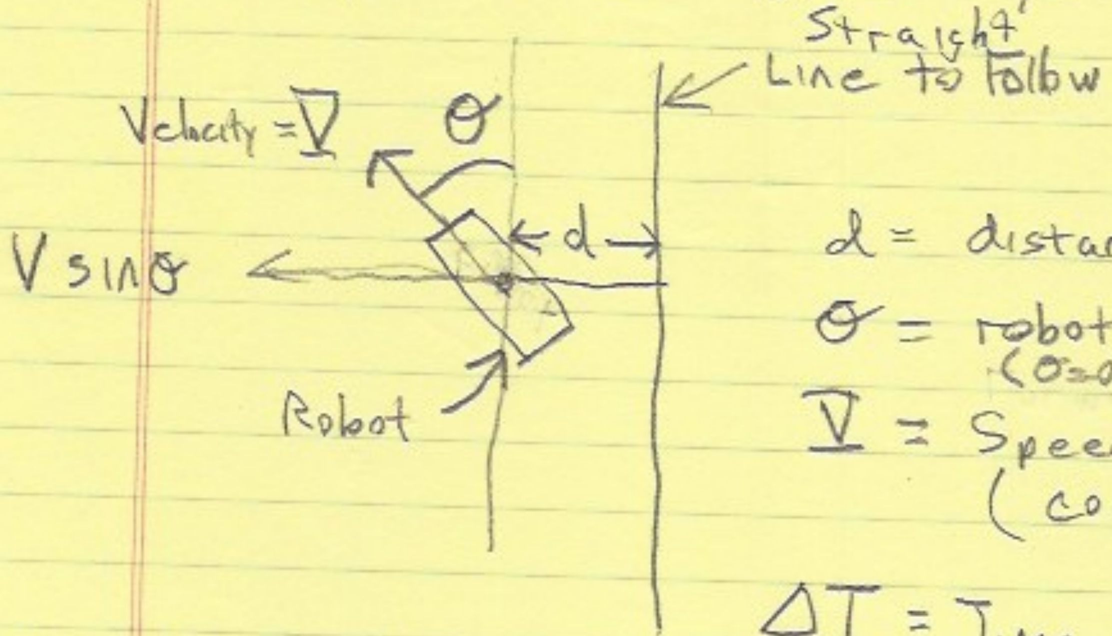
9/18/23 6.3100/2 1L



d = distance from line

θ = robot angle to line
($\theta = 0 \Rightarrow$ parallel to line)

$\underline{V}$ = Speed of Robot
(constant!)

