6.310 10/16/23 1R

If

$$\frac{d}{dt}y(t) = y'(t) = \beta_y\, y(t) + \gamma_y\, w(t)$$

$$w'(t) = \beta_w\, w(t) + \gamma_w\, r(t)$$

$$r'(t) = \beta_r\, r(t) + \gamma_r\, \underbrace{u(t)}_{\text{known}}$$

<u>Assume</u> $u(t) = \mathbb{U} e^{st}$

<u>Guess</u> $y(t) = \mathbb{Y} e^{st}$ $w(t) = \mathbb{W} e^{st}$ $r(t) = \mathbb{R} e^{st}$

$$\Rightarrow \quad s\mathbb{Y}\cancel{e^{st}} = \beta_y \mathbb{Y} \cancel{e^{st}} + \gamma_y \mathbb{W} \cancel{e^{st}}$$

$$\Rightarrow \mathbb{Y} = \frac{\gamma_y}{s+\beta_y}\mathbb{W}$$

$$s\mathbb{W} = \beta_w \mathbb{W} + \gamma_w \mathbb{R} \Rightarrow \mathbb{W} = \frac{\gamma_w}{s+\beta_w}\mathbb{R}$$

$$s\mathbb{R} = \beta_r \mathbb{R} + \gamma_r \mathbb{U} \qquad \mathbb{R} = \frac{\gamma_r}{s+\beta_r}\mathbb{U}$$

<u>Subbing</u>

$$\mathbb{Y} = \left(\frac{\gamma_y}{s+\beta_y}\cdot\frac{\gamma_w}{s+\beta_w}\cdot\frac{\gamma_r}{s+\beta_r}\right)\mathbb{U} = \frac{\gamma_y\gamma_w\gamma_r}{\left(s^3+(\beta_y+\beta_w+\beta_r)s^2 + (\beta_r\beta_y+\beta_y\beta_w+\beta_r\beta_w)s + \beta_y\beta_w\beta_r\right)}\mathbb{U}$$

N(s) (numerator)

$$\mathbb{Y} = \underbrace{\left(\frac{\sum_{\tilde{l}=0}^{\tilde{L}} b_{\tilde{l}} s^{\tilde{l}}}{\sum_{l=0}^{L} a_l s^l}\right)}_{H(s)}\mathbb{U}$$

(denom) D(s)

ZR

$$\overbrace{(a_\ell s^\ell + a_{\ell-1}s^{\ell-1} + \dots + a_0)}^{D(s)} \underline{Y} = \overbrace{(b_\ell s^\ell + \dots + b_0)}^{N(s)} \underline{U}$$

$$a_\ell \frac{d^\ell y(t)}{dt^\ell} + \dots a_1 \frac{dy}{dt} + a_0 = b_\ell \frac{d^\ell}{dt^\ell} u(t) + \dots + b_1 \frac{d}{dt} u(t) + b_0 u(t)$$

If $\underline{U} = 0$ then either

$$\underbrace{D(s) = 0}_{\text{roots of } D(s)} \quad \text{or} \quad \underline{Y} = 0$$

natural freqs = roots of $D(s)$

Natural Response (Homog Soln) (ZIR) $\rightarrow y_N(t) = \underbrace{\underline{Y}_1 e^{s_1 t} + \dots + \underline{Y}_L e^{s_L t}}$

Decays to zero **If** $Re(s_\ell) < 0$ $\ell \in \{0, \dots, L\}$

particular solution **If** $u(t) = \underline{U} e^{s_0 t}$

Note
ZS.R
$y_N(t) \neq 0$
even if $y(0) = y'(0) = y''(0) = \dots = y^{(L)}(0) = 0$

$$y(t) = y_N(t) + \underbrace{H(s_0)\underline{U} e^{s_0 t}}_{\text{particular solution}}$$

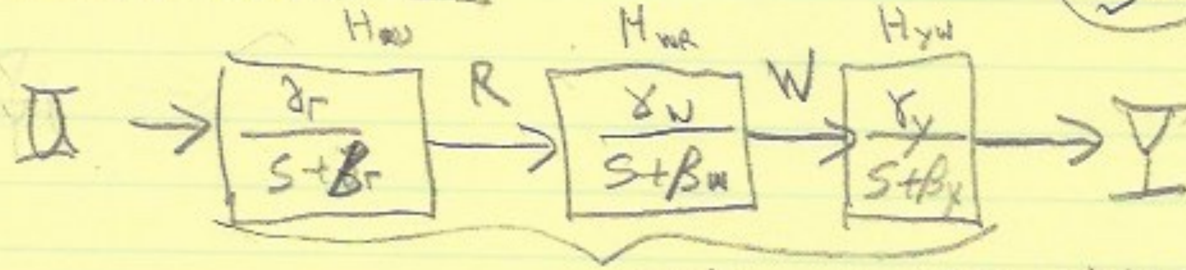
Used to match initial conditions

$$H(s) = \frac{N(s)}{D(s)}$$

roots are zeros

roots are poles aka nat freqs.

