09/20/23

* Review of 2nd Order System
* Complex natural frequencies, oscillation and parameter determination.

1. 2nd Order Linear Constant Coefficient Ordinary Difference Equation.

$$\begin{cases} y[n] = a_{-1} y[n-1] + a_{-2} y[n-2] & \text{(only consider homogeneous one for now).} \\ y[0] = y_0, \quad y[1] = y_1 & \text{(initial condition).} \end{cases}$$

Guessed solution $y[n] = c\lambda^n$, substitute into the equation:

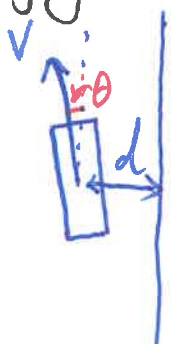
$\lambda^2 - a_{-1}\lambda - a_{-2} = 0$ (characteristic equation).

Two solutions of natural frequency: $\lambda_{1,2} = \dfrac{a_{-1} \pm \sqrt{a_{-1}^2 + 4a_{-2}}}{2}$ (complex numb in general)

General form of solution: $y[n] = c_1\lambda_1^n + c_2\lambda_2^n$

Determine c_1, c_2 with initial conditions: $\begin{cases} y[0] = c_1 + c_2 \\ y[1] = c_1\lambda_1 + c_2\lambda_2 \end{cases}$

A physical example introduced last time: Line follower.



$$\begin{cases} d[n] = d[n-1] + \Delta T\, V\theta[n-1] & (\sin\theta \simeq \theta). \\ \theta[n] = \theta[n-1] + \Delta T\, \omega[n-1] & (\omega[n] \text{ is the angular velocity}) \end{cases}$$

Control Command: set $\omega[n] = -K_p \gamma d[n]$.

"−" for negative feedback. γ is newly added, to represent control "efficiency".

Re-arrange equations $\Rightarrow d[n] - 2d[n-1] + (1+\Delta T^2 \gamma V K_p)\, d[n-2] = 0$.

Natural frequency: $\lambda_{1,2} = 1 \pm j\sqrt{\Delta T^2 V \gamma K_p}$. (complex roots).

- solution: $d[n] = C_1 \lambda_1^n + C_2 \lambda_2^n$.

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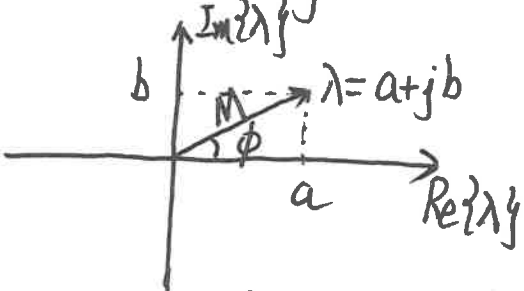
2. Evolution of $d[n]$.

$d[n] \sim \lambda^n = (1 \pm j\sqrt{\Delta T^2 V \gamma K_p})^n = ?$ How to see the trend?

Review of Polar Form of Complex number:

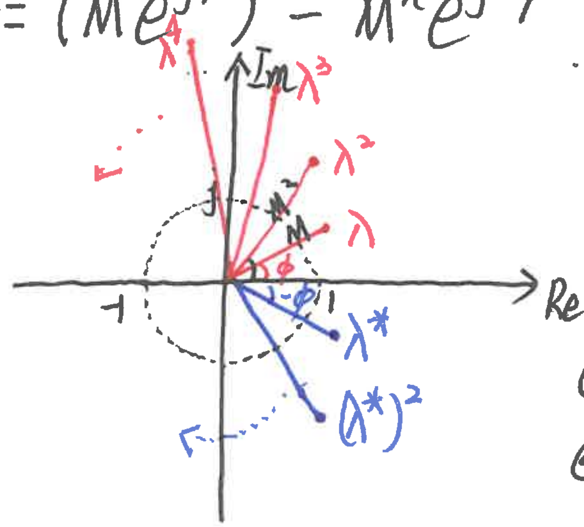
$$\lambda = a + jb = M e^{j\phi}$$



$$\begin{cases} M = |\lambda| = \sqrt{a^2+b^2} \\ \phi = \measuredangle\lambda = \arctan\frac{b}{a} \end{cases} \quad \text{or} \quad \begin{cases} a = M\cos\phi \\ b = M\sin\phi \end{cases}$$

It is much easier to calculate λ^n using polar form:

$$\lambda^n = (M e^{j\phi})^n = M^n e^{jn\phi}$$

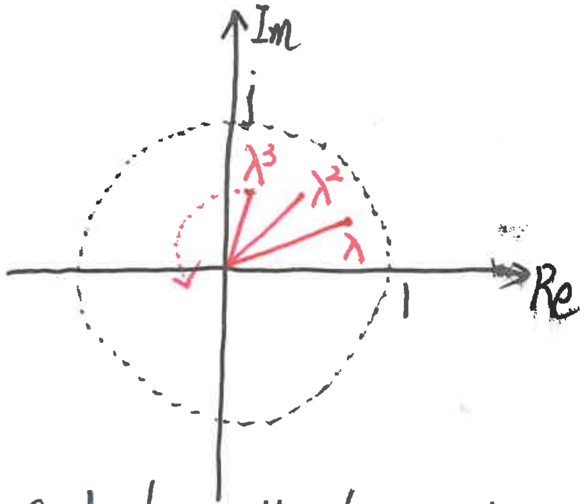


① $|\lambda| > 1$, as $n \to \infty$, $|\lambda^n| \to \infty$.
"a spiral that diverges".

