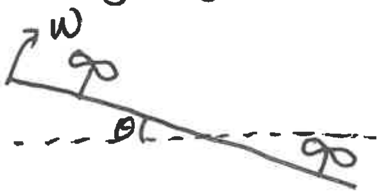
10/18/23

* Review of Transfer function.

Example: 

β_w : friction, air resistance, etc.
(different from β parameter of our lab).

A. Differential Equation: (Under input $\theta_d(t) = \Theta_d e^{st}$
Guess $\theta(t) = \Theta e^{st}$, $w(t) = \Omega e^{st}$)

$$\frac{d}{dt}\theta(t) = w(t) \longrightarrow \Theta = \frac{1}{s}\Omega \quad ①$$

$$\frac{d}{dt}w(t) = \beta_w w(t) + \gamma c(t) \longrightarrow \Omega = \frac{\gamma}{s-\beta_w} C \quad ②$$

$$c(t) = K_p(\theta_d(t) - \theta(t)) + K_d \frac{d}{dt}(\theta_d(t) - \theta(t)) + K_i \int_0^t (\theta_d(\tau) - \theta(\tau))d\tau$$

$$\longrightarrow C = K_p(\Theta_d - \Theta) + K_d s(\Theta_d - \Theta) + K_i \frac{1}{s}(\Theta_d - \Theta). \quad ③$$

$$① ② ③ \Longrightarrow \Theta = \frac{\gamma}{s(s-\beta_w)}\left(K_p + K_d s + \frac{K_i}{s}\right)(\Theta_d - \Theta).$$

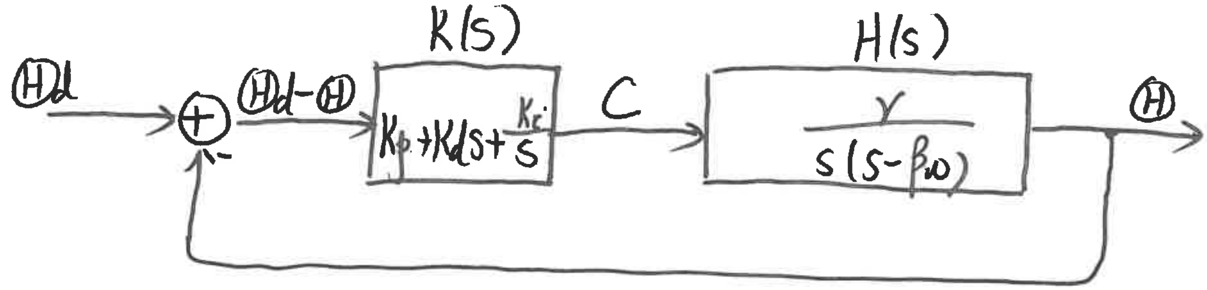
B. Transfer function

$$\boxed{\Theta = \frac{\left(K_p + K_d s + \frac{K_i}{s}\right)\frac{\gamma}{s(s-\beta_w)}}{1 + \left(K_p + K_d s + \frac{K_i}{s}\right)\frac{\gamma}{s(s-\beta_w)}}\Theta_d.}$$

C. Block Diagram.

First define $\Theta = \frac{1}{s}\cdot\frac{\gamma}{s-\beta_w} C = H(s)\,C$ $\quad$ $H(s)$: transfer function of physical system.

$C = \left(K_p + K_d s + \frac{K_i}{s}\right)(\Theta_d - \Theta) = K(s)(\Theta_d - \Theta)$ $\quad$ $K(s)$: transfer function of controller



$$\Theta(s) = \frac{K(s)H(s)}{1+K(s)H(s)}\Theta_d(s)$$ — Black's formula.

Closed loop gain: $$\frac{\Theta(s)}{\Theta_d(s)} = G(s) = \frac{K(s)H(s)}{1+K(s)H(s)}$$ transfer function of feedforward branch.

* From transfer function to time domain response.

Use simple example of "P" control, i.e. $K_p \neq 0$, $K_d = K_i = 0$.

$$\Theta(s) = \frac{K_p \gamma}{s^2 - \beta\omega s + K_p\gamma}\Theta_d(s).$$

Two cases: 1) denominator $s^2 = 0$

characteristic equation: $s^2 - \beta\omega s + K_p\gamma = 0$.

