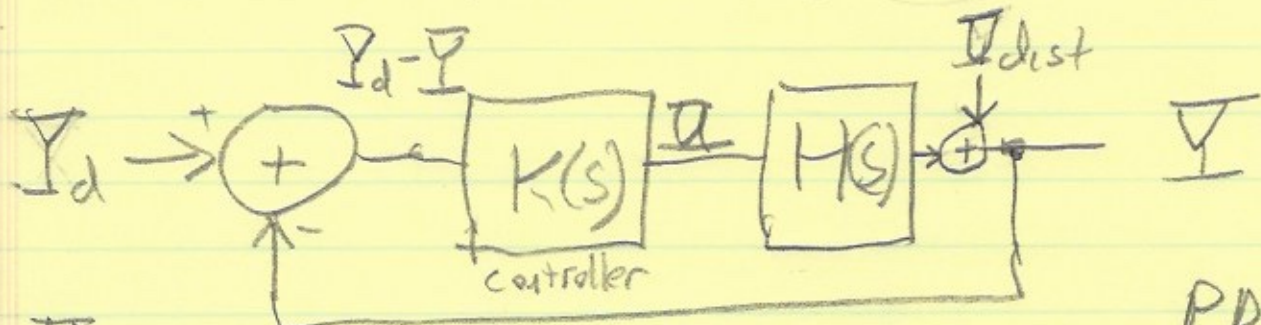


6.310 10/23/23

AKA System  
Feedback System using Transfer Functions



$U_{dist} = 0$

$$Y = \frac{K(s)H(s)}{1 + K(s)H(s)} Y_d$$

$G(s)$

PD  
 $K(s) = k_p + k_d s$   
e.g.  
 $H(s) = \frac{1}{M}$   
e.g.  $\frac{1}{s(s + B/M)}$   
Friction Block  
 $= 0$

$Y_d = 0$

$$Y = U_{dist} + K(s)H(s)(Y_d - Y)$$

$$Y = \frac{1}{1 + K(s)H(s)} U_{dist}$$

$G_{dist}(s)$

Re (Not freq)  $< 0$

IF stable then we can exam s. s. s.

When  $K(j\omega)H(j\omega) \gg 1$

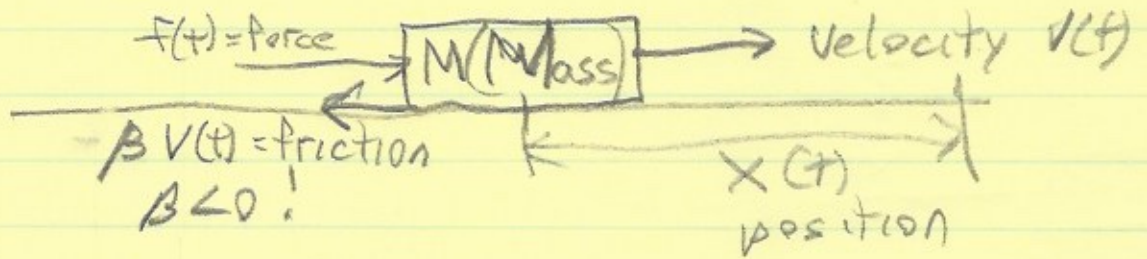
$G(j\omega) \approx 1$  (so  $Y \approx Y_d$  Good Tracking!)

$G_{dist}(j\omega) \approx 0$  (Good disturbance rejection)



2

# Sliding Block



## Diff Eqs

$$sV \quad \frac{d}{dt} V(t) = \frac{1}{M} (F(t) + B V(t)) \quad BIV$$

$$sX \quad \frac{d}{dt} X(t) = V(t)$$

$$F(t) = K_p (X_d(t) - X(t))$$

$$F = K_p (X_d - X)$$

$$V = \left( \frac{1/M}{s - B/M} \right) F_p (X_d - X)$$

$$X = \frac{1}{s} V = \frac{1}{s} \underbrace{\frac{1/M}{s - B/M}}_{H(s)} F$$

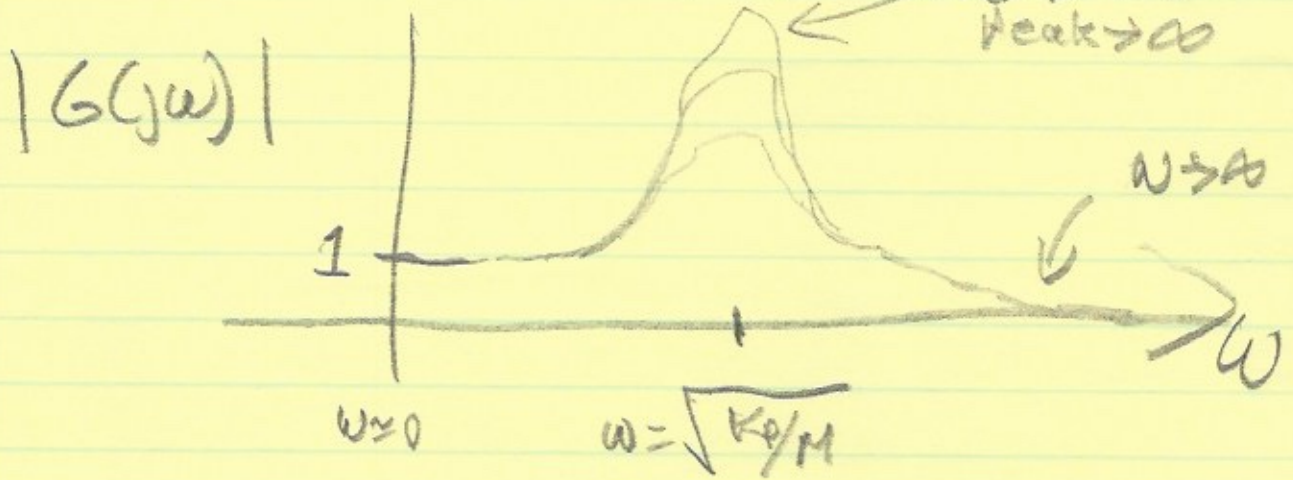
$$X = \frac{K_p H(s)}{1 + K_p H(s)} X_d$$

$$= \frac{K_p/M}{s^2 - B/Ms + K_p/M} \Big|_{s=j\omega} X_d$$

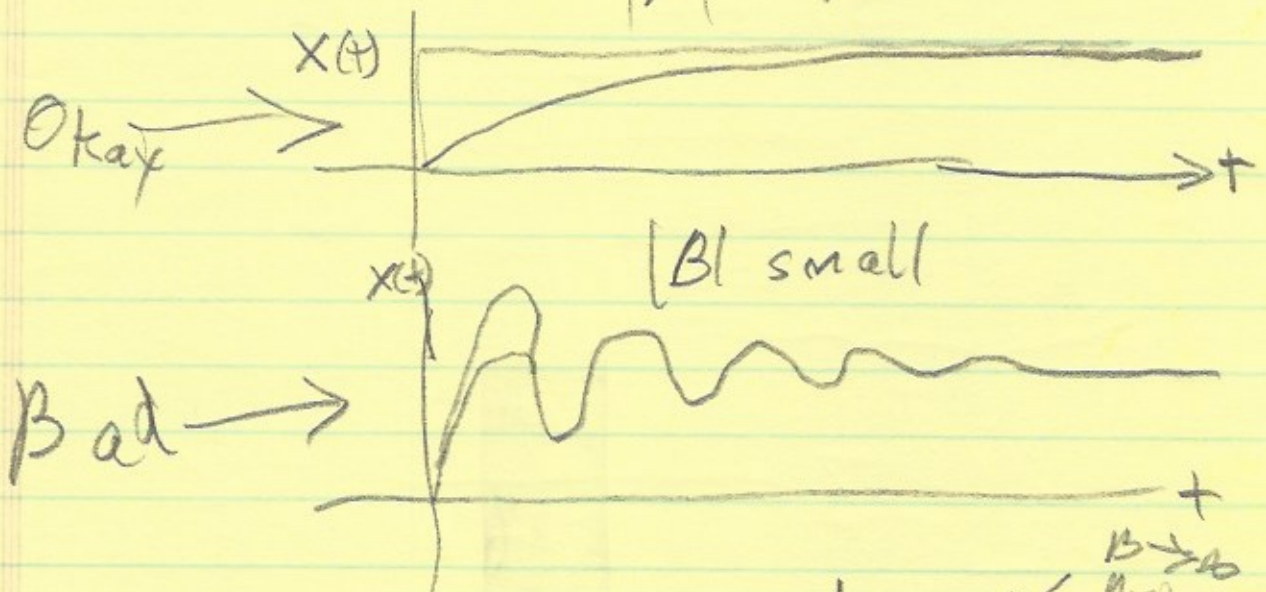
3

$$\bar{X} = \frac{K_p/M}{-w^2 - j(\beta/M)w + K_p/M} \bar{X}_d$$

$G(jw)$



Step Response



Disturbance Rejection (Wind Blowing)

$$1 + K(s)H(s)$$

goes to zero as  $\beta \rightarrow 0$   
 Bad Bad Bad

$\beta \rightarrow \infty$   
 Best Gain  
 $\rightarrow \infty$



(4)