ΔT = Time between distance samples

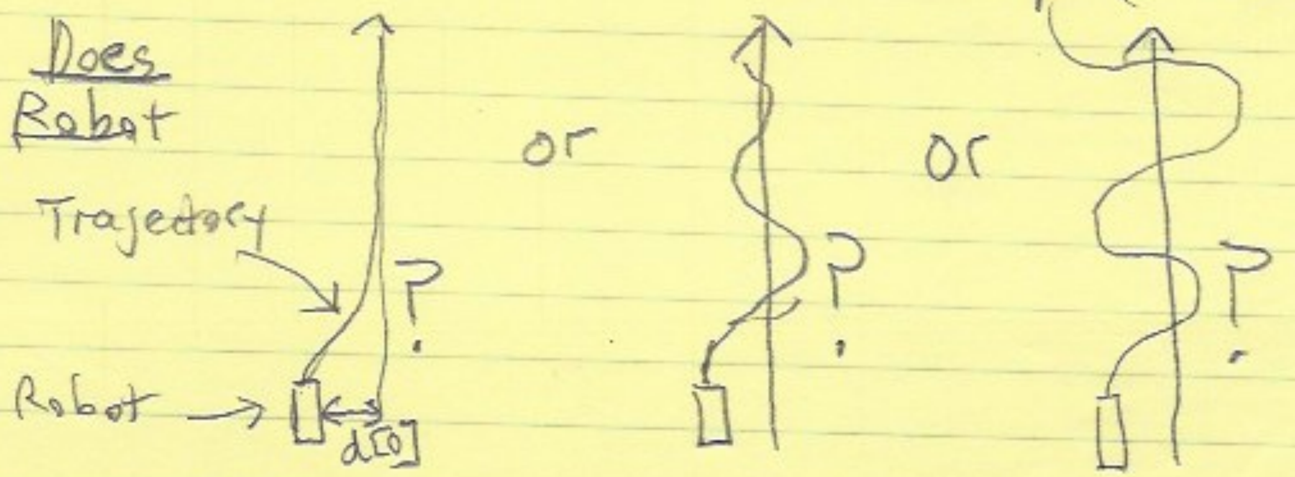
We measure distance to line

$$d[n] = d[n-1] + \Delta T \, \underline{V} \sin\theta$$

$$\theta[n] = \theta[n-1] + \Delta T \, \omega[n-1]$$

We control "rate of robot rotation"

Problem suppose $\omega[n] = K_p(-d[n])$



(2L)

$\sin\theta \simeq \theta$ if theta in radians!

(1) $d[n] \simeq d[n-1] + \Delta T V \theta[n-1]$

(2) $d[n-1] \simeq d[n-2] + \Delta T V \theta[n-2]$

(1)-(2) $d[n] - d[n-1] = d[n-1] - d[n-2] + \Delta T V(\theta[n-1] - \theta[n-2])$

$$\theta[n-1] - \theta[n-2] = \Delta T\, \omega[n-2] = \Delta T\left(-K_p\,(d[n-2])\right)$$

$$d[n] = 2d[n-1] - d[n-2] + \Delta T V\left(\Delta T\left(-K_p\, d[n-2]\right)\right)$$

$$= 2d[n-1] - \left(1 + \Delta T^2 V K_p\right) d[n-2]$$

$$\underbrace{d[n]}_{\lambda^2} - 2\underbrace{d[n-1]}_{\lambda} + \left(1 + \Delta T^2 V K_p\right)\underbrace{d[n-2]}_{1} = 0$$

$-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - ac}$

$$1 \pm \sqrt{1 - (1 + \Delta T^2 V K_p)}$$

$$1 \pm \sqrt{-\underbrace{(\Delta T)^2}_{>0}\,\underbrace{V}_{>0}\,\underbrace{K_p}_{>0}}$$ Complex Numbers.

1R

Suppose we have

$$y[n] = \underbrace{a_{-1}}\, y[n-1] + \underbrace{a_{-2}}\, y[n-2]$$

real coeffs

Note a_{-1} to go with $y[-1]$

& a_{-2} to go with $y[-2]$

$$\underbrace{y[0] \text{ and } y[1]}_{\text{initial conditions}} \text{ are given!}$$

Plug & Chug

$$y[2] = a_{-1}\, y[1] + a_{-2}\, y[0]$$

$$y[3] = a_{-1}\, y[2] + a_{-2}\, y[1]$$

$$= a_{-1}^2\, y[1] + a_{-1} a_{-2}\, y[0]$$

$$+$$

$$a_{-2} a_{-1}\, y[1] + a_{-2}^2\, y[0]$$

$$y[4] = ?$$

$$\vdots$$

$$y[n] = ?$$

Note: Solution exists and is unique!

Plug & Chug Always results in Solution

Only one plug and chug result given initial conditions

2R

<u>Guess at Solution</u>

$$y[n] = c\lambda^n \Rightarrow y[n-1] = c\lambda^{n-1} = \lambda^{-1} y[n]$$

$$y[n] = a_{-1} y[n-1] + a_{-2} y[n-2]$$

$$\cancel{y[n]} = a_{-1} \lambda^{-1} \cancel{y[n]} + a_{-2} \lambda^{-2} \cancel{y[n]}$$

$$1 - a_{-1}\lambda^{-1} - a_{-2}\lambda^{-2} = 0$$

$$\underbrace{\lambda^2 - a_{-1}\lambda - a_{-2} = 0}$$

$$\boxed{\begin{array}{c} x^2 + bx + c = 0 \\ x_1, x_2 = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c} \end{array}}$$

quadratic with two sol'ns

$$\lambda_1, \lambda_2 = \frac{a_{-1}}{2} \pm \sqrt{\frac{(a_{-1})^2}{4} + a_{-2}}$$

