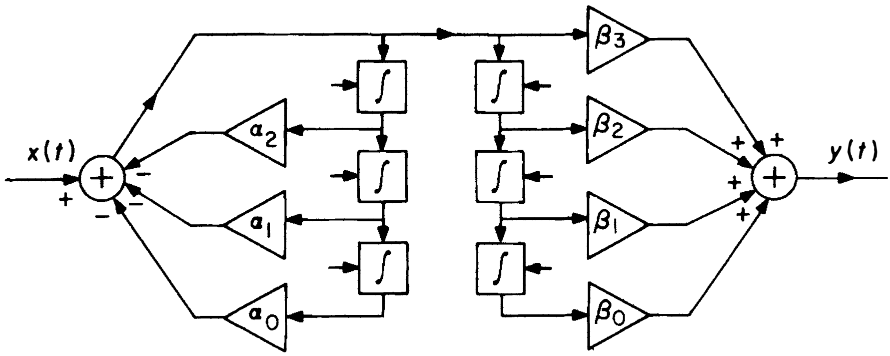
b) Show that the block diagram to the right above is described by the input-output differential equation

$$\frac{d^3y(t)}{dt^3} + \alpha_2\frac{d^2y(t)}{dt^2} + \alpha_1\frac{dy(t)}{dt} + \alpha_0 y(t) = \frac{d^3\xi(t)}{dt^3}.$$

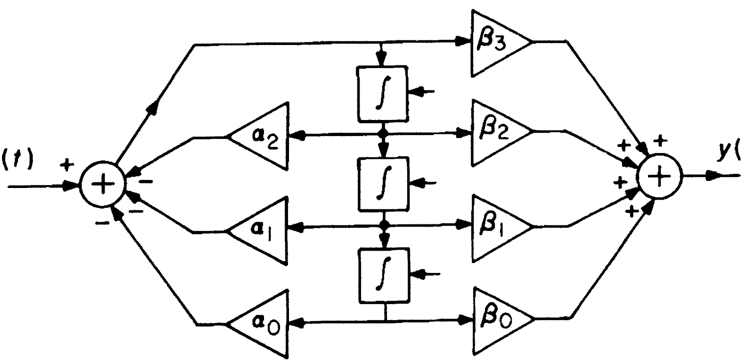
c) If the two block diagrams above are connected together in *cascade* so that $\xi(t) = \eta(t)$, the overall structure is described by

$$\frac{d^3y(t)}{dt^3} + \alpha_2\frac{d^2y(t)}{dt^2} + \alpha_1\frac{dy(t)}{dt} + \alpha_0 y(t) = \beta_3\frac{d^3x(t)}{dt^3} + \beta_2\frac{d^2x(t)}{dt^2} + \beta_1\frac{dx(t)}{dt} + \beta_0 x(t)$$

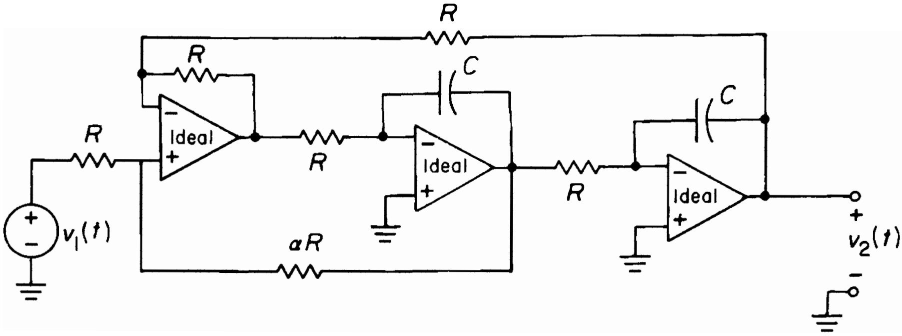
and thus with appropriate choices of gains can simulate the input-output behavior of any 3rd-order system as claimed. However, a simpler system results if the structures above are cascaded in *reversed order* as shown below. (In later chapters we shall show that the overall input-output result of cascading LTI systems is independent of the order in which they are cascaded.)