Use  $K(s) = K_p + K_d s$

$$G(s) \Big|_{s=j\omega} = \frac{(K_p + K_d s) \frac{1}{M}}{s^2 + \left(\frac{B}{M} + \frac{K_d}{M}\right) s + K_p/M}$$

$$= -\omega^2 + \left(\frac{B}{M} + \frac{K_d}{M}\right) j\omega + K_p/M$$

Stays away from zero

So adding  $K_d$  is like adding more friction, keeps

$1 + K(j\omega)H(j\omega)$  away from zero



## Alternative Design Criteria (5)

$$G(j\omega) = \frac{K(j\omega)H(j\omega)}{1 + K(j\omega)H(j\omega)}$$

Keep away from zero

OR

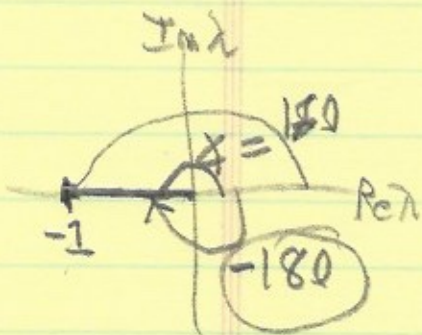
Keep  $K(j\omega)H(j\omega)$  away from -1

For  $K(j\omega)H(j\omega)$  to equal -1

a)  $|K(j\omega)H(j\omega)| = 1$

b)  $\angle (K(j\omega)H(j\omega)) = -180^\circ$

( Necessary but  
Not sufficient! )



## Important Points

We have a criteria based on examining just  $K(j\omega)H(j\omega)$

But we need to know  
 $|K(j\omega)H(j\omega)|$  &  $\angle K(j\omega)H(j\omega)$



6

Let's Look at some cases

Real Pole

$$H(s) = \frac{-p}{s-p}$$

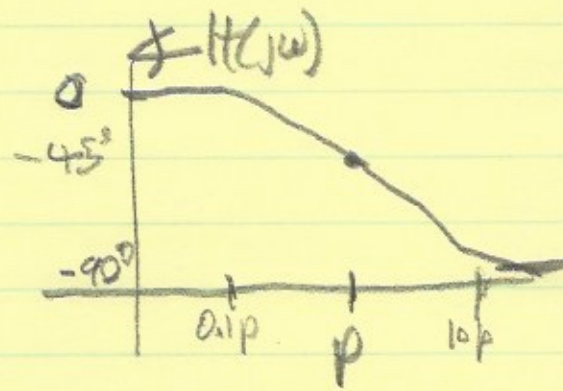
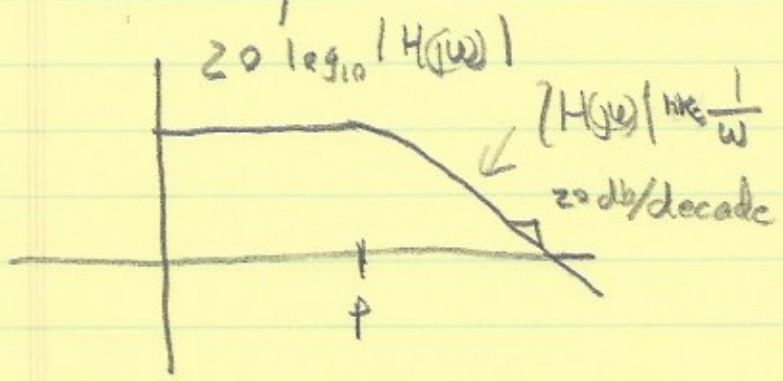
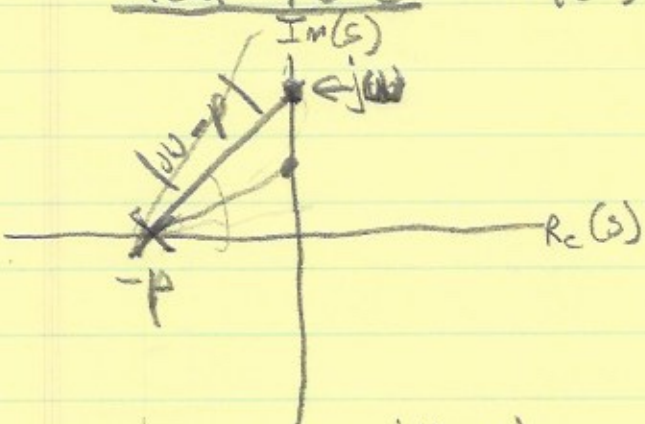
$$H(j\omega) = \frac{-p}{j\omega - p}$$

$$= \left( \frac{-p}{Me^{j\theta}} \right)$$

$$= \frac{p}{M} e^{j\theta}$$

$$|H(j\omega)| = \frac{|-p|}{|j\omega - p|}$$

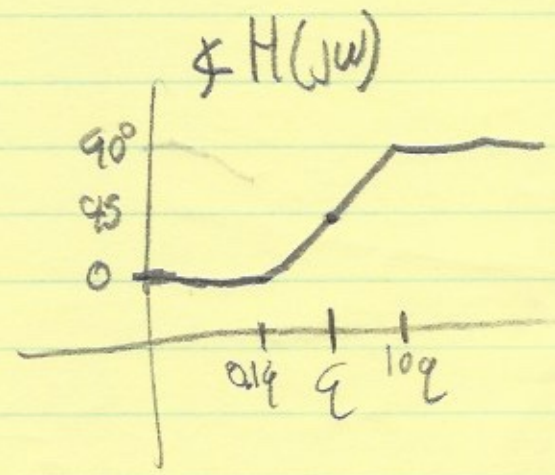
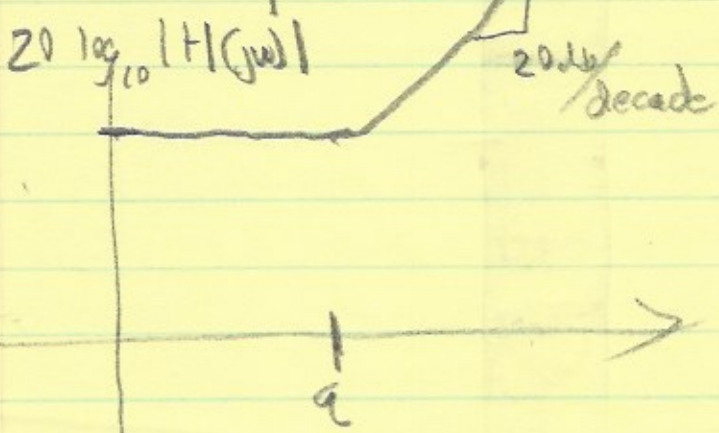
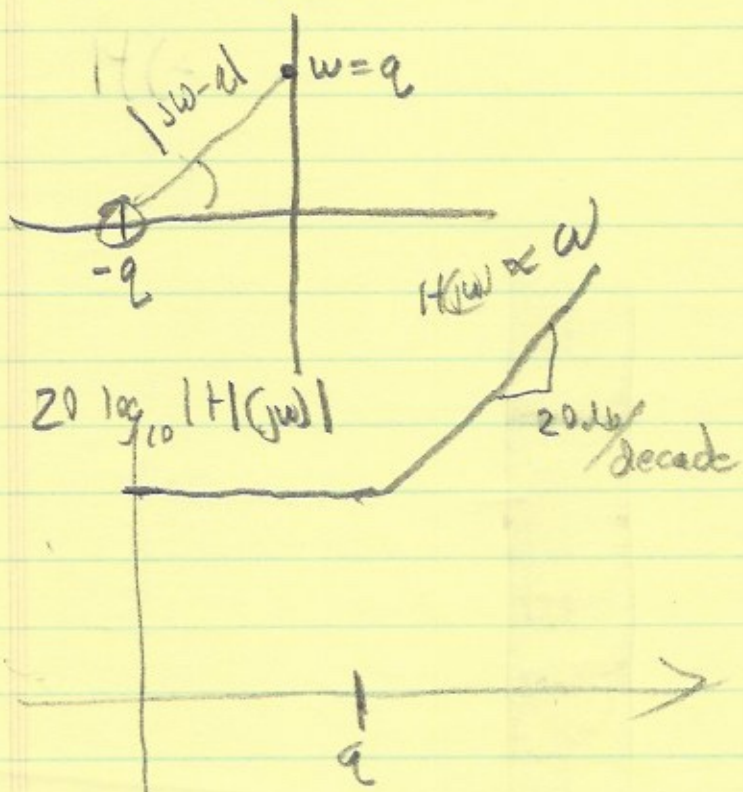
$$\angle H(j\omega) = -\angle(j\omega - p)$$



