6.3100/2 – lecture 9/27/23

Last time: State space form (vector difference equation)

$$\underset{N_s}{\updownarrow}\begin{bmatrix} x[n] \end{bmatrix} = \underset{N_s}{\updownarrow}\overset{\leftarrow N_s \rightarrow}{\begin{bmatrix} A \end{bmatrix}}\begin{bmatrix} x[n-1] \end{bmatrix} + \underset{N_s}{\updownarrow}\overset{\leftarrow 1 \rightarrow}{\begin{bmatrix} B \end{bmatrix}} u[n-1]$$

General solution: $\vec{x}[n] = A^n \vec{x}[0] + \sum_{m=0}^{n-1} A^{(n-1)-m} B\, u[m]$

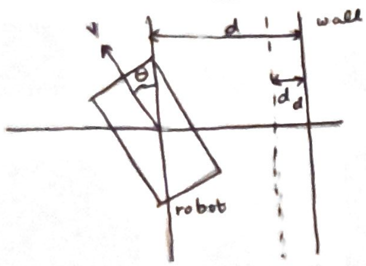
Let $(\lambda_i, \vec{s}_i)$ be eigenvals and eigenvecs of A (for $i = 1, \ldots, N_s$)

Then: $\vec{x}[n] = c_1 \lambda_1^n \vec{s}_1 + \cdots + c_{N_s} \lambda_{N_s}^n \vec{s}_{N_s}$

and: $\lim_{n \to \infty} \vec{x}[n] = 0$ if $|\lambda_i| < 1 \quad \forall i$

Eigenvalues are natural frequencies!

Line following example



$d[n] \approx d$ [illegible] $\qquad (\sin\theta \approx \theta)$

$\theta[n] = \theta[n-1] +$ [illegible] $\qquad$ ($\omega[n]$ rate of angle change)

Assume $d_d[n]$ known

① Last time: Choose $\omega[n] = K_p(d_d[n] - d[n])$

Note: $d_d[n]$ explicitly added from last time

$$\underbrace{\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix}}_{\vec{x}[n]} = \underbrace{\begin{bmatrix} 1 & \Delta T V \\ -\Delta T K_p & 1 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix} d_d[n-1]$$

Find eigenvals: $\det(\lambda I - \hat{A}) = 0 \Rightarrow \lambda^2 - 2\lambda + (1 + \Delta T^2 V K_p) = 0$

$\Rightarrow \lambda = 1 \pm j \Delta T \sqrt{V K_p}$

$|\lambda| \geq 1$ $\qquad$ never stable !!

② New controller: $\omega[n] = K_p(d_d[n] - d[n]) + K_\theta(-\theta[n])$

$$\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta T V \\ -\Delta T K_p & 1 - \Delta T K_\theta \end{bmatrix}}_{\hat{A}} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix} d_d[n-1]$$

Find eigenvals: $\det(\lambda I - \hat{A}) = 0 \Rightarrow \det\left(\begin{bmatrix} \lambda - 1 & -\Delta T V \\ \Delta T K_p & \lambda - (1 - \Delta T K_\theta) \end{bmatrix}\right) = 0$

$\Rightarrow (\lambda - 1)(\lambda - (1 - \Delta T K_\theta)) + \Delta T^2 V K_p = 0$

$\Rightarrow \lambda^2 - (2 - \Delta T K_\theta)\lambda + (1 - \Delta T K_\theta + \Delta T^2 V K_p) = 0$

$\Rightarrow \lambda_1, \lambda_2 = \left(1 - \frac{\Delta T K_\theta}{2}\right) \pm \Delta T \sqrt{\frac{K_\theta^2}{4} - V K_p}$

Note: Same as for controller ① when $K_\theta = 0$

Controller ②, ctd.

Plotting different combinations of K_p and K_θ – there exist choices that will cause the system to converge! (Requires $K_p \neq 0$ and $K_\theta \neq 0$)

(See plots attached.)

③ What if we're only able to measure $d[n]$ (not $\theta[n]$)?