Block Diagrams



$$H(s) = H_{yw}(s)\, H_{wr}(s)\, H_{ru}(s)$$

Feedback

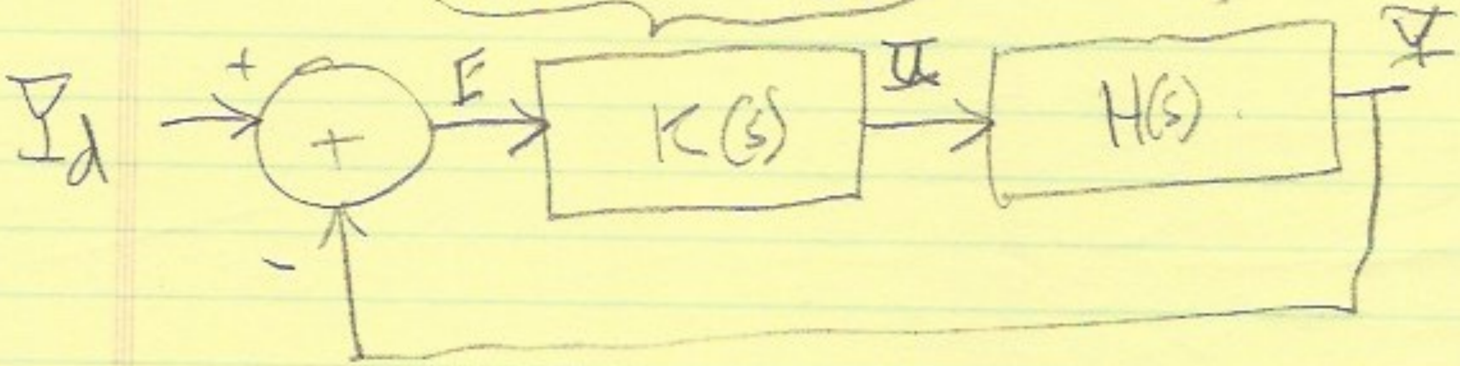
$K_p e(t) + K_d \frac{d}{dt} e(t) + K_i \int e(t)$

K(s)?

$K_p + K_d s$

$$\underbrace{\left(K_p + K_d s + K_i \frac{1}{s}\right) E}$$

$$\frac{N(s)}{D(s)}$$



$$Y = H(s)\,K(s)\,(Y_d - Y)$$

$$(1 + H(s)K(s))\,Y = H(s)\,K(s)\,Y_d$$

$$Y = \frac{H(s)\,K(s)}{1 + H(s)\,K(s)}\,Y_d$$

$\underline{G(s)}$

Blacks Formula

Aside on Root locus

$$Y = \frac{\overbrace{\frac{N(s)}{D(s)} K(s)}^{G(s)}}{1 + \frac{N(s)}{D(s)} K(s)} U$$

$$= \frac{N(s)K(s)}{D(s) + N(s)K(s)} U$$

$$K(s) = K_0 \left(K_p + K_d s + K_i \frac{1}{s} \right)$$

As $K_0 \to 0$ poles $G(s) \to$ roots of $D(s)$

$K_0 \to \infty$ poles $G(s) \to$ roots of $N(s)$

5R

In sinusoidal steady state

$$Y_d(t) = \underline{Y}_d e^{j\omega t}$$

$$\underline{Y} = \left.\overbrace{\frac{H(s)\,K(s)}{1+K(s)H(s)}}^{G(s)}\right|_{s=j\omega} \underline{Y}_d$$

Assume Re(poles $(G(s))) < 0$

$$y(t) = G(j\omega)\cdot \underline{Y}_d e^{j\omega t} \Big\} \text{ for } t \text{ large enough}$$

If $\omega = 0$ (steady state)

$$\lim_{t\to\infty} Y(t) = Y_{ss} \quad Y_{ss} = G(j\omega)\Big|_{j\omega=0} Y_{d_{ss}} \quad Y_{d_{ss}} \equiv \lim_{t\to\infty} Y_d(t)$$

PID (Assume Stability Re(roots(G(s)) < 0)

$H(s) = \frac{N(s)}{D(s)}$

$K(s) = K_p + K_d s + K_i/s$

$$\Rightarrow G(s)\Big|_{s=0} = \left.\frac{N(s)\left(K_p + K_d s + K_i \frac{1}{s}\right)}{D(s) + N(s)\left(K_p + K_d s + \frac{K_i}{s}\right)}\right|_{s=0} = \left.\frac{N(s)\left(K_p s + K_d s^2 + K_i\right)}{sD(s) + N(s)\left(K_p s + K_d s^2 + K_i\right)}\right|_{s=0}$$