Real Zero

$$H(s) = \frac{s-q}{-q}$$

$$H(j\omega) = \frac{j\omega - q}{-q}$$

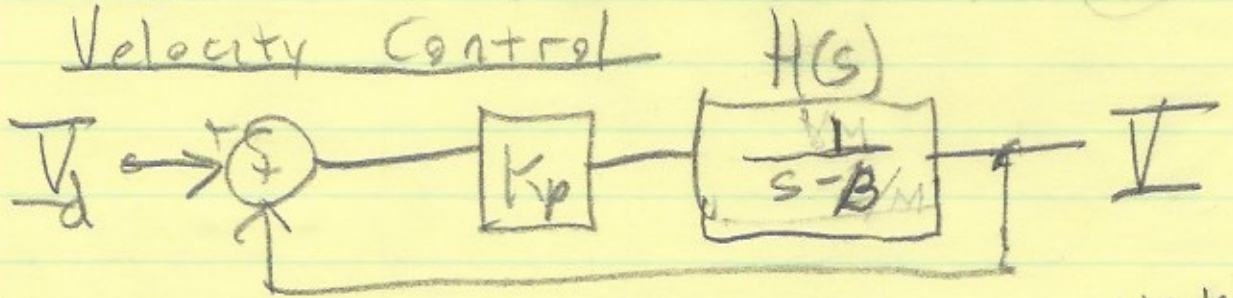




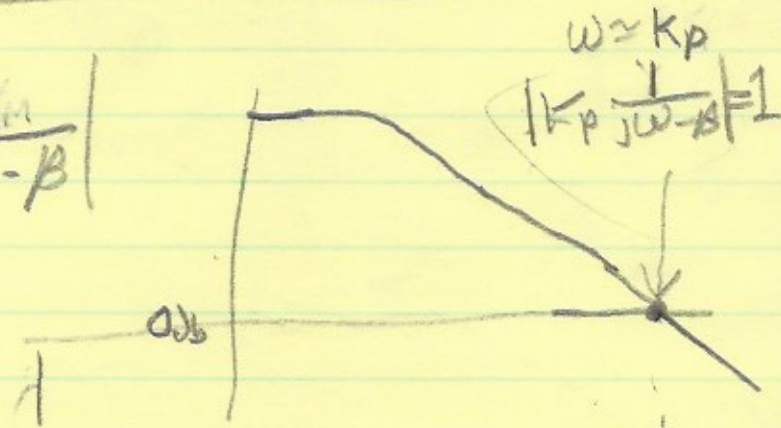
7

Let  $M=1$  to simplify

$$\frac{d}{dt}V(t) = \frac{1}{M}(f(t) + BV(t)) = \frac{1}{M}(K_p(V_d(t) - V(t)) + BV(t))$$



$$(K_p H(j\omega)) = \left| K_p \frac{1}{j\omega - \beta} \right|$$

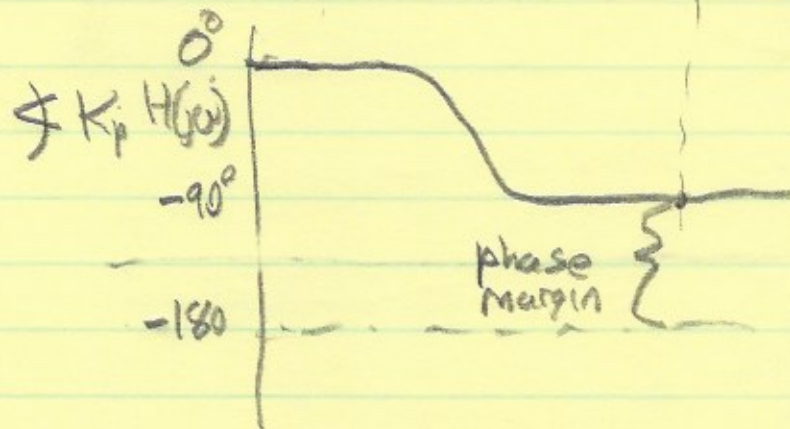


When

$$|K(j\omega)H(j\omega)| = 1$$

$$\angle K(j\omega)H(j\omega) = -90^\circ$$

far from  
-180!



Position control (No friction) (8)

$$\frac{d}{dt} x(t) = v(t)$$

$$\frac{d}{dt} v(t) = f(t) \quad (\text{Mass} = 1)$$



$$20 \log_{10} |K_p \frac{1}{(j\omega)^2}|$$



$$\frac{K_p}{(j\omega)^2}$$

when  $|K_p \frac{1}{(j\omega)^2}| = 1$

$$K_p \frac{1}{\omega^2} = 1$$

-180

$$1 + \frac{K_p}{j\omega} - 1$$

$\Rightarrow$  Bad  $1 + K(j\omega)H(j\omega)$



## 6.3100: Dynamic System Modeling and Control Design

### Systems, Subsystems, and Basic Building Blocks

#### Modularity

Last Time: Reasoning with **transfer (system) functions**

Today: Reasoning with **frequency responses**

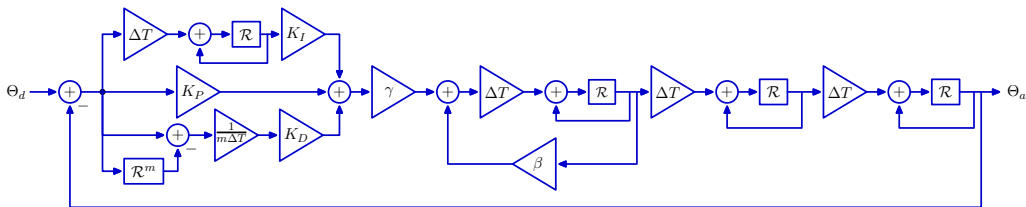
*March 15, 2023*

# Modularity

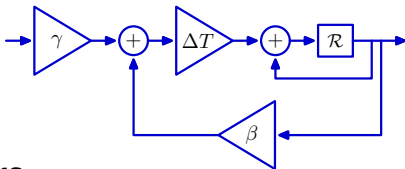
Last time: breaking complicated systems into modules.

→ Black's equation and the importance of transfer (system) functions.

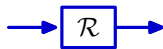
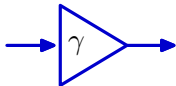
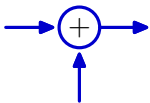
## Systems



## Modules



## Basic Building Blocks





## Modularity

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Today: Modular construction of frequency responses using Bode plots.

## Modularity

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Today: Modular construction of frequency responses using Bode plots.

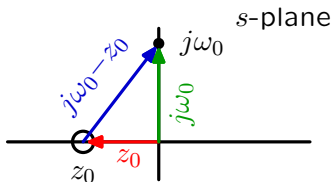
→ builds on the vector analysis method from last week.