Since the inputs to the integrators at the top of each chain of integrators are the same, their outputs are the same, and so on down the chain. Hence only a single chain of integrators is really necessary, as shown below. Show directly for the system below that the input-output differential equation is the same as that given above.



Problem 1.7

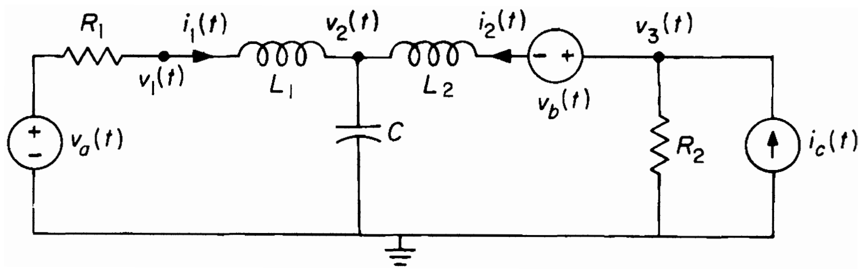


a) Pick an appropriate set of variables and write differential equations describing the system above in normal state form.

b) Derive from these equations a single input-output differential equation relating $v_1(t)$ and $v_2(t)$.

c) If R, C, and α may take on any positive values, what range of natural frequencies may be realized with this circuit?

Problem 1.8



In Example 1.3–1, node equations were written for the above circuit. With $v_a(t) = v_b(t) = i_c(t) = 0$ these equations become:

$$\frac{v_1(t)}{R_1} + \frac{1}{L_1}\int_0^t [v_1(\tau) - v_2(\tau)]\, d\tau + i_1(0) = 0$$

$$\frac{1}{L_1}\int_0^t [v_2(\tau) - v_1(\tau)]\, d\tau - i_1(0) + C\frac{dv_2(t)}{dt} + \frac{1}{L_2}\int_0^t [v_2(\tau) - v_3(\tau)]\, d\tau - i_2(0) = 0$$

$$\frac{1}{L_2}\int_0^t [v_3(\tau) - v_2(\tau)]\, d\tau + i_2(0) + \frac{v_3(t)}{R_2} = 0\,.$$

Assuming $R_1 = R_2 = 1$ kΩ, $L_1 = L_2 = 0.1$ H, $C = 0.2\ \mu$F, parallel the development of Example 1.6–1 to find the natural frequencies and the normal modes of this circuit. (HINT: The analysis will be simplified if you first differentiate all three equations to clear out intergrals and constants. As a check you should find that the natural frequencies are $10^{-4}s = -1, -1/2 \pm j\sqrt{3}/2$).

Problem 1.9

The dynamic state equations for an LTI system under ZIR conditions may be written in matrix form as

$$\dot{\lambda}] = [A] \times \lambda] .$$

Here $\lambda]$ is a column matrix of the state variables $\lambda_i(t)$, $\dot{\lambda}]$ is a column matrix of the time derivatives $d\lambda_i(t)/dt$, and $[A]$ is a square matrix of coefficients a_{ij}. To find the natural frequencies, as in Example 1.6–1, set

$$\lambda_i(t) = \Lambda_i e^{st}$$

and substitute to obtain

$$s\Lambda]e^{st} = [A] \times \Lambda]e^{st}$$

where $\Lambda]$ is the column matrix of amplitudes Λ_i. Cancelling the nonzero factor e^{st} leads to

$$([A] - s[I]) \times \Lambda] = 0$$

where $[I]$ is the identity matrix. The natural frequencies are thus the s-roots of the equation

$$\det([A] - s[I]) = 0$$

which are called the *eigenvalues* of the matrix $[A]$.