Define: $e[n] = d_d[n] - d[n]$

Choose: $\omega[n] = K_p e[n] + K_d \left(\frac{e[n] - e[n-1]}{\Delta T}\right)$

What is the state space model?

$$d[n] = d[n-1] + \Delta T V \theta[n-1]$$

$$\theta[n] = \theta[n-1] + \Delta T \left(K_p e[n-1] + K_d \frac{e[n-1] - e[n-2]}{\Delta T} \right)$$

$$= \theta[n-1] + \Delta T K_p (d_d[n-1] - d[n-1]) + K_d (d_d[n-1] - d[n-1] - d_d[n-2] + d[n-2])$$

$$\begin{bmatrix} d[n] \\ d[n-1] \\ \theta[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \Delta T V \\ 1 & 0 & 0 \\ -(\Delta T K_p + K_d) & K_d & 1 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} d[n-1] \\ d[n-2] \\ \theta[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ (\Delta T K_p + K_d) d_d[n-1] - K_d d_d[n-2] \end{bmatrix}$$

Nat freqs = eigvals($\hat{A}$)

Plotting different combinations of K_p and K_d – we can still get the system to converge! (Requires $K_p \neq 0$ and $K_d \neq 0$)

(See plots attached.)

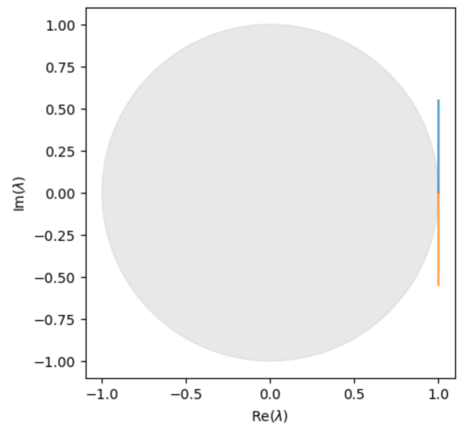
Code to generate plots:

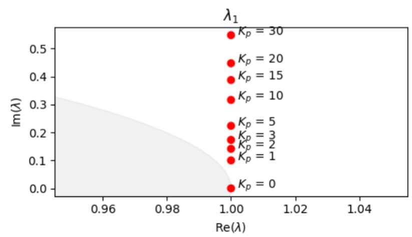
https://colab.research.google.com/drive/1sBG-LTzw0AmxWcTKHPF_CQIOwtzwytMg

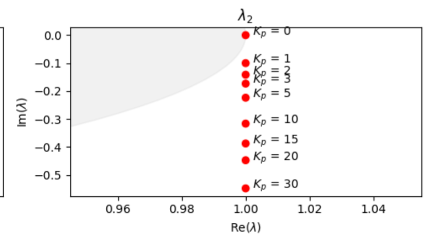
Plots over different values of K_θ

(for $V = 1$, $\Delta T = 1$, and K_p varying from 0 to 30)