② phase increase by ϕ when n increase by 1

③ $(\lambda^*)^n$ spirals in opposite direction.

④ unstable!



① $|\lambda|<1$, as $n\to\infty$, $|\lambda^n|\to 0$.

② stable.

Go back to the line follower example: $\lambda_1 = 1 + j\sqrt{\Delta T^2 V \gamma K_p} = \lambda$, $\lambda_2 = \lambda^*$.

$\lambda_1 = M e^{j\phi}$ $\quad M = \sqrt{1+\Delta T^2 V \gamma K_p}$, $\quad \phi = \arctan\sqrt{\Delta T^2 V \gamma K_p}$

$\lambda_2 = M e^{-j\phi}$

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3. Match initial conditions.

$$\begin{cases} d[0] = d_0 \\ \theta[0] = \theta_0 \end{cases} \Longrightarrow \begin{cases} d[0] = d_0 \\ d[1] = d_0 + \Delta T V \theta_0 \end{cases}$$

substitute $d[n] = c_1 \lambda^n + c_2 \lambda^{*n}$ $\quad \begin{cases} c_1 + c_2 = d[0] \\ c_1\lambda + c_2\lambda^* = d[1] \end{cases}$

solution: $c_1 = \dfrac{d[0]\lambda^* - d[1]}{\lambda^* - \lambda}$

$$c_2 = \frac{d[0]\lambda - d[1]}{\lambda - \lambda^*} = \left(\frac{d[0]\lambda^* - d[1]}{\lambda^* - \lambda}\right)^* = c_1^*.$$

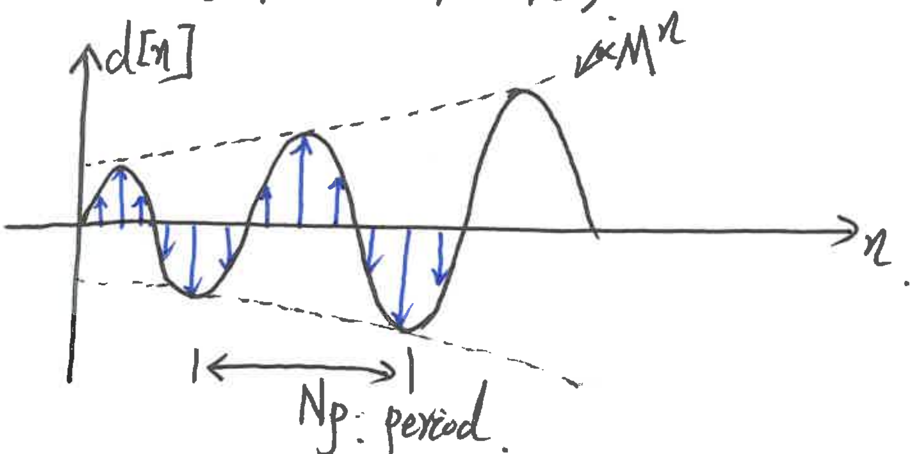
The two natural frequencies are complex conjugate, their coefficients are complex conjugate, too! (ensures real $d[n]$)

Assume $c_1 = M_c e^{j\phi_c}$, $\quad c_2 = c_1^* = M_c e^{-j\phi_c}$

$$d[n] = \underbrace{M_c e^{j\phi_c}}_{c_1} \underbrace{\left(M e^{j\phi}\right)^n}_{\lambda^n} + \underbrace{M_c e^{-j\phi_c}}_{c_2} \underbrace{\left(M e^{-j\phi}\right)^n}_{(\lambda^*)^n}$$

$$= M_c M^n \left(e^{j(n\phi+\phi_c)} + e^{-j(n\phi+\phi_c)}\right)$$

$$= 2 M_c M^n \cos(n\phi + \phi_c)$$



Determine system parameter (e.g. γ) from $d[n]$:

a. From period of oscillation: N_p

$$N_p = \frac{2\pi}{\phi} = \frac{2\pi}{\arctan\sqrt{\Delta T^2 V \gamma K_p}} \approx \frac{2\pi}{\sqrt{\Delta T^2 V \gamma K_p}}$$

($\arctan x \approx x$ for small x if ΔT is short)

$$\gamma = \left(\frac{2\pi}{N_p \Delta T \sqrt{K_p V}}\right)^2.$$

b. From amplitude envelope: Assume at $n = n_d$, Amplitude doubles:

$$M^{n_d} = 2 \cdot M^1 \qquad M = 2^{\frac{1}{n_d - 1}} \Rightarrow \sqrt{1 + \Delta T^2 V \gamma K_p} = 2^{\frac{1}{n_d - 1}}$$