Natural frequency: $S_{1,2} = \frac{\beta\omega \pm \sqrt{\beta\omega^2 - 4K_p\gamma}}{2}$

Even with $\Theta_d = 0$, $\theta(t)$ can be non-zero

$\theta(t) = \Theta_1 e^{s_1 t} + \Theta_2 e^{s_2 t}$ — zero input response, natural response.

2). denominator $\neq 0$.

$$\theta_d(t) = \Theta_d e^{st} \longrightarrow \boxed{G(s)} \longrightarrow \theta(t) = \Theta e^{st}.$$

The output and input have the same exponential function "e^{st}"
only the prefactor is scaled by $G(s)$.
Recall in lab, we saw $\sin wt$ in, $\sin wt$ out, with scaled magnitude and phase!

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* Frequency Response / Sinusoidal Steady state.

Input: $\theta_d(t) = \Theta_d \cos wt$ (assume phase to be "0" for simplicity).

Output: $\theta(t) = ?$

Euler's formula: $\theta_d(t) = \frac{1}{2}\Theta_d\left(e^{jwt} + e^{-jwt}\right)$

$$\frac{1}{2}\Theta_d e^{jwt} \longrightarrow \boxed{G(s)} \longrightarrow \frac{1}{2}\Theta_d G(s)\Big|_{s=jw} e^{jwt}$$

$$\frac{1}{2}\Theta_d e^{-jwt} \longrightarrow \boxed{G(s)} \longrightarrow \frac{1}{2}\Theta_d G(s)\Big|_{s=-jw} e^{-jwt}.$$

Superposition: $\theta(t) = \frac{1}{2}\Theta_d\left(G(jw)e^{jwt} + G(-jw)e^{-jwt}\right)$

$$= \mathrm{Re}\left\{\Theta_d G(jw) e^{jwt}\right\}. \qquad \left(G(jw) = |G(jw)|\, e^{j\measuredangle G(jw)}\right)$$

$$= \Theta_d |G(jw)| \cos\left(wt + \measuredangle G(jw)\right).$$

Compared with $\theta_d(t) = \Theta_d \cos wt$, output $\theta(t)$'s magnitude changes by $\underline{|G(jw)|}$, its phase by $\underline{\measuredangle G(jw)}$.

Example: $\beta_\omega = -0.01$, $K_p = 1$, $\gamma = 36$.

$S_{1,2} \approx -0.005 \pm j \cdot 6$. $Re\{S_{1,2}\} < 0$, stable, $Im\{S_{1,2}\} = 6$ — oscillation frequency in natural response

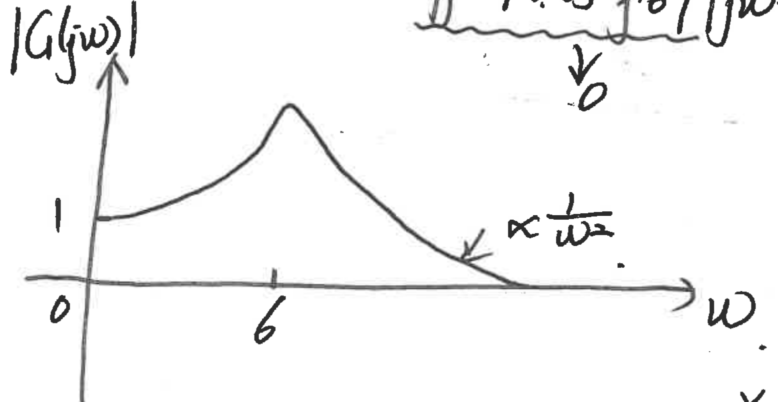
$$G(s) = \frac{K_p \gamma}{S^2 - \beta_\omega S + K_p \gamma} = \frac{K_p \gamma}{(S - S_1)(S - S_2)}$$

$$|G(j\omega)| = \left| \frac{K_p \gamma}{(j\omega)^2 - \beta_\omega j\omega + K_p \gamma} \right| = \frac{K_p \gamma}{|j\omega - S_1||j\omega - S_2|}$$

① $\omega = 0$, $|G(j\omega)| = 1$

② $\omega \to \infty$, $|G(j\omega)| \Rightarrow \frac{K_p \gamma}{\omega^2} \to 0$.

