6.3100 ~~10/2/23~~ 10/4/23 1c

Last Time: State-Space can model anything
PID control, More detailed models disturbances

$$\begin{bmatrix} \\ x[n] \\ \\ \end{bmatrix} = \underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{} \begin{bmatrix} \\ x[n-1] \\ \\ \end{bmatrix} + \begin{bmatrix} \\ B \\ \\ \end{bmatrix} u[n-1] + \begin{bmatrix} \\ B_d \\ \\ \end{bmatrix} u_d[n-1]$$

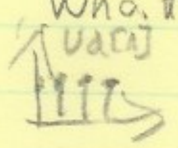
eig(A) = nat freqs
|eig(A)| < 1 for stability

Given sets of difference equations

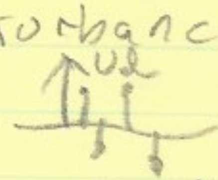
1) A's & B's easy to construct, State Space easy to simulate

2) But how do you decide

Do you need P + I + D control
What disturbances are important



or



Is the model detailed enough

Easiest in CT

Need and approach that provides ways to reason about the system
Transfer func, Block diagrams, frequency Response

1L

D.T.

$$y[n] = \lambda y[n-1] + \gamma u[n-1]$$

General Solution

$$y[n] = \lambda^n y[0] + \sum_{m=0}^{n-1} \lambda^{(n-1)-m} \gamma u[m]$$

(Proof by induction)

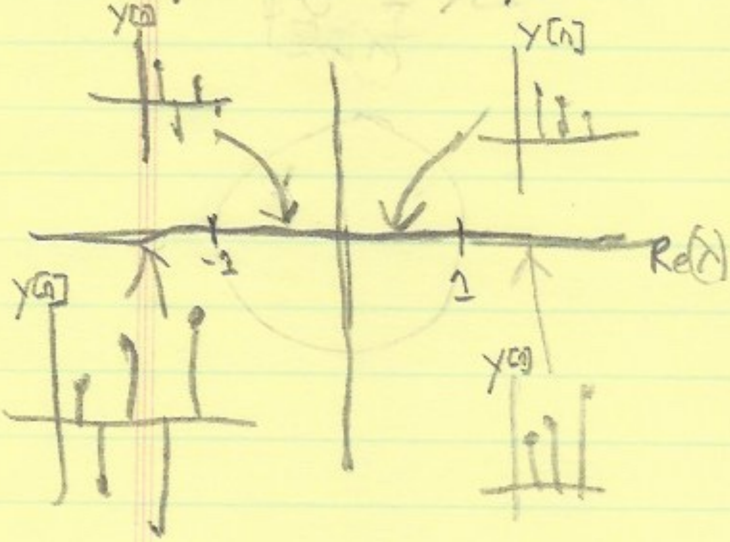
Implies Linearity

ZIR Case

$$y[n] = \lambda y[n-1]$$

$$y[n-1] = Y\lambda^{n-1} \quad \text{Scalar}$$

$$y[n] = Y\lambda^n = \lambda y[n-1]$$



Stability

$$|\lambda| < 1$$

C.T.

$$\frac{d}{dt} y(t) = \lambda y(t) + \gamma u(t)$$

General Solution

$$y(t) = e^{\lambda t} y(0) + \int_0^t e^{\lambda(t-\tau)} \gamma u(\tau)\, d\tau$$

(Proof by differentiating and using integration by parts)