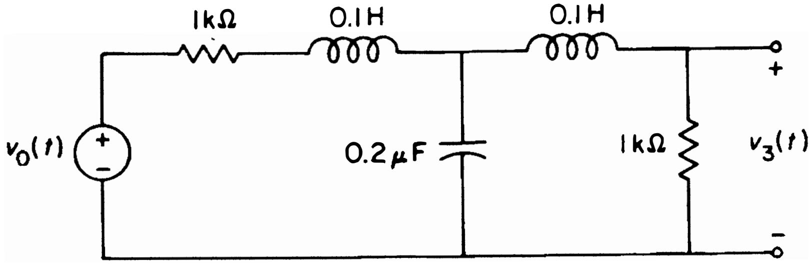
For the circuit of Examples 1.3–1 and 1.3–3, as well as Problem 1.8, the $[A]$ matrix is (Example 1.3–3)

$$[A] = \begin{bmatrix} \frac{-R_1}{L_1} & 0 & \frac{-1}{L_1} \\ 0 & \frac{-R_2}{L_2} & \frac{-1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} .$$

For the element values of Problem 1.8, find the eigenvalues of $[A]$ and show that they are the same as the natural frequencies found in Problem 1.8. (The eigenvalues of a matrix with real elements are generally easier to find numerically than the roots of a polynomial of equivalent degree.)

Problem 1.10

Continuing the analysis of the circuit of Problems 1.8 and 1.9, suppose we seek the response $v_3(t)$ to the input $v_0(t) = 2u(t)$. (Recall that a step drive implies that the circuit is at rest for $t < 0$.)



a) From the fact that the current in the right-hand inductor is zero at $t = 0$, conclude that $v_3(0+) = 0$.

b) From the fact that the voltage on the capacitor is zero at $t = 0$ conclude that $\dot{v}_3(0+) = \left.\dfrac{dv_3(t)}{dt}\right|_{t=0+} = 0$.

c) From the fact that the current in the left-hand inductor is zero at $t = 0$, conclude that $\ddot{v}_3(0+) = \left.\dfrac{d^2 v_3(t)}{dt^2}\right|_{t=0+} = 0$.

d) From the results of problem 1.8 and a consideration of the steady-state value of $v_3(t)$ conclude that

$$v_3(t) = A + Be^{-10^4 t} + Ce^{-10^4 t/2} e^{j\sqrt{3}\times 10^4 t/2} + C^* e^{-10^4 t/2} e^{-j\sqrt{3}\times 10^4 t/2}, \quad t > 0$$

and find the (real) values of A and B and the (complex) value of C or C^*. (HINT: compare Example 1.7–2.)

Problem 1.11

In the circuit of Exercise 1.4, let $C_1 = 1\ \mu\text{F}$, $C_2 = 1.5\ \mu\text{F}$, $R_1 = 1\ \text{k}\Omega$, $R_2 = 333\ \Omega$, $\alpha = 3$.

a) Find the natural frequencies and the form of the ZIR.

b) Suppose $v_0(t) = 3e^{-400t}$ volts in the interval $t > 0$. Find the component of the response $v_2(t)$ that has the form $V_2 e^{-400t}$.

c) Find the complete solution for $v_2(t)$ in the interval $t > 0$ if it is known that $v_2(0+) = -15$ volts, $\dot{v}_2(0+) = \left.\dfrac{dv_2(t)}{dt}\right|_{t=0+} = 5 \times 10^3$ volts/sec.

d) What values of the initial capacitor voltages $v_a(t)$ and $v_b(t)$ at $t = 0+$ correspond to the values of $v_2(0+)$ and $\dot{v}_2(0+)$ given in (c)?