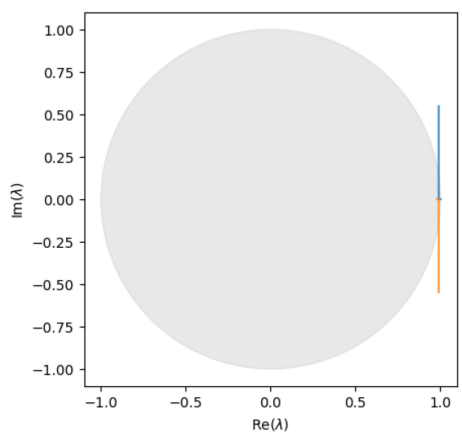
$K_\theta = 0$

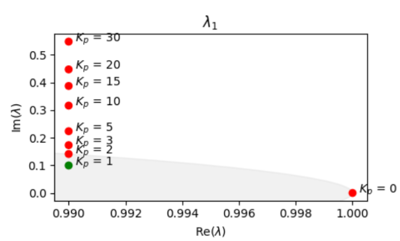


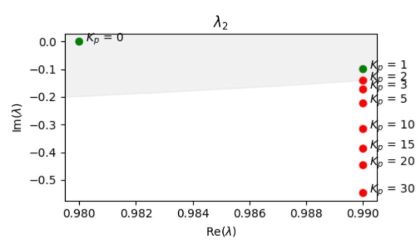




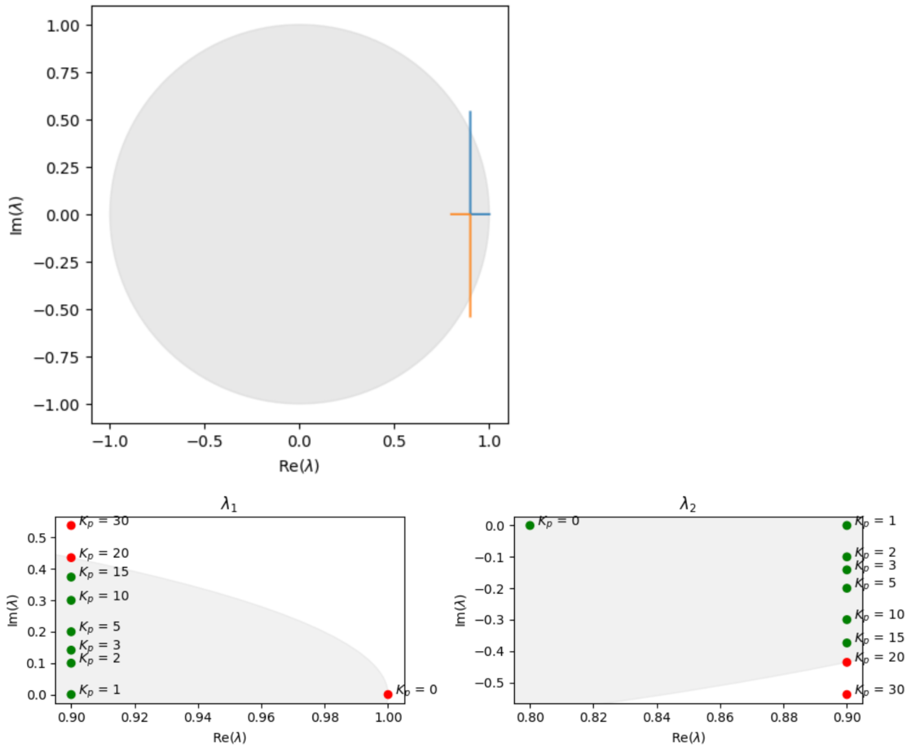
$K_\theta = 0.2$



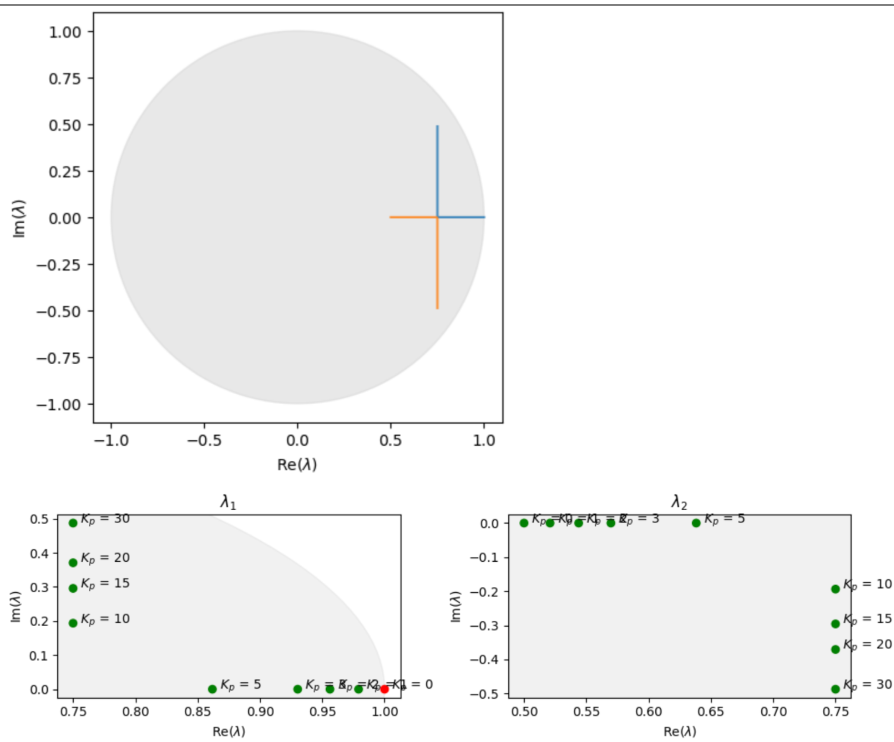