The **frequency response** of a system composed of adders, gains, and integrators

$$\cos(\omega_0 t) \rightarrow \boxed{H(j\omega_0)} \rightarrow |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))$$

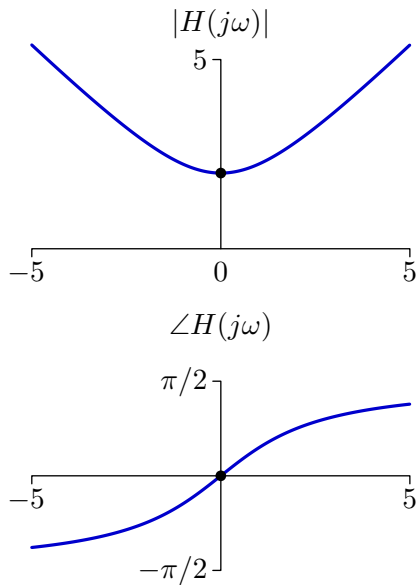
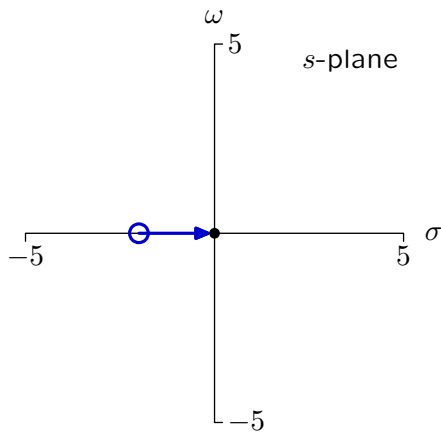
can be determined from **vectors** associated with the system's poles/zeros.

$$H(j\omega_0) = K \frac{(j\omega_0 - z_0)(j\omega_0 - z_1)(j\omega_0 - z_2) \cdots}{(j\omega_0 - p_0)(j\omega_0 - p_1)(j\omega_0 - p_2) \cdots}$$



## Frequency Response: Single Zero

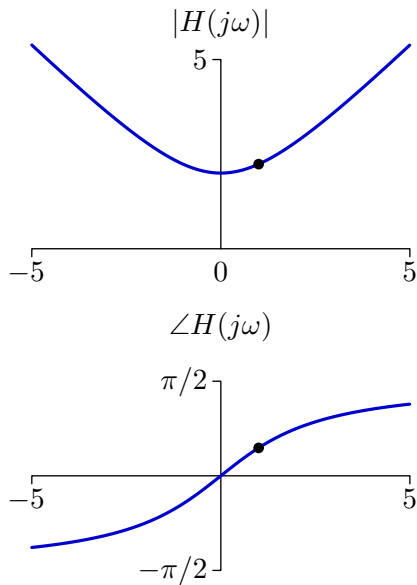
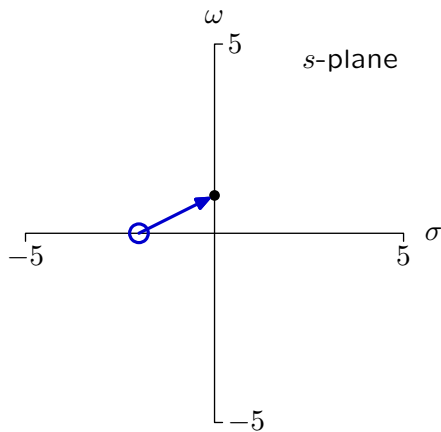
$$H(s) = s - z_1$$





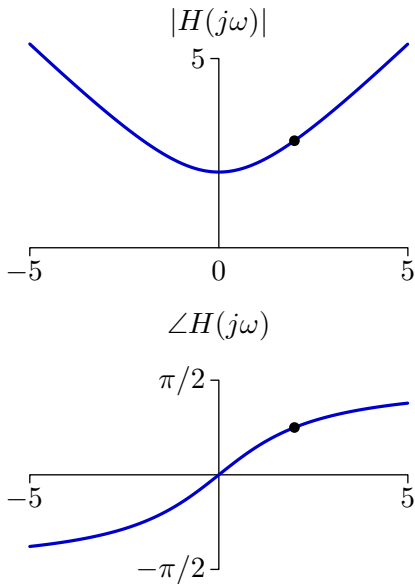
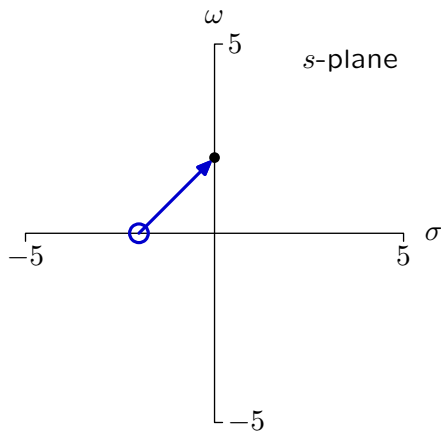
## Frequency Response: Single Zero

$$H(s) = s - z_1$$



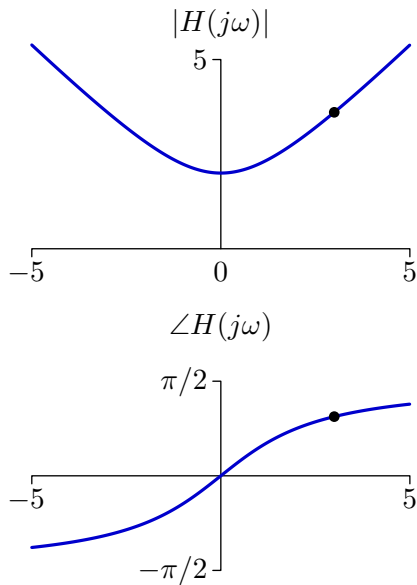
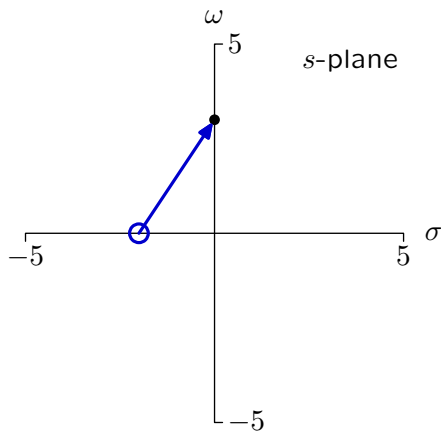
## Frequency Response: Single Zero

$$H(s) = s - z_1$$



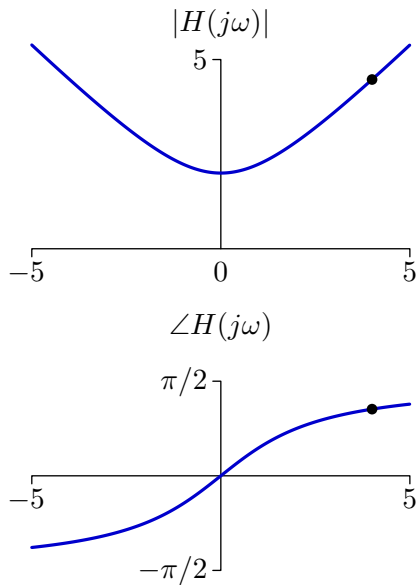
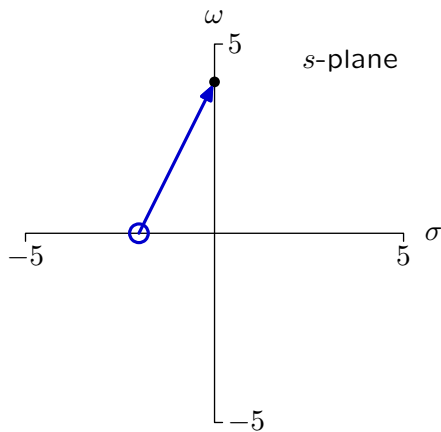
## Frequency Response: Single Zero

$$H(s) = s - z_1$$



## Frequency Response: Single Zero

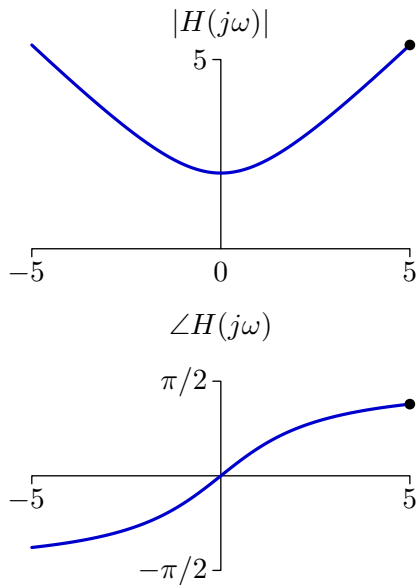
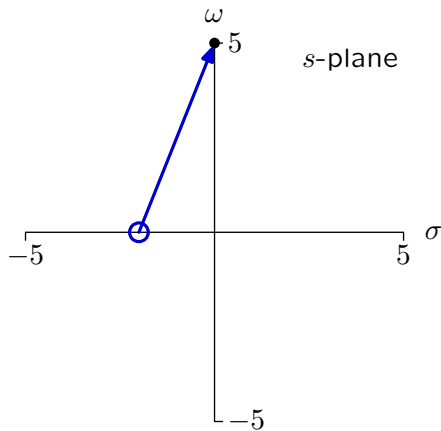
$$H(s) = s - z_1$$





