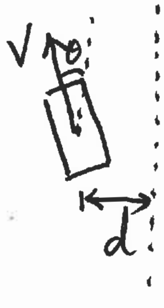
10/10/23.

*. Last week, we learned to go from D.T. to C.T.

Use the line follower example:



C.T. model (instant).

$$\frac{d}{dt}d(t) = V\theta(t). \quad ① \quad \left(\text{from } \frac{d[n]-d[n-1]}{\Delta T} = V\theta[n-1]\right)$$

$$\frac{d}{dt}\theta(t) = \gamma c(t) \quad ② \quad \left(\text{from } \frac{\theta[n]-\theta[n-1]}{\Delta T} = \gamma c[n-1]\right)$$

$$c(t) = -K_p d(t) - K_D \frac{d}{dt} d(t) \quad ③. \quad \left(\text{from } c[n] = -K_p d[n] - K_D \frac{d[n]-d[n-1]}{\Delta T}\right).$$

(set desired value $d_d(t) = 0$).

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*. Solve the C.T. model.

Approach 1: "$\frac{d}{dt}$" both sides of Eq ①, convert into a 2nd order differential equation. use what you learned in 18.03/6.200 to solve it.

Approach 2: directly plug in trial solutions into Eq ① ②.

Assume $d(t) = De^{st}$, $\theta(t) = \Theta e^{st}$, $c(t) = Ce^{st}$.

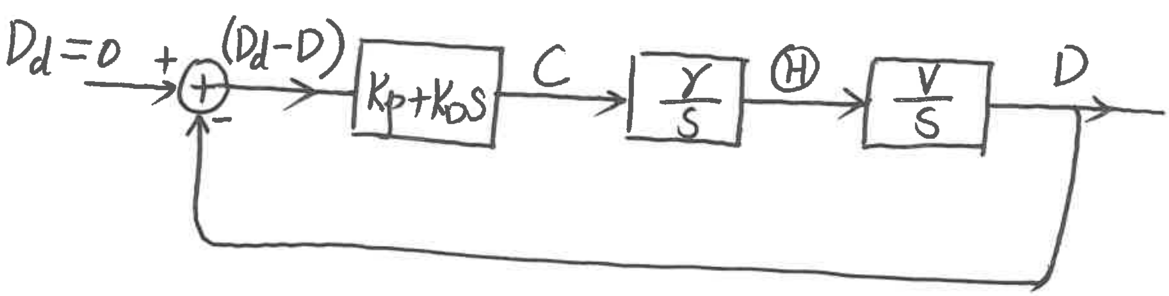
(why choose the same "s" in d, θ and c?)

Eq.① $\rightarrow$ $sD\cancel{e^{st}} = V\Theta\cancel{e^{st}}$ $\rightarrow$ $D = \frac{V}{s}\Theta$

Eq.② $\rightarrow$ $s\Theta\cancel{e^{st}} = \gamma C\cancel{e^{st}}$ $\rightarrow$ $\Theta = \frac{\gamma}{s}C$

Eq.③ $\rightarrow$ $C\cancel{e^{st}} = -(K_P + K_D s)D\cancel{e^{st}}$ $\rightarrow$ $C = -(K_P + K_D s)D$.

The system can be equivalently described by the block diagram:



$$D = \frac{V}{s}\cdot\frac{\gamma}{s}\cdot(-K_P \mp K_D s)D \Rightarrow (s^2 + V\gamma K_D s + V\gamma K_P)D = 0.$$

Non-trivial solution: $s^2 + V\gamma K_D s + V\gamma K_P = 0$.

$$S_{1,2} = \frac{-V\gamma K_D \pm \sqrt{V^2\gamma^2 K_D^2 - 4V\gamma K_P}}{2}.$$

$$d(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}.$$

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* Discuss the form of the solution in the following cases:

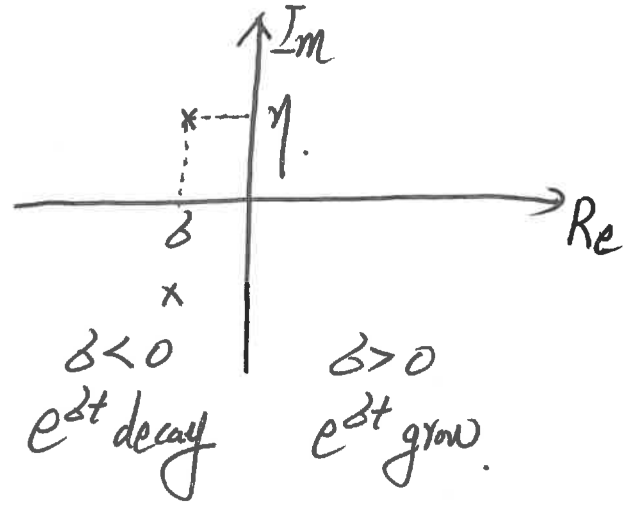
1. $\gamma^2 K_D^2 - 4\gamma K_P < 0$ (assume $V = 1$). Two complex roots.

$$S_{1,2} = \underbrace{-\tfrac{1}{2}\cancel{V}\gamma K_D}_{\sigma} \pm j\underbrace{\sqrt{\gamma K_P - \tfrac{1}{4}\gamma^2 K_D^2}}_{\eta}.$$

$\sigma = \mathrm{Re}\{S_{1,2}\}$, $\eta = \mathrm{Im}\{S_1\} = -\mathrm{Im}\{S_2\}$

$S_{1,2} = \sigma \pm j\eta$. $S_1 = S_2^*$.

$$d(t) \sim e^{st} = \underbrace{e^{\delta t}}_{\text{growing or decaying}} \underbrace{e^{j\eta t}}_{\text{oscillation}}$$



$$d(t) = C_1 e^{(\delta + j\eta)t} + C_2 e^{(\delta - j\eta)t}$$

Use initial condition to determine C_1, C_2, $d(0) = d_0$, $\theta(0) = \theta_0 \Rightarrow d'(0) = V\theta_0$.

$$\begin{cases} C_1 + C_2 = d_0 \\ (\delta + j\eta)C_1 + (\delta - j\eta)C_2 = V\theta_0 \end{cases} \Rightarrow C_1 = C_2^*$$

(the same as in D.T. case)

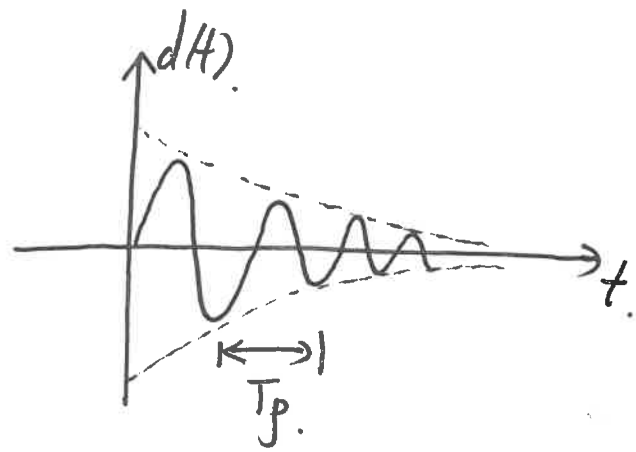
Assume $C_1 = C_2^* = M_c e^{j\phi_c}$ ← phase

↑ magnitude

$$d(t) = e^{\delta t}\left(M_c e^{j\phi_c} e^{j\eta t} + M_c e^{-j\phi_c} e^{-j\eta t}\right)$$

$$= 2M_c e^{\delta t} \cos(\eta t + \phi_c)$$

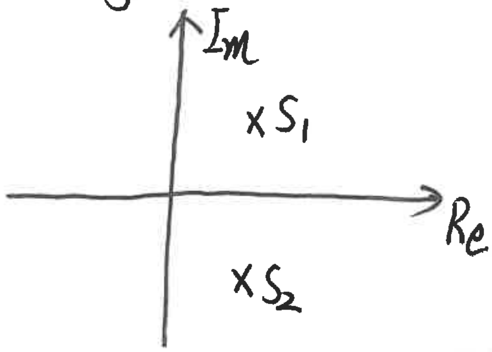
period: $T_p = \dfrac{2\pi}{\eta} = \dfrac{2\pi}{\sqrt{\gamma K_p - \frac{1}{4}\gamma^2 K_D^2}}$