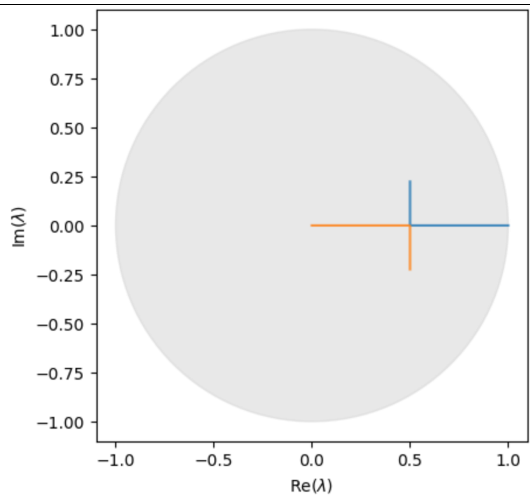
$K_\theta = 2$

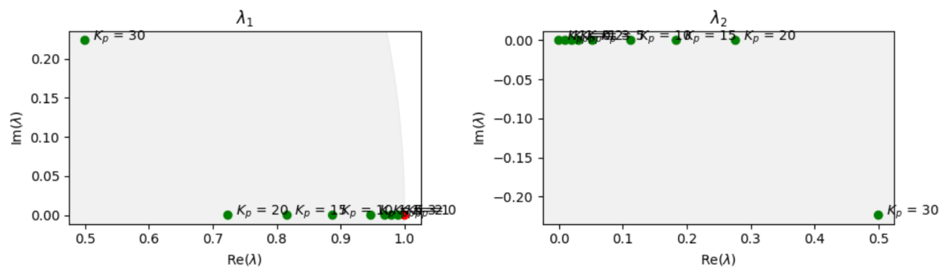


$K_\theta = 5$

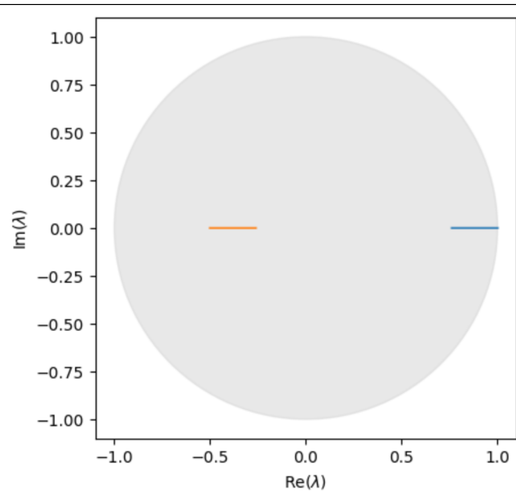


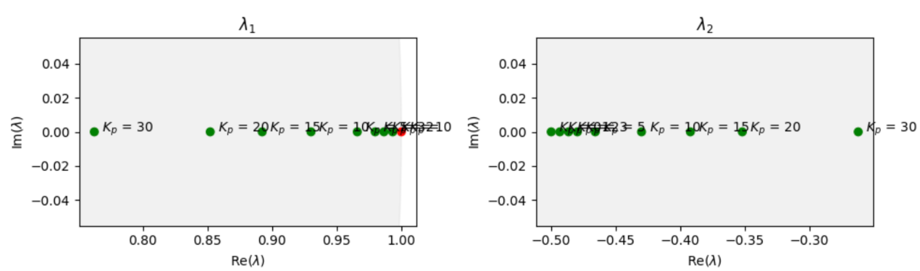
$K_\theta = 10$



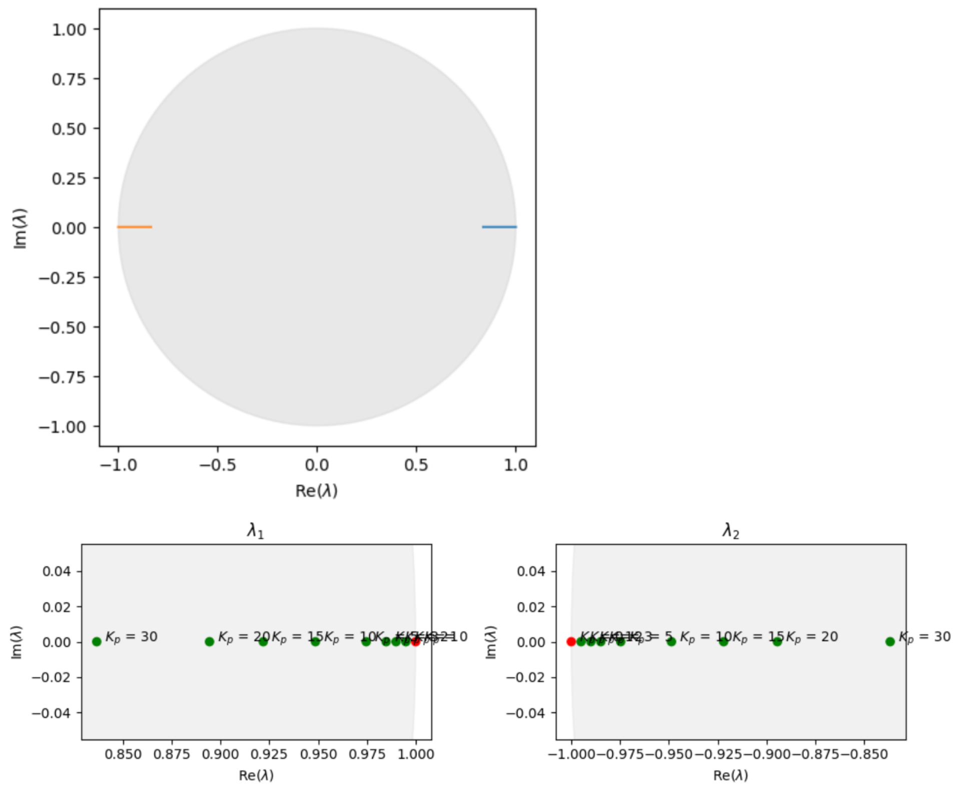