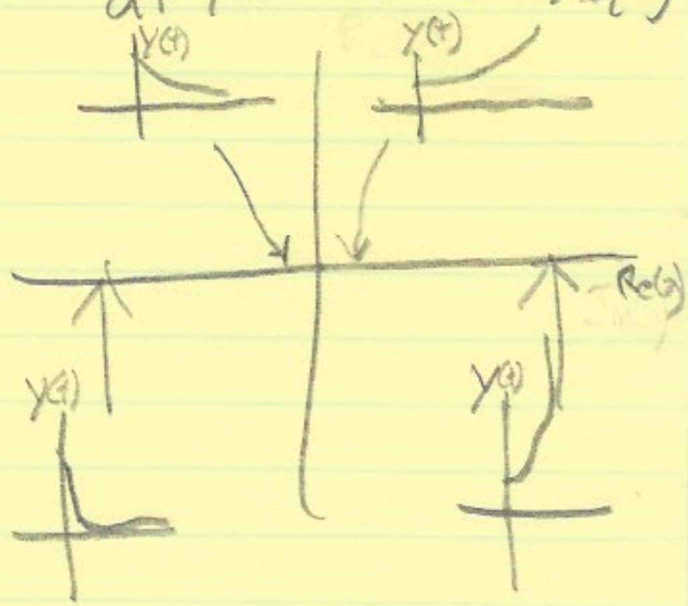
Implies Linearity

ZIR Case

$$\frac{d}{dt} y(t) = \lambda y(t)$$

$$y(t) = Y e^{\lambda t}$$

$$\frac{d}{dt} y(t) = \lambda Y e^{\lambda t} = \lambda y(t)$$



Stability

$$Re(\lambda) < 0$$

2L

C. T. Case

$$y'(t) = a_{11}\,y(t) + a_{12}\,w(t)$$

$$w'(t) = a_{21}\,y(t) + a_{22}\,w(t)$$

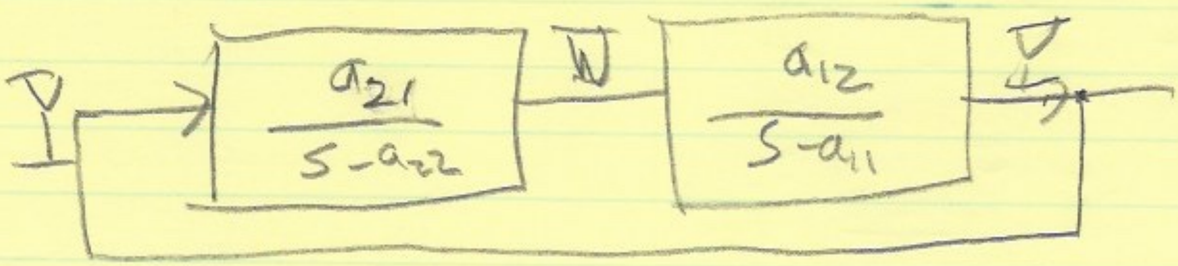
Guess $y(t) = \underline{Y}\,e^{st} \quad w(t) = \underline{W}\,e^{st}$

$$s\,\underline{Y}\,\cancel{e^{st}} = a_{11}\,\underline{Y}\,\cancel{e^{st}} + a_{12}\,\underline{W}\,\cancel{e^{st}}$$

$$s\,\underline{W}\,\cancel{e^{st}} = a_{21}\,\underline{Y}\,\cancel{e^{st}} + a_{22}\,\underline{W}\,\cancel{e^{st}}$$

$$s\,\underline{Y} = a_{11}\,\underline{Y} + a_{12}\,\underline{W} \Rightarrow \underline{Y} = \frac{a_{12}}{s-a_{11}}\,\underline{W}$$

$$s\,\underline{W} = a_{21}\,\underline{Y} + a_{22}\,\underline{W} \Rightarrow \underline{W} = \frac{a_{21}}{s-a_{22}}\,\underline{Y}$$



$$\underline{Y} = \frac{a_{12}\,a_{21}}{(s-a_{22})(s-a_{11})}\,\underline{Y}$$

$$\underline{Y}\,\underbrace{\left(s^2 - (a_{11}+a_{22})s + a_{22}a_{11} - a_{21}a_{12}\right)}_{\text{Quadratic for } s} = 0$$

$$y(t) = \underline{Y}_1\,e^{s_1 t} + \underline{Y}_2\,e^{s_2 t}$$

1R

Speed Control DT

perforbation in speed

$$\omega[n] = \omega[n-1] + \Delta T(\beta\omega[n-1] + \gamma c[n-1])$$

$$c[n] = K_p(\omega_d[n] - \omega[n])$$

$\omega_d = 0$

or

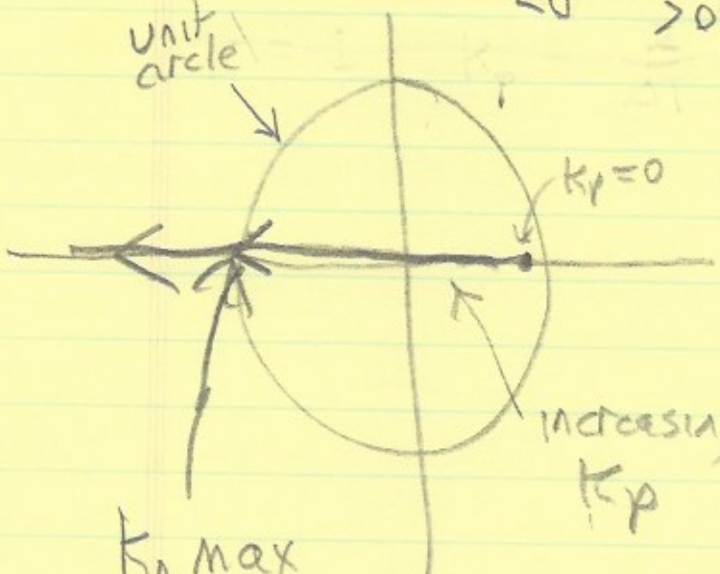
$$\frac{\omega[n]-\omega[n-1]}{\Delta T} = \beta\omega[n-1] - \gamma K_p\omega[n-1]$$