<u>What if</u>

$$y[n] = c_1 \lambda_1^n + c_2 \lambda_2^n$$

<u>Satisfies Diff Eqn</u>

$$c_1\lambda_1^n + c_2\lambda_2^n \stackrel{?}{=} a_{-1}(c_1\lambda_1^{n-1} + c_2\lambda_2^{n-1}) + a_{-2}(c_1\lambda_1^{n-2} + c_2\lambda_2^{n-2})$$

<u>or</u>

$$c_1\lambda_1^n - a_{-1}c_1\lambda_1^{n-1} - a_{-2}c_1\lambda_1^{n-2} = 0$$
$$+$$
$$c_2\lambda_2^n - a_{-1}c_2\lambda_2^{n-1} - a_{-2}c_2\lambda_2^{n-2} = 0$$

$$= 0 ? \checkmark$$

Determine λ_1, λ_2 from Diff Eqn
Determine C_1, C_2 from initial conditions

Two eqns two unknowns
$$\begin{cases} C_1 \underbrace{\cancel{\lambda_1^0}}_{=1} + C_2 \underbrace{\cancel{\lambda_2^0}}_{=1} = y[0] \\ C_1 \lambda_1 + C_2 \lambda_2 = y[1] \end{cases}$$

natural frequencies
(Not like Hertz or cycles/sec)

Then $\boxed{y[n] = C_1 \lambda_1^n + \lambda_2 \lambda^n \text{ must be solution}}$ $\Leftrightarrow$ $\boxed{\text{Satisfies Difference Eqn, Matches Initial Conditions}}$

Because Solution Exists and is unique

if $|\lambda_1| < 1$ & $|\lambda_2| < 1$

$y[n] \xrightarrow[n\to\infty]{} 0$

if $|\lambda_1| > 1$ and $|\lambda_2| > 1$

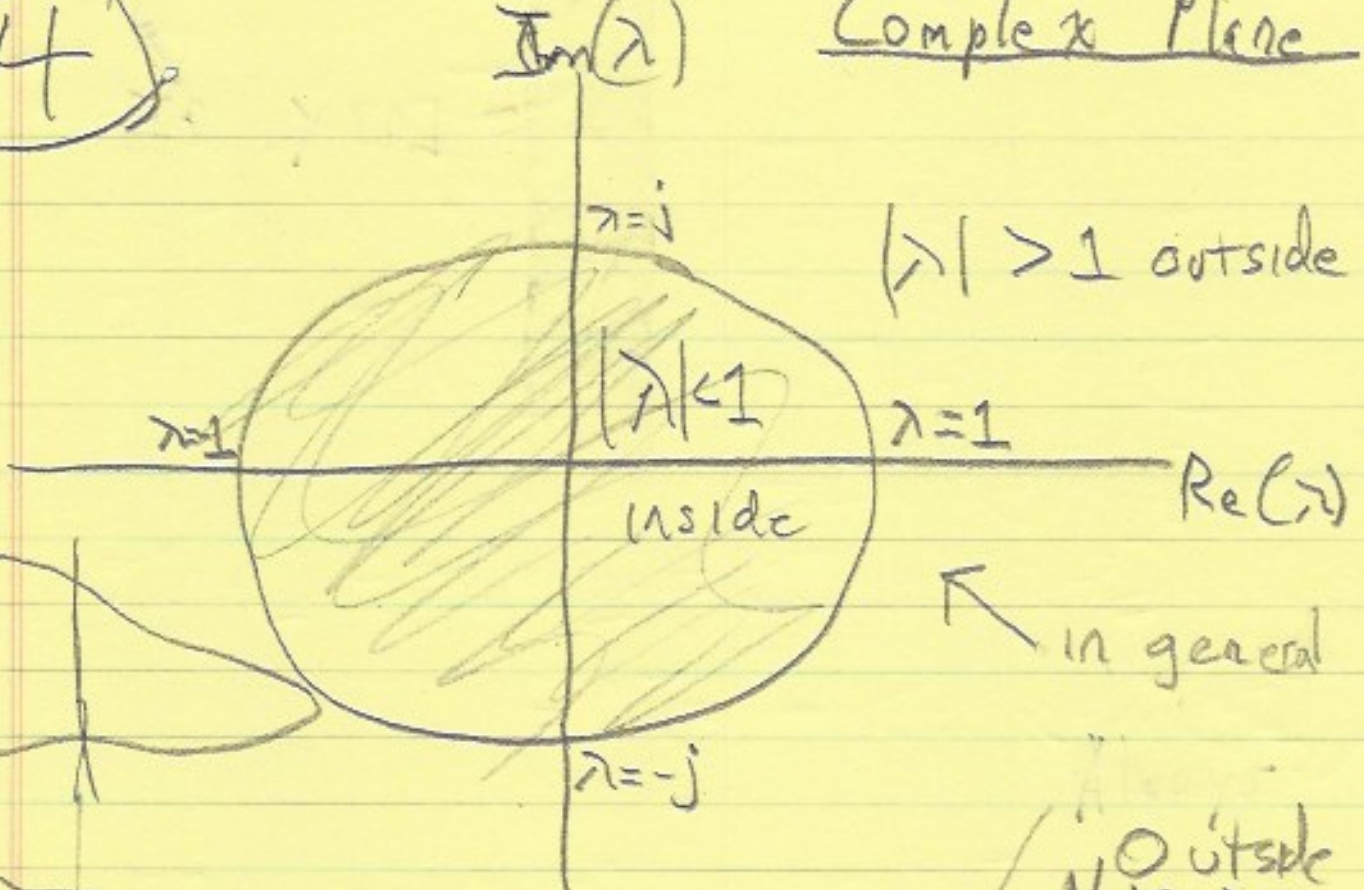
$y[n] \xrightarrow[n\to\infty]{} \pm\infty$