$K_\theta = 15$

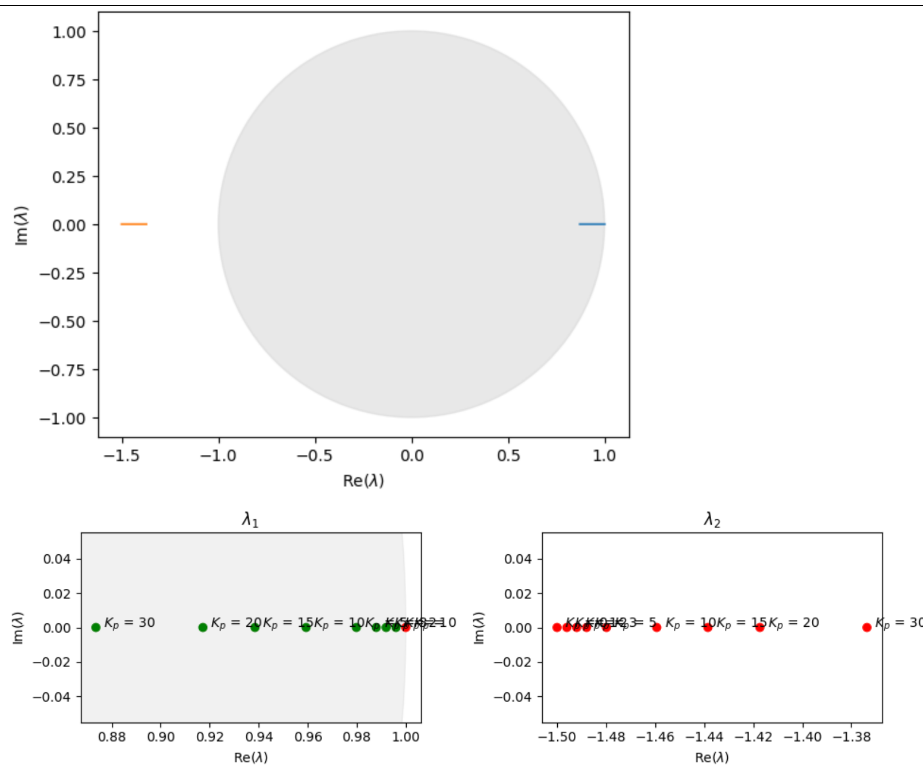




$K_\theta = 20$



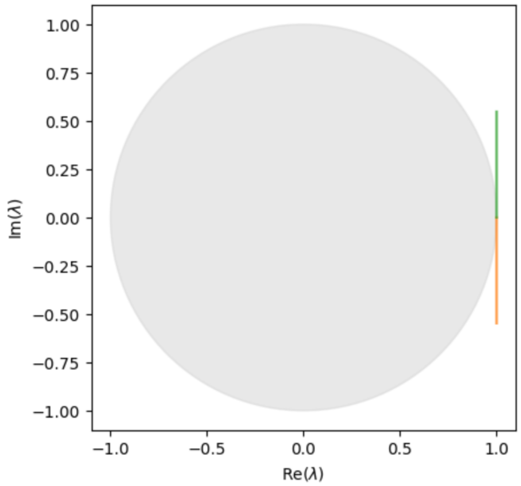
$K_\theta = 25$



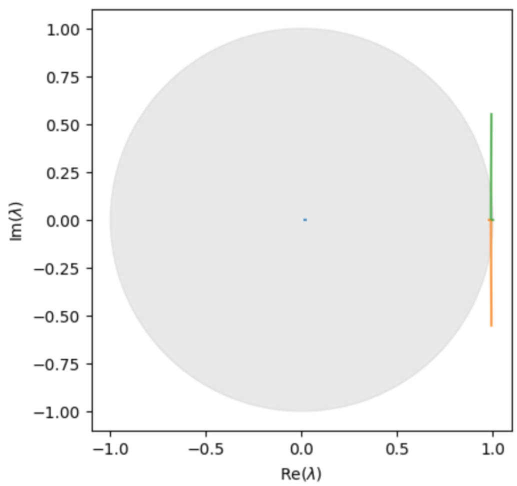
Plots over different values of K_d