Problem 1.12

The equation

$$\frac{d^2y_1(t)}{dt^2} + \epsilon\big[y_1^2(t) - 1\big]\frac{dy_1(t)}{dt} + y_1(t) = x(t)$$

is an example of a non-linear inhomogeneous second-order differential equation of the *Van der Pol* type. Such equations have been extensively studied, and (for $\epsilon \ll 1$) are often proposed as approximate descriptions of a variety of slightly non-linear systems with known input $x(t)$ and output $y_1(t)$.

a) Define another variable $y_2(t)$ by the equation

$$y_2(t) = \frac{dy_1(t)}{dt}.$$

Using $y_1(t)$ and $y_2(t)$ as state variables, write a pair of first-order equations in state form that are equivalent to Van der Pol's equation.

b) For $\epsilon = 0$, the Van der Pol equation describes an LTI system. Find the general (real) form of the zero-input response.

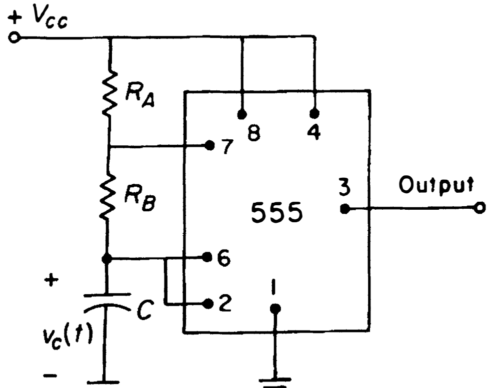
c) For $\epsilon \ll 1$, and starting at rest so that initially $|y_1(t)| \ll 1$, the equation describes an approximately LTI 2^{nd}-order system with negative damping:

$$\frac{d^2y_1(t)}{dt^2} - \epsilon\frac{dy_1(t)}{dt} + y_1(t) = x(t).$$

Find the general (real) form of the zero-input response under these conditions. Sketch $y_1(t)$.

The results of (c) should be an exponentially increasing sinusoid of frequency about 1 rad/sec and time constant $2/\epsilon$. As $|y_1(t)|$ increases, it obviously becomes less reasonable to neglect the $y_1^2(t)$ term compared with 1. Very roughly, we may expect the "average damping" to be less negative as $|y_1(t)|$ grows; eventually the oscillation stabilizes near the point where the average of $y_1^2(t)$ is 1, so that the "average damping" is zero. For small ϵ, the oscillation is nearly sinusoidal with period 2π; for larger ϵ, the waveform becomes markedly non-sinusoidal with an increased period.

Problem 1.13



The *555-type timer* is a classic integrated circuit with many uses. The figure on page 39 shows the 555 connected to form a free-running relaxation-oscillator or multivibrator. Under these conditions the 555 behaves as a voltage-controlled switch. When power is applied with C discharged, current flows from the supply V_{CC} through R_A and R_B to charge C. During this period the output voltage is near V_{CC}. When $v_C(t)$, the voltage across C, reaches $(2/3)V_{CC}$, the 555 effectively switches terminals 7 and 3 to ground. C thus discharges through R_B, and the output voltage drops to near zero. When $v_C(t)$ reaches $(1/3)V_{CC}$, the 555 switches again—disconnecting terminal 7 so that C recharges through R_A and R_B from V_{CC}, and connecting the output to V_{CC}. When $v_C(t)$ reaches $(2/3)V_{CC}$, the cycle repeats.

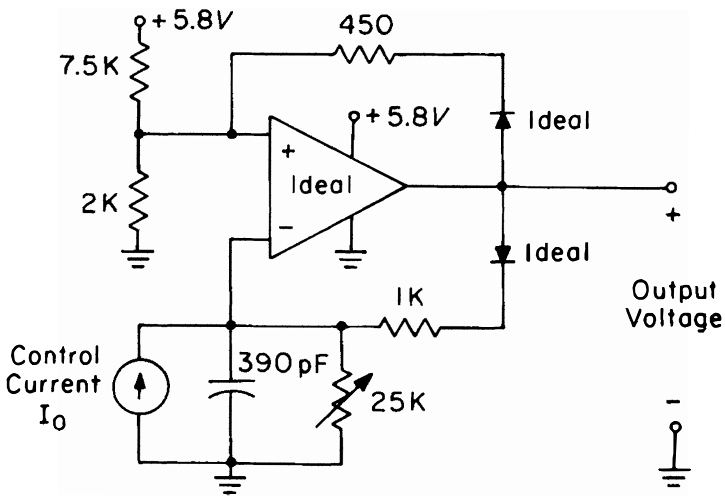
a) Sketch the output voltage and the voltage $v_C(t)$ across C more or less to scale in accordance with the description above, assuming that C is discharged at the moment power is applied.