$$\omega[n] = \left(1+\Delta T(\beta-\gamma K_p)\right)\omega[n-1]$$

$$\omega[n] = \lambda^n\,\omega[0]$$

$$\lambda = 1+\Delta T(\beta - \gamma K_p)$$

$\beta < 0$, $\gamma > 0$, $K_p > 0$



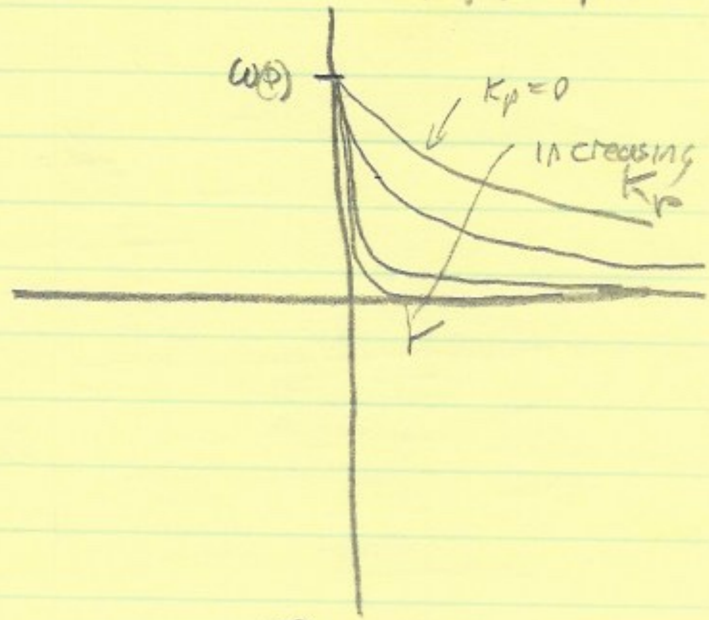
$$\Delta T(\beta - \gamma K_p) = -2$$

$$K_{p_{max}} = \left(\frac{2}{\Delta T} + \beta\right)/\gamma$$

$$\frac{d}{dt}\omega(t) = (\beta - \gamma K_p)\,\omega(t)$$

$$\omega(t) = e^{(\beta-\gamma K_p)t}\,\omega(0)$$

neg pos pos



Why?

2R

Wall Follower

D.T.

$d[n] = d[n-1] + \Delta T\, V \theta[n-1]$

$\theta[n] = \theta[n-1] + \Delta T\, \gamma\, c[n-1]$

↑ control

$c[n] = K_p(-d[n])$

↓

$$\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta T V \\ \Delta T \gamma K_p & 1 \end{bmatrix}}_{\text{eig}(A)} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}$$

$'$ ← means d/dt

$d'(t) = V\theta(t)$

$\theta'(t) = \gamma c(t)$

$c(t) = K_p(-d(t))$

guess $d(t) = D e^{st}$ — scalar

guess $\theta(t) = \Theta e^{st}$

guess $c(t) = C e^{st}$ $\quad c(t) = C e^{st}$

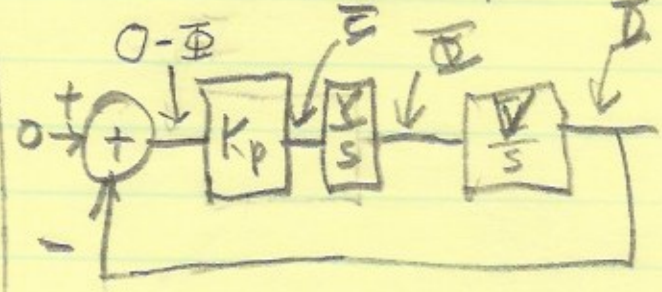
$s D \cancel{e^{st}} = V \Theta \cancel{e^{st}}$