③. $\omega = 6$ $|G(j\omega)| = \frac{K_p \gamma}{\underbrace{|j\omega + 0.005 - j \cdot 6|}_{\to 0} |j\omega + 0.005 + j \cdot 6|}$ reaches maximum.

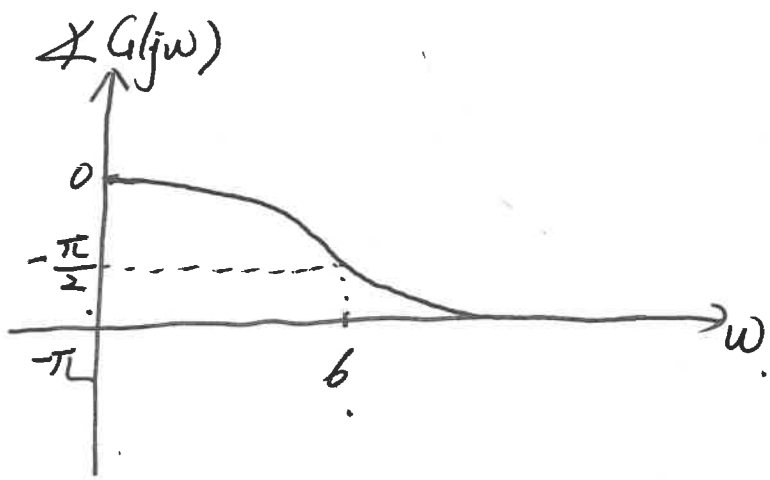


$\measuredangle G(j\omega) = \measuredangle num - \measuredangle den$

① $\omega = 0$, $\measuredangle G(j\omega) = 0$.

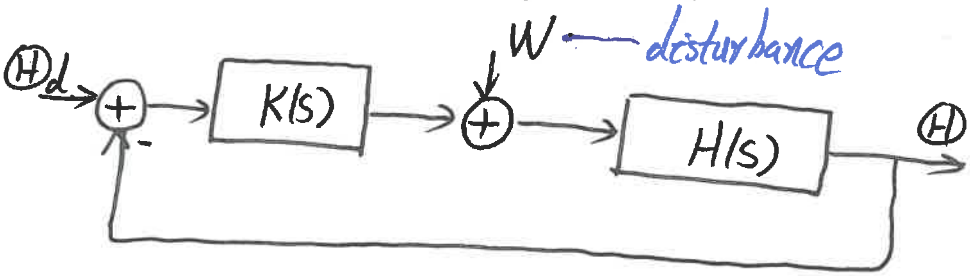
② $\omega \to \infty$, $\measuredangle G(j\omega) = -\pi$

③. $\omega = 6$, $\measuredangle G(j\omega) \approx -\frac{\pi}{2}$.



For general case $K_d \neq 0$, $K_i \neq 0$ and higher order, use software.

* Disturbance Rejection from Transfer Function.



$$\Theta = \underbrace{\frac{K(s)H(s)}{1+K(s)H(s)}}_{G(s)}\Theta_d + \underbrace{\frac{H(s)}{1+K(s)H(s)}}_{G_d(s)} W.$$

$G_d(s)$ — closed loop t.f. of disturbance.

consider constant disturbance: $S \to 0$ $(K_p \neq 0, K_d \neq 0)$.

$$G_d(s) = \frac{\cancel{K_p+K_d}\ \frac{\gamma}{s(s-\beta_\omega)}}{1+(K_p+K_d s)\frac{\gamma}{s(s-\beta_\omega)}} = \frac{\gamma}{s^2+(-\beta_\omega+K_d\gamma)s+K_p\gamma}$$

$\lim_{s\to 0} G_d(s) = \frac{1}{K_p}$: The disturbance is not completely rejected.