b) Show that the period of the oscillation is given by

$$T = 0.693(R_A + 2R_B)C.$$

Problem 1.14

An oscillator whose frequency is a function of the value of some voltage (or current) is usually called a *voltage-* (or *current-) controlled oscillator* (abbreviated VCO). VCO's have a variety of uses in intrumentation, modulation, phase-locked loops, etc., and are often included as subcomponents of integrated circuits. The LM1800, for example, is widely used as an FM stereo demodulator. It contains as part of a phased-locked loop the VCO described by the following diagram:



a) Assume a control current $I_0 = 0$. If the device is turned on from rest, the capacitor holds the inverting op-amp input momentarily at ground while the non-inverting input is above ground. The output voltage thus goes to the supply voltage, 5.8 volts, and both diodes are forward-biased. The non-inverting op-amp input remains at a voltage $V_+(max)$ as the capacitor starts to charge. What is $V_+(max)$?

b) The output voltage and the non-inverting op-amp input voltage will stay at 5.8 volts and $V_+(max)$, respectively, until the capacitor voltage and the inverting op-amp input voltage exceed $V_+(max)$, at which point the output voltage will suddenly fall to zero, both diodes will become reverse-biased, the non-inverting op-amp input voltage will fall to $V_+(min)$, and the capacitor will start to discharge through the 25 kΩ resistor (which is variable in the actual circuit but which we will assume fixed for this problem). What is the value of $V_+(min)$?

c) The output voltage and the non-inverting op-amp input voltage will stay at 0 volts and $V_+(min)$, respectively, until the capacitor voltage and the inverting op-amp input voltage fall below $V_+(min)$, at which point the output suddenly rises to 5.8 volts and the cycle repeats. How long does it take the capacitor voltage to fall from $V_+(max)$ to $V_+(min)$?

d) How long does it take the capacitor voltage to rise from $V_+(min)$ to $V_+(max)$?

e) Sketch approximately to scale the output voltage and the inverting and non-inverting op-amp input voltages.

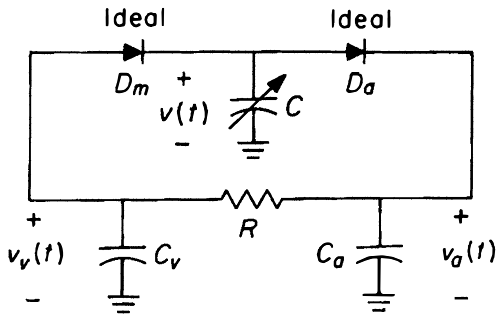
f) What is the period of the output voltage?

g) Describe with simple sketches the qualitative effect of the control current if it is *small* ($-40\,\mu$amp $< I_0 < 40\,\mu$amp). (Note that for this range the effect of I_0 during the charging interval can be ignored.)

h) Compute the period of the output voltage for 4–5 points in the range $-40\,\mu$amp $< I_0 < 40\,\mu$amp and sketch the output frequency (= 1/period) vs. I_0. This is the control characteristic for the VCO (although in this case the control is assumed to be a current, so it might be more appropriate to call it a CCO).

Problem 1.15

An electrical analog of the heart and circulatory system provides an interesting example of a system that is both non-linear and time-varying, and yet can be analyzed entirely using the simple techniques described in Example 1.7–1.