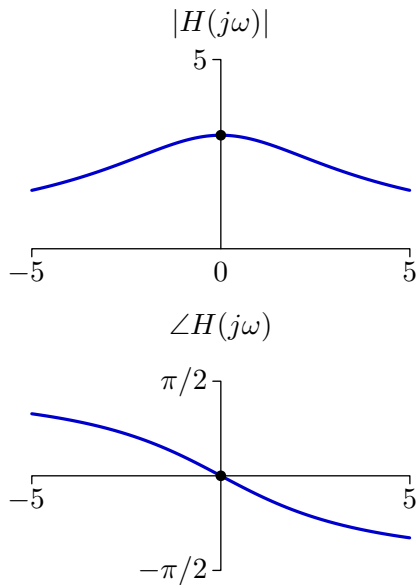
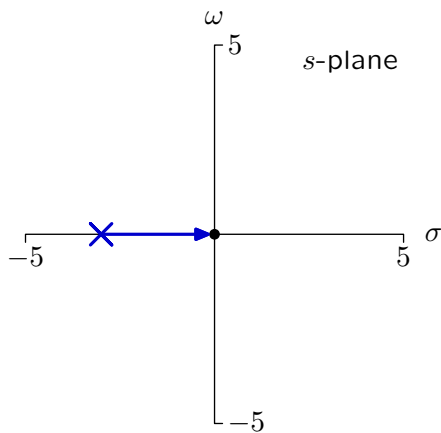
## Frequency Response: Single Zero

$$H(s) = s - z_1$$



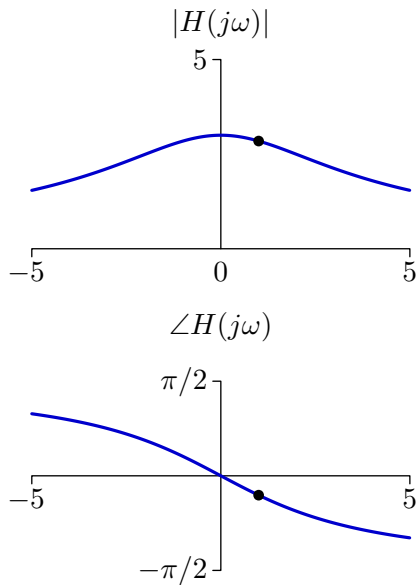
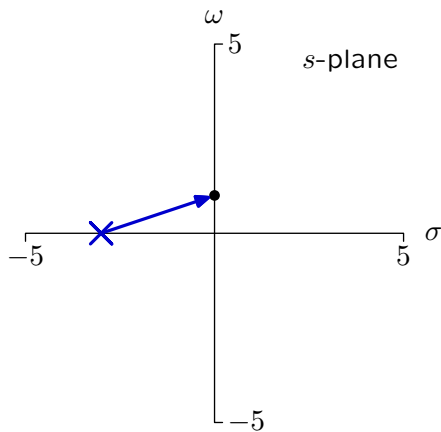
## Frequency Response: Single Pole

$$H(s) = \frac{9}{s - p_1}$$



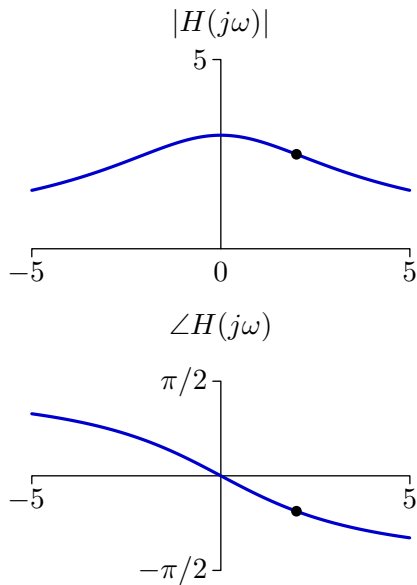
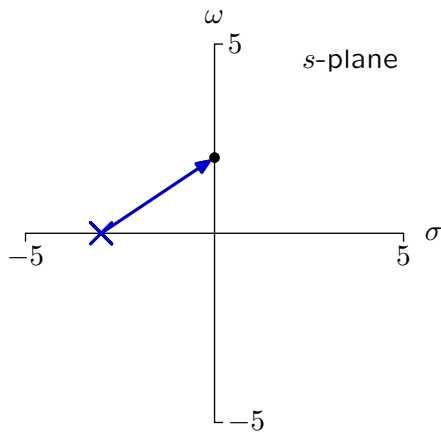
## Frequency Response: Single Pole

$$H(s) = \frac{9}{s - p_1}$$



## Frequency Response: Single Pole

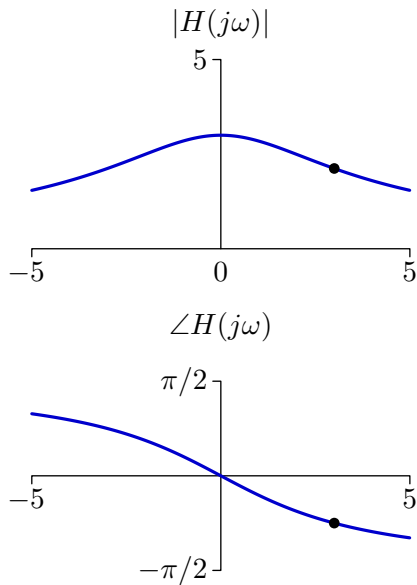
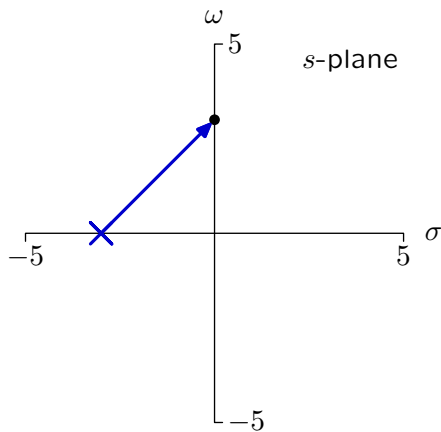
$$H(s) = \frac{9}{s - p_1}$$