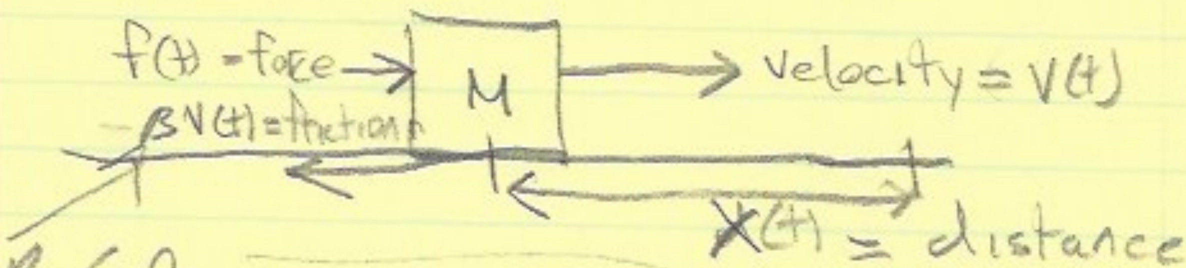
$$= \underbrace{\frac{N(0)K_i}{N(0)K_i}}_{=1}$$

If $K_i > 0$ & $G(0) = 1$

If $K_i = 0$

$$G(s)\Big|_{s=0} = \frac{N(0)\,K_p}{D(0) + N(0)\,K_p} \underset{\lim K_p\to\infty}{\Longrightarrow} 1$$

Example



$\beta < 0$

$$\frac{d}{dt} V(t) = \frac{1}{M}\left(f(t) + \beta V(t)\right)$$

$$\frac{d}{dt} X(t) = V(t)$$

$$f(t) = K_p\left(X_d(t) - X(t)\right)$$

Assume $V(t) = \mathbf{V} e^{st}$, etc

$$s\mathbf{V} = \frac{1}{M}\mathbf{F} + \frac{\beta}{M}\mathbf{V}$$

$$\mathbf{V} = \frac{1/M}{s - \beta/M}\mathbf{F}$$

$$s\mathbf{X} = \mathbf{V} \qquad \mathbf{X} = \frac{1}{s}\mathbf{V}$$

$$x(t) = \int v(t)\, dt$$

$$\mathbf{X} = \frac{1/M}{s(s - \beta/M)}\mathbf{F} \quad (\text{in s.s.s}$$

$$\mathbf{F} = K_p(\mathbf{X}_d - \mathbf{X})$$

2L

$$\mathbb{X} = \frac{K_p \, \frac{1}{M} / (s^2 - \frac{B}{M} s)}{1 + K_p \left(\frac{1}{M} / (s^2 - \frac{B}{M} s) \right)} \mathbb{X}_d$$

$$= \frac{K_p/M}{s^2 - (B/M) s + K_p/M} \mathbb{X}_d$$

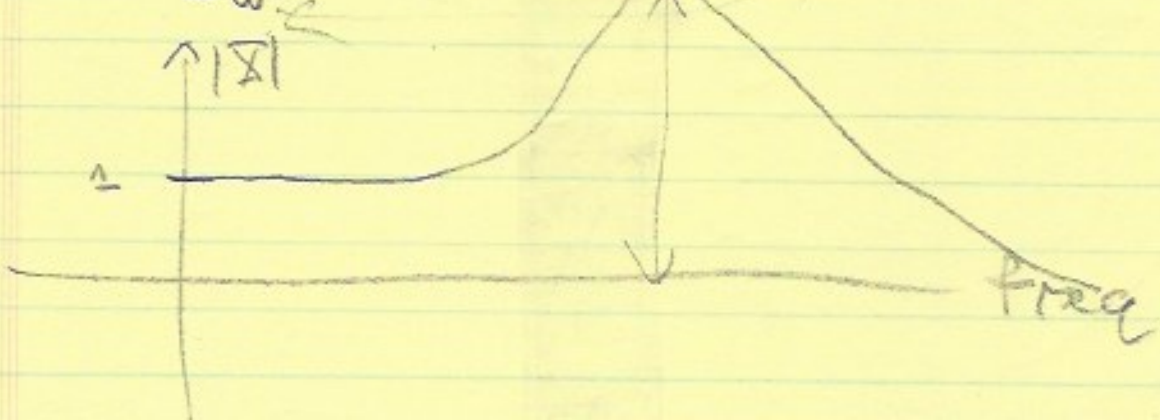
$$= \frac{K_p/M}{s^2 + (|B|/M) s + K_p/M} \mathbb{X}_d$$

$B < 0, \; |B| > 0$

Nat. freqs. $Re\left(\frac{\frac{|B|}{M} \pm \sqrt{\left(\frac{|B|}{M}\right)^2 - 4\left(\frac{K_p}{M}\right)}}{2} \right) < 0$

S. S. S.

$$\mathbb{X} = \frac{K_p/M}{\underbrace{(j\omega)^2}_{-\omega^2} + (|B|/M)\, j\omega + K_p/M} \overset{=1}{\mathbb{X}_d}$$



3L

$$K(s) = K_p + \frac{K_i}{s}$$

$$X = \frac{(K_p + K_i/s)/M}{s^2 + (|B|/M)s + (K_p + K_i/s)/M}$$

$$= \frac{K_p s + K_i}{M s^3 + (|B|)s^2 + K_p s + K_i} X_d$$

4L

$$K(s) = K_p + K_d s$$

$$\underline{X} = \frac{(K_p + K_d s)}{M s^2 + |B| s + (K_p + K_d s)}$$