Turn K_i on:

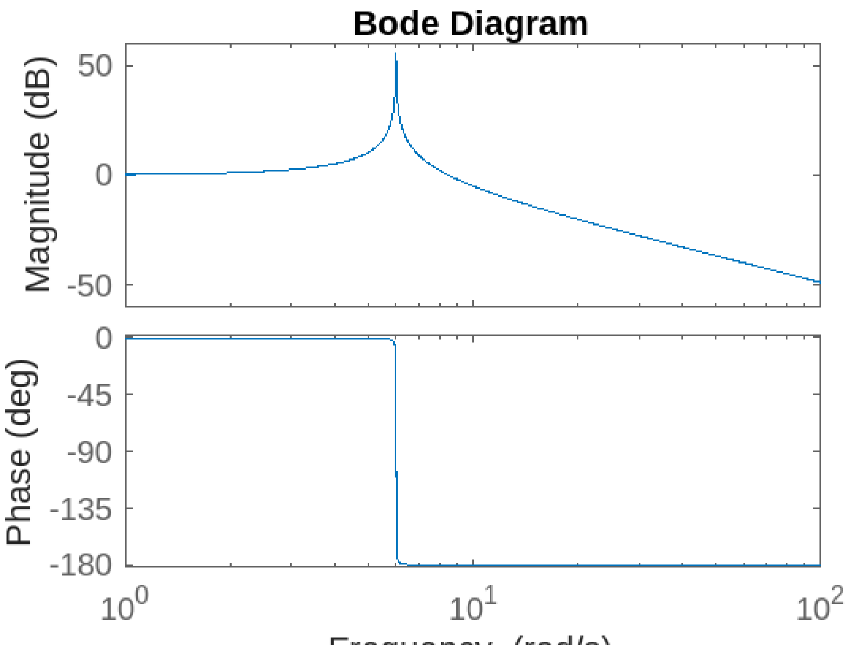
$$G_d(s) = \frac{\frac{\gamma}{s(s-\beta_\omega)}}{1+(K_p+K_d s+\frac{K_i}{s})\frac{\gamma}{s(s-\beta_\omega)}} = \frac{s\gamma}{s^3+(-\beta_\omega+K_d\gamma)s^2+K_p\gamma s+K_i\gamma}$$

$\lim_{s\to 0} G_d(s) = 0$: Integral control removes disturbance.

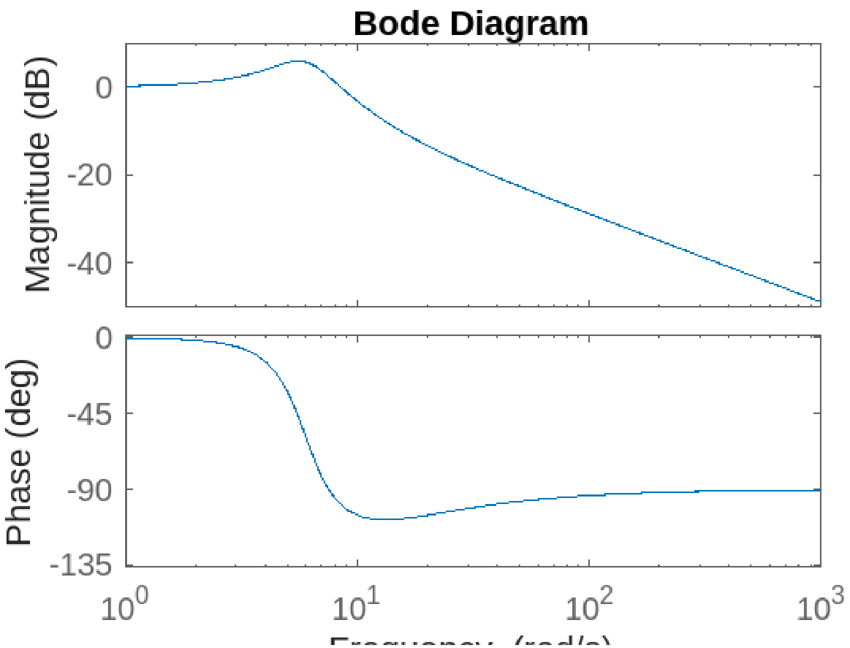
$$G(s) = \frac{(K_p + K_D s + K_I)\,\dfrac{\gamma}{s(s-\beta_\omega)}}{1 + (K_p + K_D s + K_I)\,\dfrac{\gamma}{s(s-\beta_\omega)}}$$

γ=36; β=-0.01

Kp = 1;
Kd = Ki = 0;



Kp = 1;
Kd = 0.1; Ki = 0;



Kp = 1;
Kd = 0.1; Ki = 2;