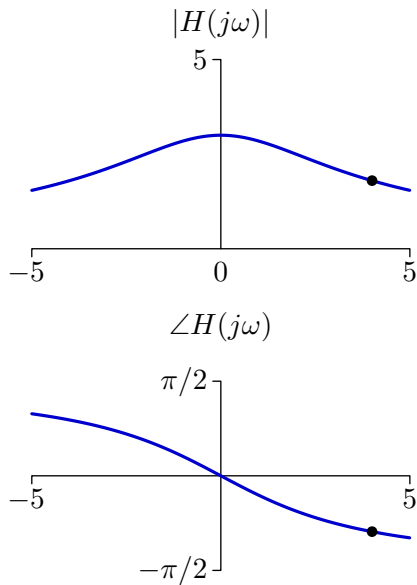
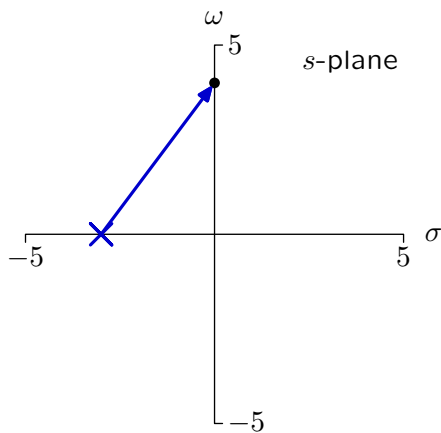
## Frequency Response: Single Pole

$$H(s) = \frac{9}{s - p_1}$$



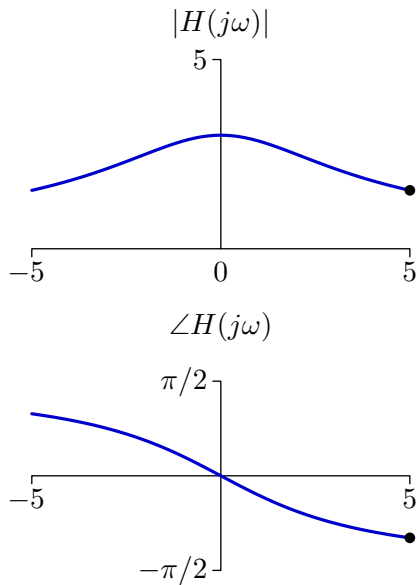
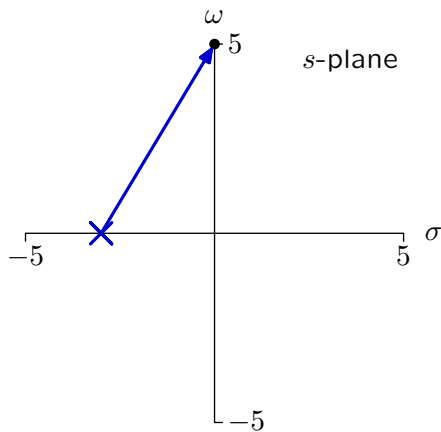
## Frequency Response: Single Pole

$$H(s) = \frac{9}{s - p_1}$$



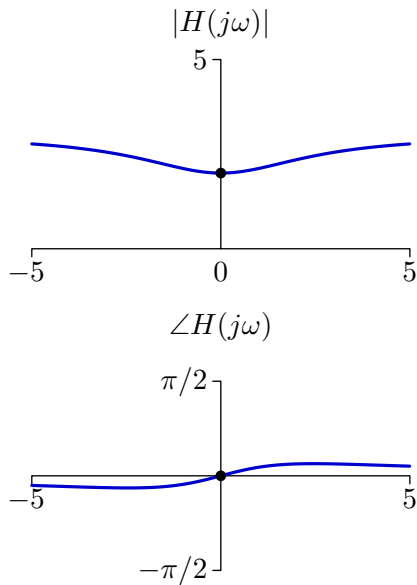
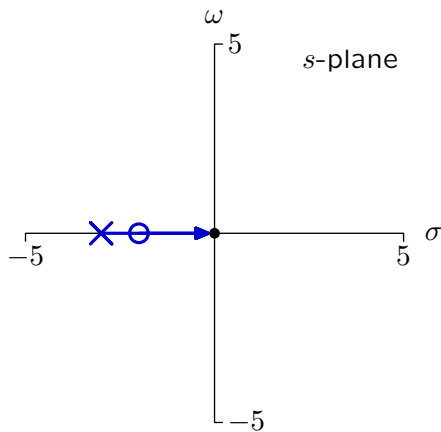
## Frequency Response: Single Pole

$$H(s) = \frac{9}{s - p_1}$$



## Frequency Response: One Zero and One Pole

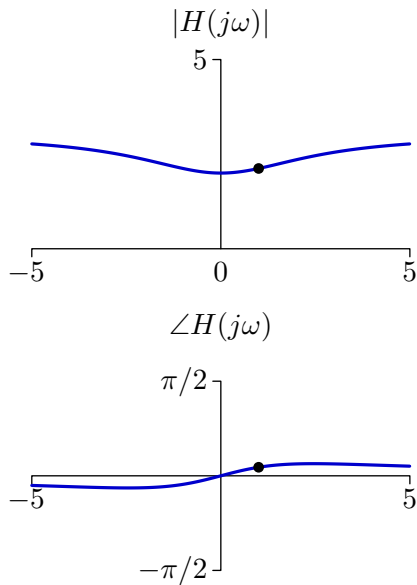
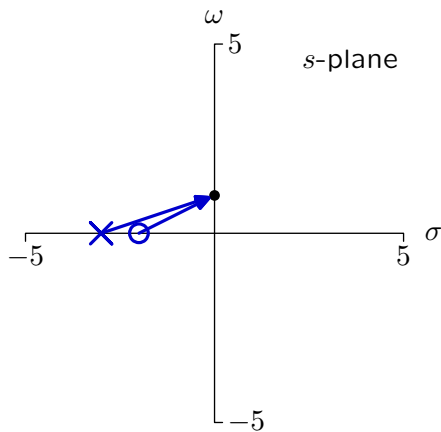
$$H(s) = 3 \frac{s - z_1}{s - p_1}$$





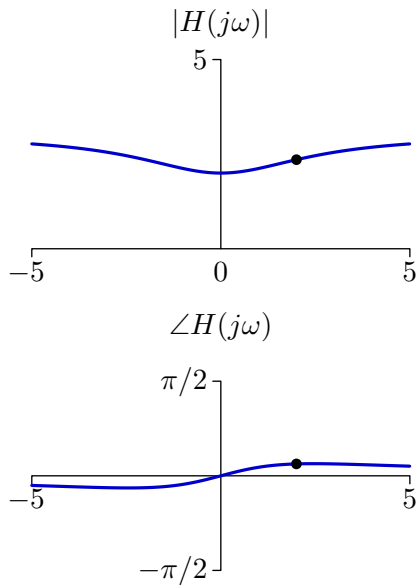
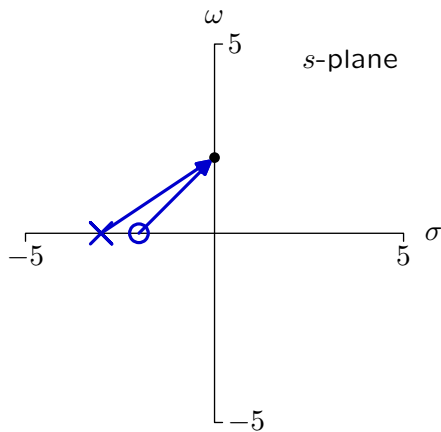
## Frequency Response: One Zero and One Pole

$$H(s) = 3 \frac{s - z_1}{s - p_1}$$



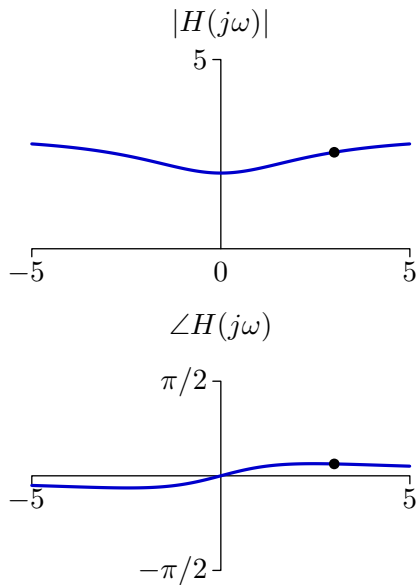
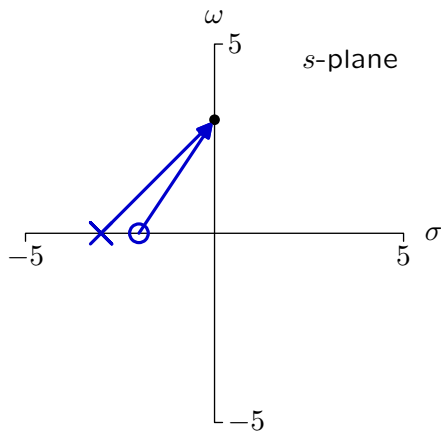
## Frequency Response: One Zero and One Pole