$s \Theta \cancel{e^{st}} = \gamma C \cancel{e^{st}}$

$C \cancel{e^{st}} = -K_p D \cancel{e^{st}}$

$D = \frac{V}{s}\Theta$

$\Theta = \frac{-1}{s}\gamma K_p D$



$D = \frac{V}{s}\Theta = \frac{V}{s}\frac{\gamma}{s} C$

$= -\frac{V}{s}\frac{\gamma}{s} K_p D$

$\left(1 + \frac{V\gamma K_p}{s^2}\right) D = 0$

$s^2 + V\gamma K_p = 0$

$s_{1,2} = \pm\sqrt{-V\gamma K_p}$

3 R

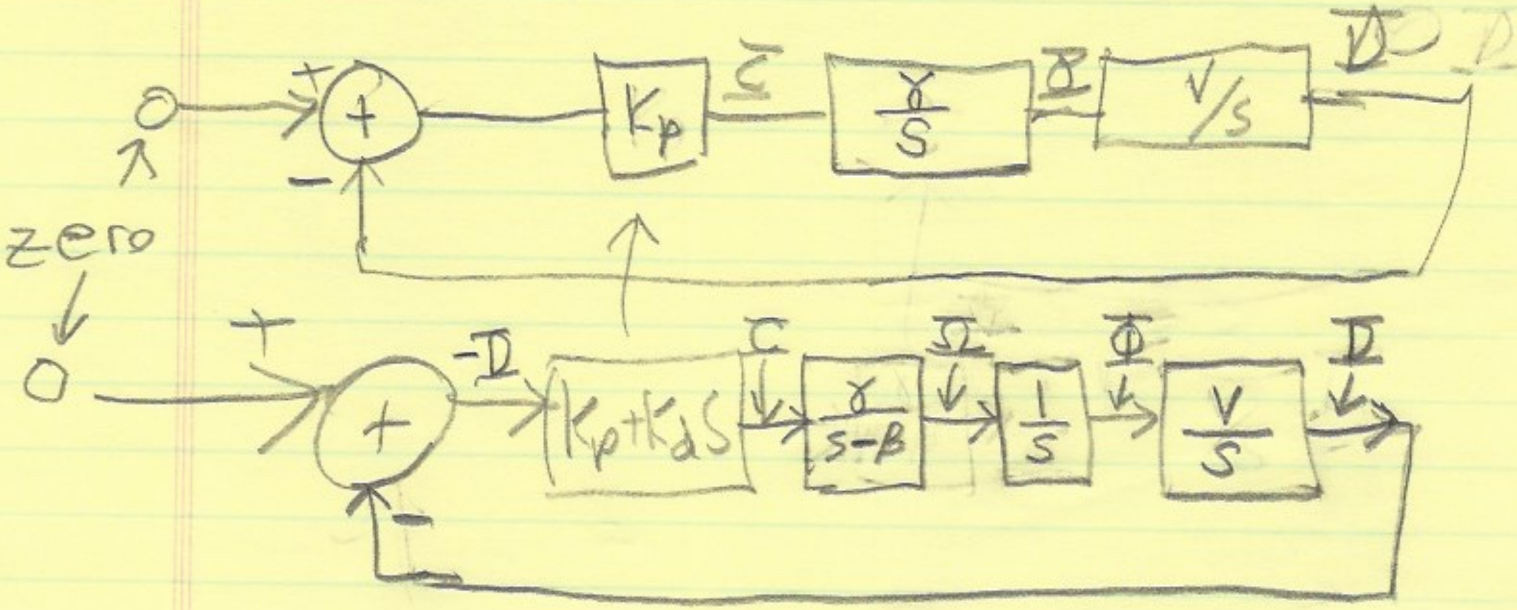
More Complete Model PD Control

$$d'(t) = V\theta(t) \Rightarrow sD = V\Theta$$

$$\theta'(t) = \omega(t) \qquad s\Theta = \Omega +$$

$$\omega'(t) = \gamma c(t) + \beta\omega(t) \qquad s\Omega = \gamma C + \beta\Omega$$

$$c(t) = K_p(-d(t)) + K_d(-d'(t))$$



$$D = -\frac{(K_p + Ks)\gamma}{(s-\beta)\,s\,s} D$$

$$\underbrace{\left((s-\beta)s^2 + \gamma K_p + \gamma K s\right)}_{\text{cubic}} D = 0$$