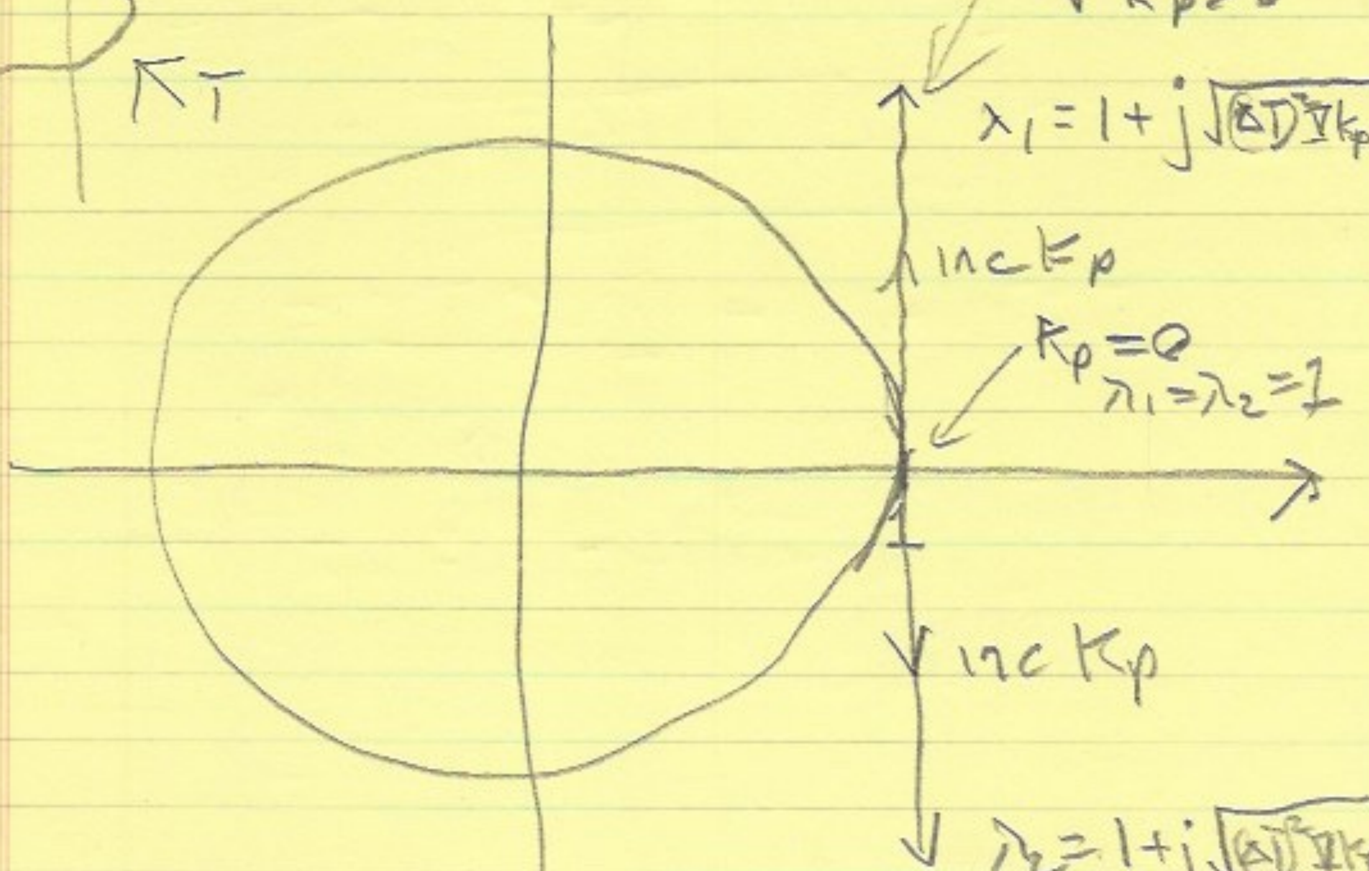
if $|\lambda_1| > 1$ and $|\lambda_2| < 1$