$$H(s) = 3 \frac{s - z_1}{s - p_1}$$



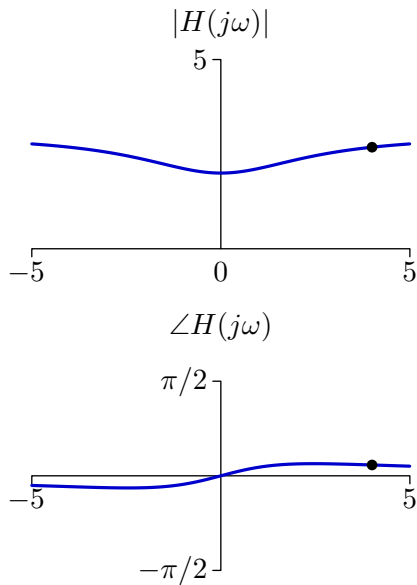
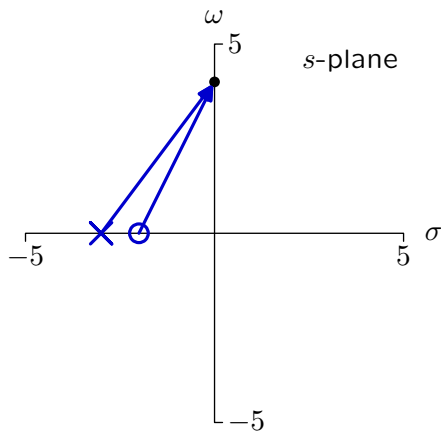
## Frequency Response: One Zero and One Pole

$$H(s) = 3 \frac{s - z_1}{s - p_1}$$



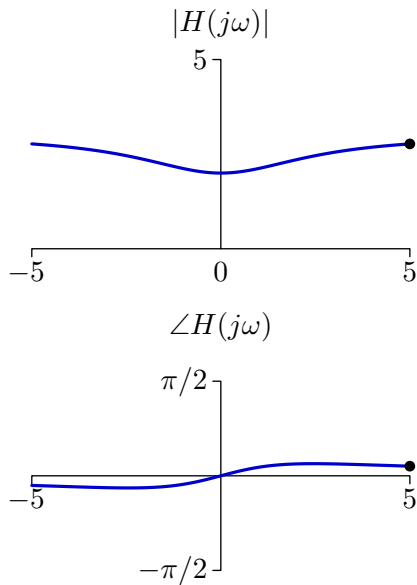
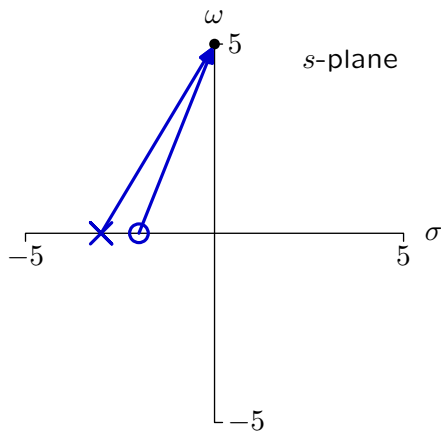
## Frequency Response: One Zero and One Pole

$$H(s) = 3 \frac{s - z_1}{s - p_1}$$



## Frequency Response: One Zero and One Pole

$$H(s) = 3 \frac{s - z_1}{s - p_1}$$





## Bode Plots

---

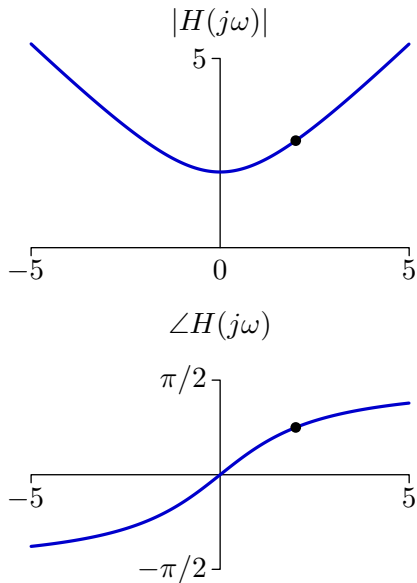
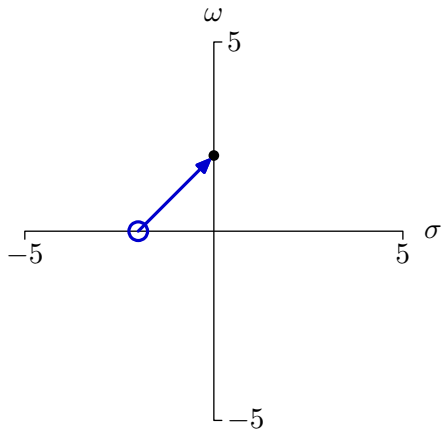
Vector diagrams: simple and powerful way to characterize frequency response with just a few vectors.

Bode plots are even simpler.

## Asymptotic Behavior: Single Zero

The magnitude response is simple at low and high frequencies.

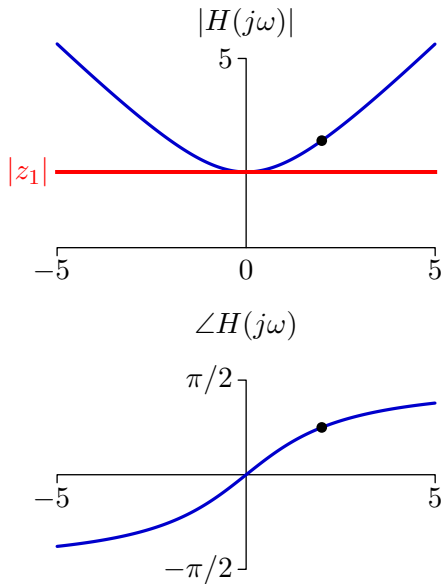
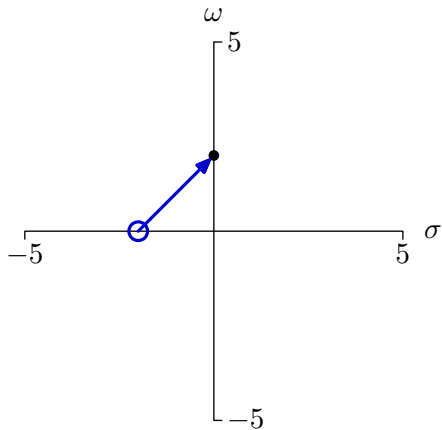
$$H(j\omega) = j\omega - z_1$$



## Asymptotic Behavior: Single Zero

The magnitude response is simple at low and high frequencies.

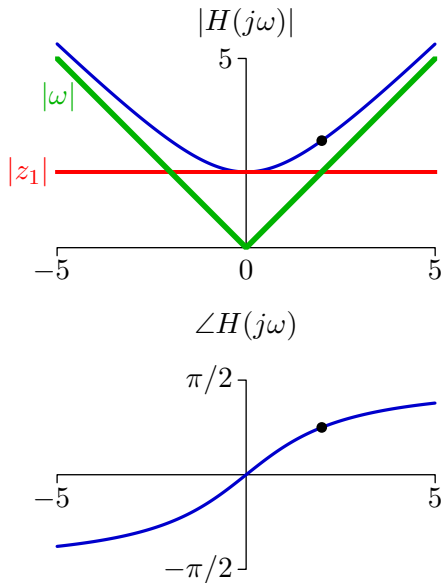
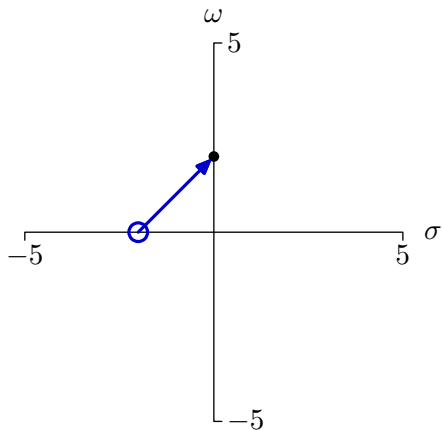
$$H(j\omega) = j\omega - z_1$$



## Asymptotic Behavior: Single Zero

The magnitude response is simple at low and high frequencies.