(for $V = 1$, $\Delta T = 1$, and K_p varying from 0 to 30)

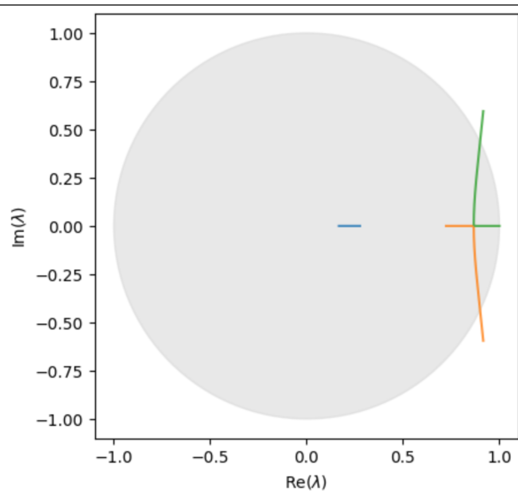
$K_d = 0$



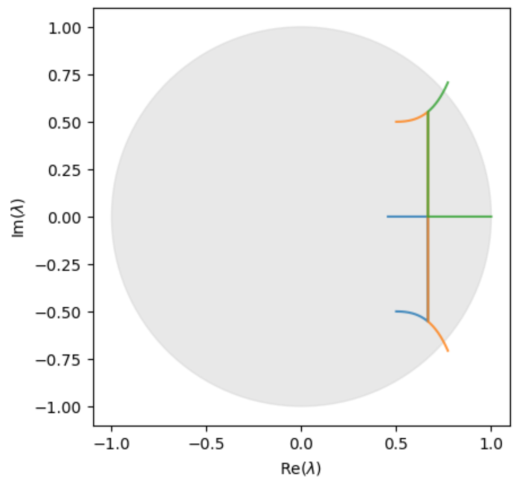
$K_d = 0.2$



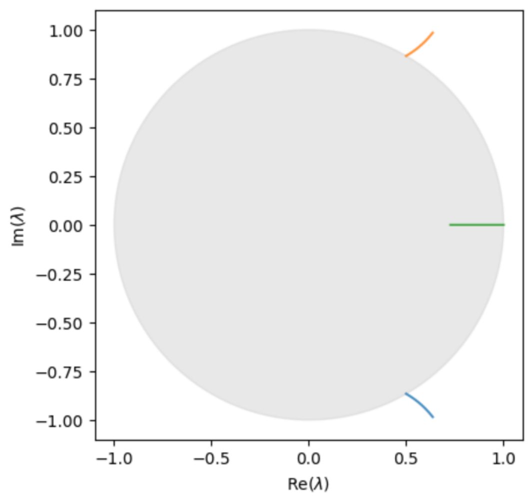
$K_d = 2$



$K_d = 5$



$K_d = 10$



$K_d = 20$