$y[n] \xrightarrow[n\to\infty]{} \pm\infty$ for almost all initial conditions!

4

Complex Plane





February 21, 2023

6.3100 Lecture 5 Notes – Spring 2023

Second order DT system, Proportional control, and PD control

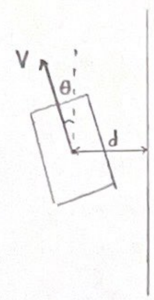
Dennis Freeman and Kevin Chen

Outline:

1. Second order DT system: line following example
2. Stability of a second order DT system under a proportional controller
3. Proportional derivative controllers

1. Second order DT system: line following example

Thus far, we studied first order systems in the previous lectures. We are going to use an example to study second order systems, where new stability properties arise and the proportional controller becomes insufficient. Let's consider a line following example illustrated below.



$$d[n] = d[n-1] + \Delta T\, V \sin\theta[n-1]$$
$$\theta[n] = \theta[n-1] + \Delta T\, \omega[n-1]$$
$$\omega[n] = \gamma u[n]$$

Suppose the robot wants to move along a straight line. The robot has a constant velocity V, and we can control its rotation speed w[n] through an input u[n]:

$$w[n] = \gamma u[n]$$

In this problem, we have an optical sensor that can measure the distance between the robot and the line. Our goal is to design a controller that minimizes the distance between the desired position $d_d[n]$ (line) and the measured position $d_m[n]$.

The discrete time kinematic equation is given by:

$$d_m[n] = d_m[n-1] + \Delta TV \sin\theta[n-1]$$

This is a nonlinear equation, and we need to linearize the term $\sin\theta$. We have $\sin\theta \approx \theta$ for small θ. The system equation becomes:

$$d_m[n] = d_m[n-1] + \Delta TV\theta[n-1]$$

We need to write $\theta[n]$ in terms of $d_m[n]$, where $\theta[n]$ is given by:

$$\theta[n] = \theta[n-1] + \Delta T w[n-1] = \theta[n-1] + \Delta T\gamma u[n-1]$$

Since we can measure the distance $d_m[n]$, we can set up a proportional controller relative to the measured distance:

$$u[n] = K_p(d_d[n] - d_m[n])$$

Substituting this controller into our system equation, we obtain:

$$\theta[n] = \theta[n-1] + \Delta T K_p \gamma (d_d[n-1] - d_m[n-1])$$

Now let's write the system equations and simplify:

$$d_m[n] = d_m[n-1] + \Delta T V \theta[n-1]$$

$$d_m[n-1] = d_m[n-2] + \Delta T V \theta[n-2]$$

Subtracting these two equations, we obtain:

$$d_m[n] - d_m[n-1] = d_m[n-1] - d_m[n-2] + \Delta T V(\theta[n-1] - \theta[n-2])$$

Next, we substitute the difference of θ:

$$d_m[n] - d_m[n-1] = d_m[n-1] - d_m[n-2] + \Delta T V(\Delta T K_p \gamma (d_d[n-2] - d_m[n-2]))$$

We can simplify this equation, collect terms, and get the control system equation:

$$d_m[n] - 2d_m[n-1] + (1 + \Delta T^2 V K_p \gamma) d_m[n-2] = \Delta T^2 V K_p \gamma d_d[n-2]$$

Note that the variable in this equation is d_m, and we have the indices n, n-1, and n-2. This is a 2nd order DT system with proportional control.