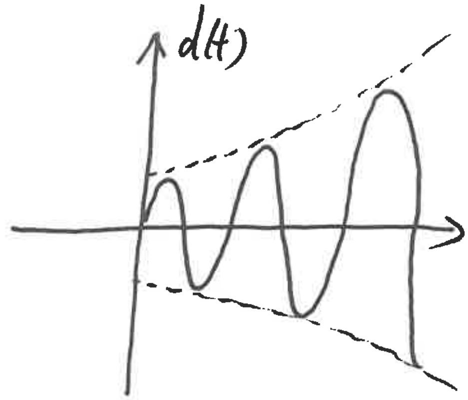


when $K_D = 0$, $T_p = \dfrac{2\pi}{\sqrt{\gamma K_p}}$

recall in Arm Lab, $N_p = \dfrac{2\pi}{\Delta T \sqrt{\gamma K_p}}$, same result!

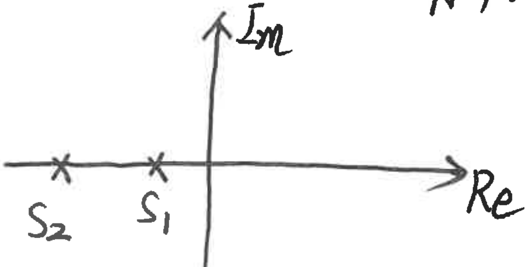
What if $\delta = Re\{S_{1,2}\} > 0$.

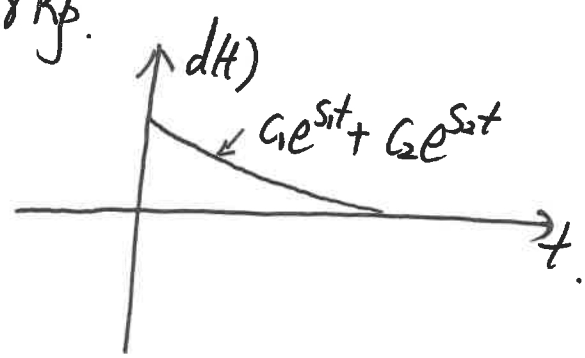




2. $\gamma^2 K_D^2 - 4\gamma K_p > 0$ Two real roots.

$$S_{1,2} = -\frac{1}{2}\gamma K_D \pm \sqrt{\frac{1}{4}\gamma^2 K_D^2 - \gamma K_p}.$$





$S_1 < 0$, $S_2 < 0$

both left plane

What if $S_1 > 0$, $S_2 > 0$? What if $S_1 > 0$, $S_2 < 0$?

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* Non-instant model:

$$\frac{d}{dt} d(t) = V\theta(t) \qquad\qquad sD = V\Theta$$

$$\frac{d}{dt}\theta(t) = w(t) \qquad \xrightarrow[\substack{w(t) = \Omega e^{st} \\ c(t) = C_1 e^{st}}]{\substack{d(t) = De^{st} \\ \theta(t) = \Theta e^{st}}} \qquad s\Theta = \Omega$$

$$\frac{d}{dt} w(t) = \beta w(t) + \gamma c(t). \qquad\qquad s\Omega = \beta\Omega + \gamma C_1$$

$$c(t) = -K_p d(t) - K_D \frac{d}{dt} d(t). \qquad\qquad C_1 = -(K_p + sK_D)D.$$

$$\Longrightarrow D = \frac{V}{s}\Theta = \frac{V}{s}\cdot\frac{1}{s}\Omega = -\frac{V}{s^2}\,\frac{\gamma}{s-\beta}(K_p + sK_D)D.$$

characteristic equation: $s^2(s-\beta)+\gamma V(K_P+sK_D)=0$.

Third order equation, complex solutions still exist in pairs of complex conjugate

See plots.

$$G(s) = \frac{1}{1 + (K_p + K_D s)\frac{V\gamma}{s^2(s-\beta)}}$$

V = 1; γ=46; β=-25