In the diagram above, C_a and C_v describe the elastic capacities of the arteries and veins, respectively (voltage is analogous to pressure, charge is analogous to volume), and R represents the resistance of the peripheral circulation (current is analogous to volume flow). The heart is modelled by the time-varying ventricular capacitance C

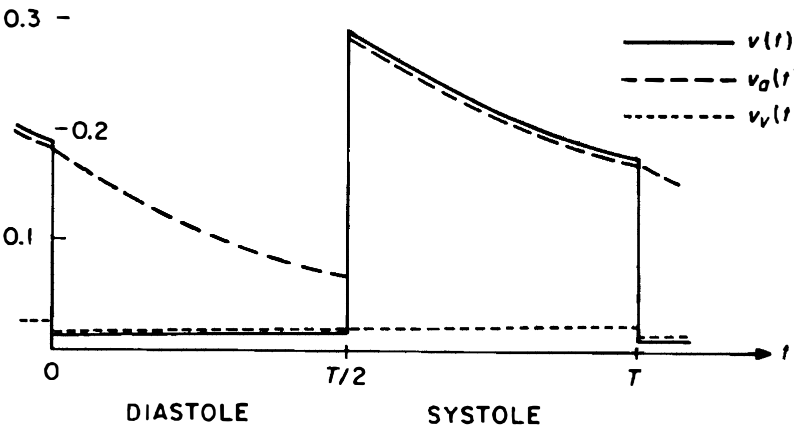
and two diodes D_m and D_a, representing the mitral and aortic valves. (Only the left side of the mammalian heart is considered here; the right heart and the pulmonary circulation, which should be part of this loop, have been omitted for simplicity.)

a) During *diastole*, the heart fills with returned blood. To model this phase, assume D_m is conducting, D_a is open, and C has the constant value C_d. At the beginning of diastole ($t = 0$), assume the charge on C_a is $q_a(0)$ and the combined charge on C_d and C_v together is $Q - q_a(0)$, where $Q =$ constant is the total charge (blood volume) in the system. Find $q_a(t)$ and $q_v(t)$. For the assumptions about the diodes to be consistent, $q_a(0)$ must satisfy some condition. What is that condition?

b) At $t = T/2$, the muscles of the heart wall contract. (Electrical correlates of the signals initiating contraction can be picked up on the external chest surface and constitute the largest peaks in the *electrocardiogram*.) During the succeeding interval, called *systole*, the mitral and aortic valves are respectively closed and opened by the build-up of pressure in the heart, and blood is squeezed out into the arteries. To model this phase, assume that the value of C is suddenly reduced from C_d to $C_s < C_d$. (Imagine suddenly pulling the plates of the capacitor further apart.) The voltage across C will quickly rise, turning D_m off and D_a on. The charge on C_v at the start of the period $T/2 < t < T$ will be the same as the previously found value $q_v(T/2)$, whereas the combined charge on C (now equal to C_s) and C_a together will be $Q - q_v(T/2)$. Find $q_a(t)$ and $q_v(t)$ for $T/2 < t < T$.

c) At $t = T$, the value of C is suddenly increased to $C = C_d$, and the cycle repeats. After a number of cycles the charge pattern will approach periodicity. How can one calculate this final periodic behavior?

d) Show that the voltages (pressures) shown below correctly describe the periodic condition for $Q = 1$, $C_a = C_s = 1$, $C_v = C_d = 20$, $T = 2RC_1$, $q_a(0) = 0.1768$, and $C_1 = C_a(C_v + C_d)/(C_a + C_v + C_d)$.



e) Note that $v_a(t) > v_v(t)$ at all times in the example of (d), corresponding to a continuous flow of blood from the arteries into the veins. The total charge in the system is, however, $Q =$ constant at all times; no charge is lost. Energy, nevertheless, is continuously "lost" to the system—dissipated as heat in the resistor R. Where does this energy come from?