2. <u>Stability of a second order DT system under a proportional controller</u>

We need to go through the same exercise again to analyze the behavior of a 2nd order DT system. The general solution of a 2nd order DT system with zero-driving ($d_d[n]=0$) is given by:

$$d_m[n] = C_1 \lambda_1^n + C_2 \lambda_2^n$$

where λ_1 and λ_2 are natural frequencies, and C_1 and C_2 are coefficients determined by the initial conditions. To analyze system stability, we need to solve for the value of λ_1 and λ_2. We can substitute the solution $d_m[n] = \lambda^n$ into the system equation and then solve for λ. Here we are interested in the homogeneous solution (when the right hand driving function is 0). We have:

$$\lambda^n - 2\lambda^{n-1} + (1 + \Delta T^2 V K_p \gamma)\lambda^{n-2} = 0$$

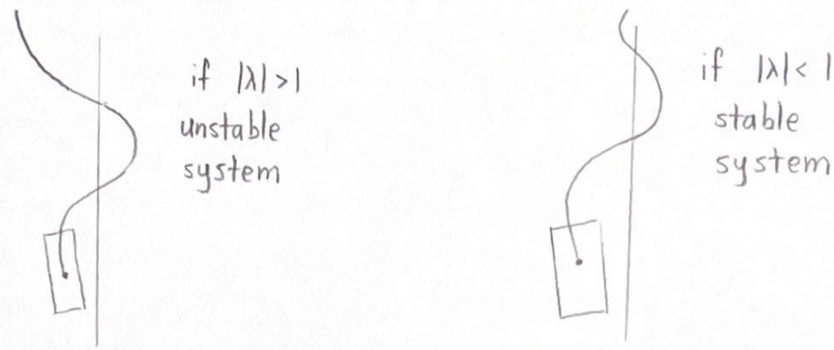
$$\lambda^2 - 2\lambda^1 + (1 + \Delta T^2 V K_p \gamma) = 0$$

We can solve for λ and obtain:

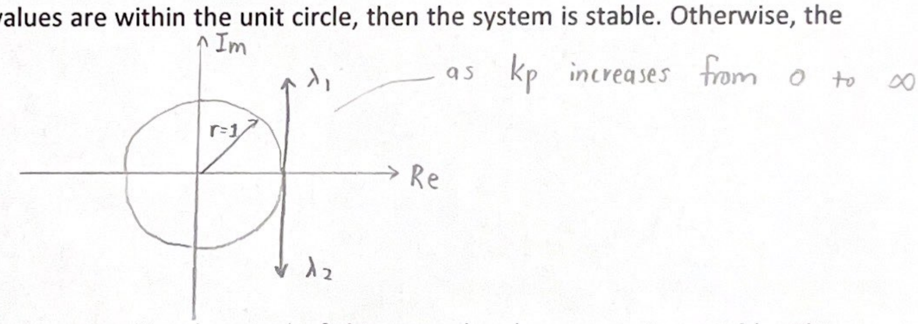
$$\lambda = 1 \pm j\sqrt{\Delta T^2 V K_p \gamma}$$

This is an interesting result that we should carefully study. First, λ is a complex number, which contains a real part and an imaginary part. We see that both have a larger than 1 amplitude. For this line following example under proportional control, the system is unstable regardless of the

K_p values we pick! This is an important result. For higher order systems, a simple proportional controller usually does not work well. Implementing a proportionally controller can lead to an unstable system. The sketch below illustrates a sample line-following experiment.



