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* Review of 2nd Order System
* Complex natural frequencies, oscillation and parameter determination.

1. 2nd Order **Linear Constant Coefficient Ordinary Difference Equation.**

\[
y[n] = a_1 y[n-1] + a_2 y[n-2] \quad \text{(only consider homogeneous one for now)}
\]
\[
y[0] = y_0, \quad y[1] = y_1, \quad \text{(initial condition)}
\]

Guessed solution \( y[n] = C \lambda^n \), substitute into the equation:

\[
\lambda^2 - a_1 \lambda - a_2 = 0 \quad \text{(characteristic equation)}
\]

Two solutions of natural frequency: \( \lambda_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2} \) (complex number, in general)

General form of solution: \( y[n] = C_1 \lambda_1^n + C_2 \lambda_2^n \)

Determine \( C_1, C_2 \) with initial conditions:
\[
\begin{cases}
y[0] = C_1 + C_2 \\
y[1] = C_1 \lambda_1 + C_2 \lambda_2
\end{cases}
\]

A physical example introduced last time: Line follower.

\[
\begin{align*}
d[n] &= d[n-1] + \Delta T V \Theta[n-1] \quad (\sin \Theta \approx \Theta) \\
\Theta[n] &= \Theta[n-1] + \Delta T W[n-1] \quad (W[n] \text{ is the angular velocity})
\end{align*}
\]

Control Command: set \( W[n] = -K_p \rho d[n] \).
Re-arrange equations \[ d[n] - 2d[n-1] + (1 + \Delta t^2 \gamma V K_p) d[n-2] = 0. \]

Natural frequency: \[ \lambda_{1,2} = 1 \pm j \sqrt{\Delta t^2 \gamma V K_p} \] (complex roots) 

Solution: \[ d[n] = C_1 \lambda_1^n + C_2 \lambda_2^n. \]

2. Evolution of \( d[n] \).

\[ d[n] \approx \lambda^n = \left(1 \pm j \sqrt{\Delta t^2 \gamma V K_p}\right)^n = ? \] How to see the trend?

Review of Polar Form of Complex number:

\[ \lambda = a + jb = M e^{i\phi} \]

It is much easier to calculate \( \lambda^n \) using polar form:

\[ \lambda^n = (Me^{i\phi})^n = M^n e^{i n\phi}. \]

1. \( |\lambda| > 1 \), as \( n \to \infty \), \( |\lambda^n| \to \infty \). "a spiral that diverges".
2. Phase increase by \( \phi \) when \( n \) increases by 1.
3. \( (\lambda^*)^n \) spirals in opposite direction.
4. Unstable!
1. \(|\lambda|<1\), as \(n \to \infty\), \(|\lambda^{|}n| \to 0\).

2. stable.

Go back to the line follower example: 
\[ \lambda_1 = 1 + j\sqrt{\tau^2 V_y K_p} = \lambda, \quad \lambda_2 = \lambda^* \]
\[ \lambda_1 = M e^{j\phi} \quad M = \sqrt{1 + j\sqrt{\tau^2 V_y K_p}}, \quad \phi = \arctan \sqrt{\tau^2 V_y K_p} \]
\[ \lambda_2 = M e^{-j\phi} \]

3. Match initial conditions.

\[ \begin{align*}
\{ d[0] &= d_0 \\
\Theta[0] &= \Theta_0 \end{align*} \quad \longrightarrow \quad \begin{align*}
\{ d[0] &= d_0 \\
\{ d[1] &= d_0 + \Delta T V \Theta_0 \\
\end{align*} \]

Substitute \( d[n] = C_1 \lambda^n + C_2 \lambda^* n \)

\[ \begin{align*}
C_1 + C_2 &= d[0] \\
C_1 \lambda + C_2 \lambda^* &= d[1] \\
\end{align*} \]

Solution:
\[ \begin{align*}
C_1 &= \frac{d[0] \lambda^* - d[1]}{\lambda^* - \lambda} \\
C_2 &= \frac{d[0] \lambda - d[1]}{\lambda - \lambda^*} = \left( \frac{d[0] \lambda^* - d[1]}{\lambda^* - \lambda} \right)^* = C_1^* \\
\end{align*} \]

The two natural frequencies are complex conjugate.
Their coefficients are complex conjugate, too! (Ensures real \( d[n^2] \))
Assume $C_1 = M_c e^{j\phi_c}$, $C_2 = C_1^*$ = $M_c e^{-j\phi_c}$

\[
d[n] = \frac{M_c e^{j\phi_c} (M e^{j\phi})^n}{C_1} + \frac{M_c e^{-j\phi_c} (M e^{-j\phi})^n}{C_2} (\lambda^n)
\]

\[= M_c M^n (e^{j(n\phi_c+\phi)} + e^{-j(n\phi+\phi_c)})
\]

\[= 2M_c M^n \cos(n\phi + \phi_c)
\]

Determine system parameter (e.g. $\gamma$) from $d[n]$:

a. From period of oscillation: $N_p$

\[N_p = \frac{2\pi}{\phi} = \frac{2\pi}{\arctan(\frac{\Delta T^2 V}{YK_p})} \approx \frac{2\pi}{\Delta T^2 VY K_p} \quad (\text{for small } \Delta T \text{ is short})
\]

\[\gamma = \left(\frac{2\pi}{N_p \Delta T \sqrt{K_y V}}\right)^2
\]

b. From amplitude envelope: Assume at $n=N_d$, Amplitude doubles:

\[M^{nd} = 2 \cdot M^1 \quad M = 2 \frac{1}{\sqrt{n+1+\Delta T^2 VY K_p}} = 2 \frac{1}{\sqrt{nd-1}}
\]