Another important concept that we want to introduce is called root locus plot. It is a plot in the complex plane that shows the location of natural frequencies as a function of our controller parameter. In this example, the only controller parameter we have is Kp. How does a change of Kp changes the two λ values? This plot is shown below, where we start with Kp = 0 and end with $K_p \to \infty$. Note that all λ values are outside of the unit circle, which means the system is unstable. For discrete time problems, the unit circle marks the stability region. For a given parameter value, if all the associated λ values are within the unit circle, then the system is stable. Otherwise, the system is unstable.



The root locus plot shows proportional control of this second order system is unstable. This seems to be a major problem. How can we stabilize a higher order system and then optimize the controller parameters according to some metrics (fastest convergence or smallest steady state error)? Next, we will introduce the proportional-derivative (PD) controller.

3. Proportional-derivative (PD) controller and 3rd order system

While a proportional (P) controller cannot stabilize the 2nd order system, we can implement a proportional-derivative (PD) controller. The intuition is that our controller should not only care about how far the car is relative to the setpoint, but also about the rate of change. The PD controller is given by:

$$c[n] = K_p(d_d[n] - d[n]) + K_d\left(\frac{d_d[n] - d_d[n-1]}{\Delta T} + \frac{d[n] - d[n-1]}{\Delta T}\right)$$

If we substitute this controller into the system equation, we will obtain:

$$d[n] - 2d[n-1] + d[n-2] \\ = \Delta T^2 V\gamma[K_p(d_d[n-2] - d[n-2]) + K_d(\frac{d_d[n-2] - d_d[n-3]}{\Delta T} + \frac{d[n-2] - d[n-3]}{\Delta T})]$$

We can rearrange this equation and obtain:

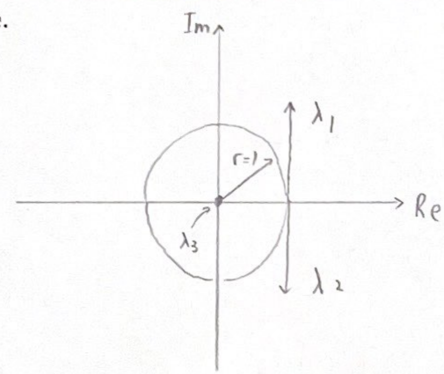
$$d[n] - 2d[n-1] + d[n-2](1 + \Delta T^2 V\gamma K_p + K_d V\gamma\Delta T) + d[n-3](-K_d V\gamma\Delta T) \\ = d_d[n-2](\Delta T^2\gamma V k_p + K_d\Delta T\, V\gamma) + d_d[n-3](-K_d\Delta T\, V\gamma)$$

This is a 3rd order difference equation, and it has the following solution:

$$d[n] = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n$$

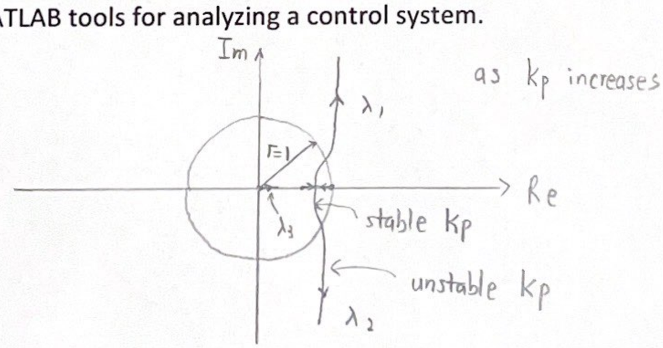
Now there are 3 natural frequencies, and they are functions of Kp and Kd. To design a good controller, we need to choose Kp and Kd such that our system is stable. Since the equation is sufficiently complex, we will use a numerical tool to generate the root locus plots.

Case 1: set Kd = 0, V=1, $\gamma = 1$, $\Delta T = 0.01$, and vary Kp. This becomes a proportional controller with two natural frequencies. We see that the system is unstable regardless of the Kp value we choose.



same picture as in pg 3.

Case 2: set Kd = 20, V=1, $\gamma = 1$, $\Delta T = 0.01$, and vary Kp. We see that now there is an optimal Kp value that corresponds to the fastest convergence. In the next lecture, we will introduce more MATLAB tools for analyzing a control system.



- we will generate the exact plot in MATLAB in next lecture