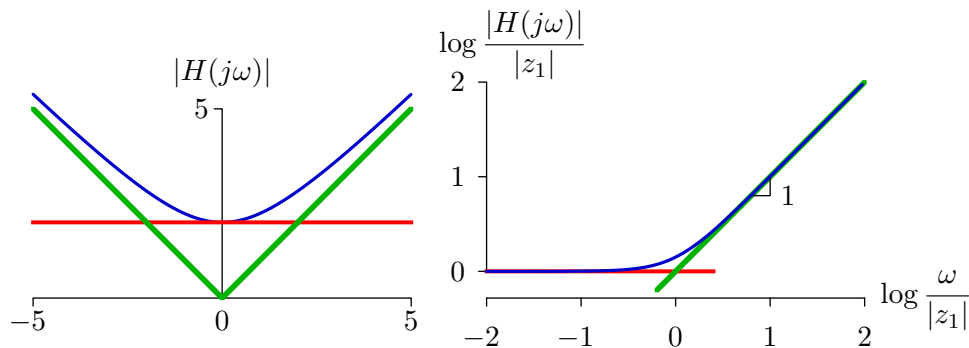
$$H(j\omega) = j\omega - z_1$$



## Asymptotic Behavior: Single Zero

Two asymptotes provide a good approximation on log-log axes.

$$H(s) = s - z_1$$



$$\lim_{\omega \rightarrow 0} |H(j\omega)| = |z_1|$$

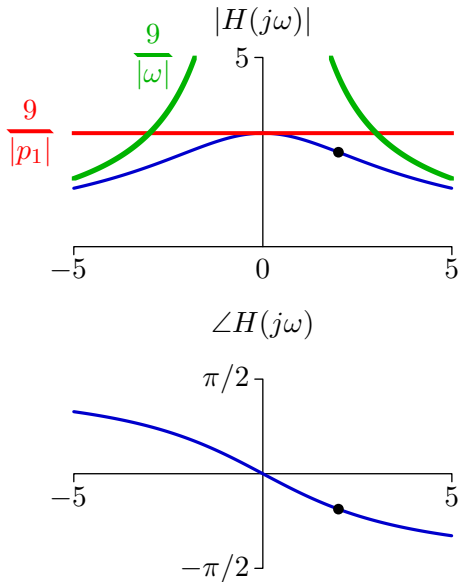
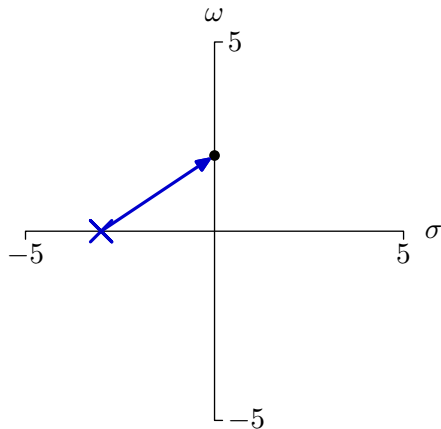
$$\lim_{\omega \rightarrow \infty} |H(j\omega)| = \omega$$



## Asymptotic Behavior: Single Pole

The magnitude response is simple at low and high frequencies.

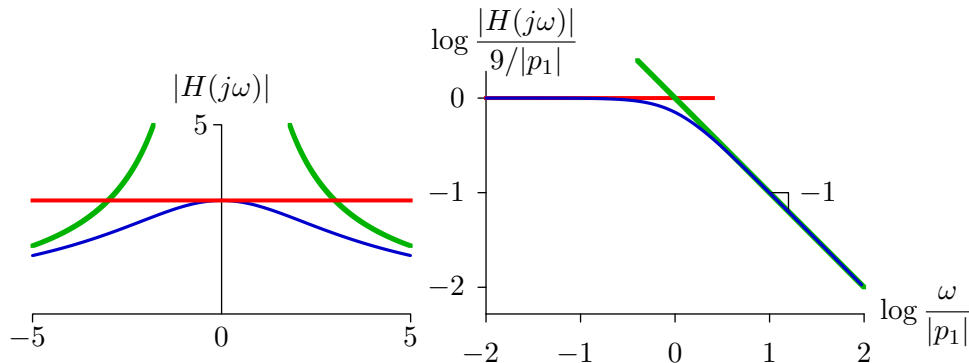
$$H(s) = \frac{9}{s - p_1}$$



## Asymptotic Behavior: Single Pole

Two asymptotes provide a good approximation on log-log axes.

$$H(s) = \frac{9}{s - p_1}$$



$$\lim_{\omega \rightarrow 0} |H(j\omega)| = \frac{9}{|p_1|}$$

$$\lim_{\omega \rightarrow \infty} |H(j\omega)| = \frac{9}{\omega}$$

## Check Yourself

---

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \quad \text{and} \quad H_2(s) = \frac{1}{s+10}$$

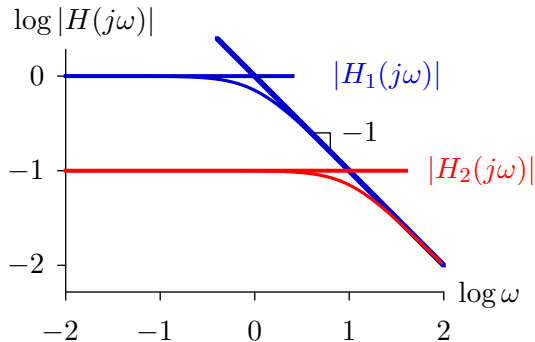
The former can be transformed into the latter by

1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

## Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \quad \text{and} \quad H_2(s) = \frac{1}{s+10}$$



## Check Yourself

---

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \quad \text{and} \quad H_2(s) = \frac{1}{s+10}$$

The former can be transformed into the latter by **3**

1. shifting horizontally
2. shifting and scaling horizontally
- 3. shifting both horizontally and vertically**
4. shifting and scaling both horizontally and vertically
5. none of the above

no scaling in either vertical or horizontal directions!

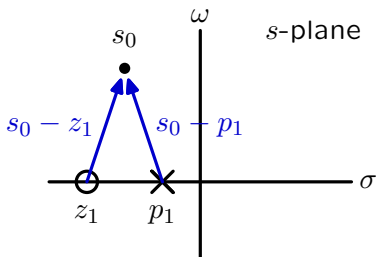


## Asymptotic Behavior of More Complicated Systems

---

Constructing  $H(s_0)$ .

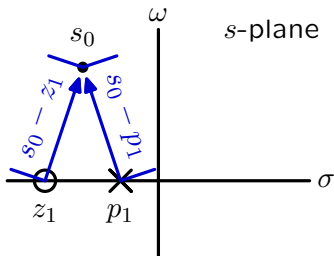
$$H(s_0) = K \frac{\prod_{q=1}^Q (s_0 - z_q)}{\prod_{p=1}^P (s_0 - p_p)} \quad \leftarrow \text{product of vectors for zeros}$$
$$\prod_{p=1}^P (s_0 - p_p) \quad \leftarrow \text{product of vectors for poles}$$



## Asymptotic Behavior of More Complicated Systems

The magnitude of a product is the product of the magnitudes.

$$|H(s_0)| = \left| K \frac{\prod_{q=1}^Q (s_0 - z_q)}{\prod_{p=1}^P (s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^Q |s_0 - z_q|}{\prod_{p=1}^P |s_0 - p_p|}$$



## Bode Plot

---

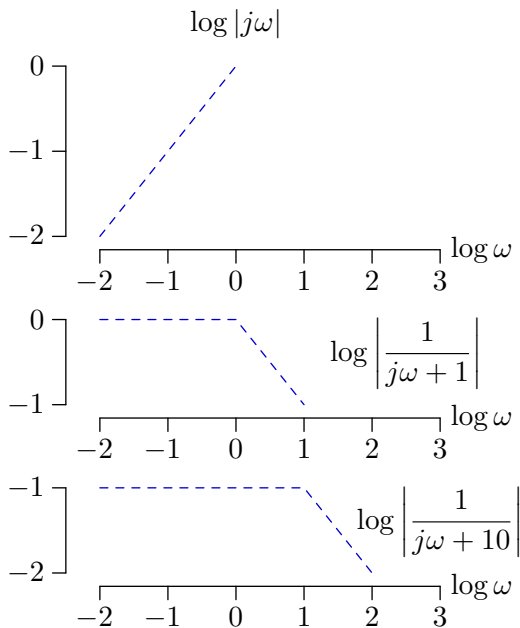
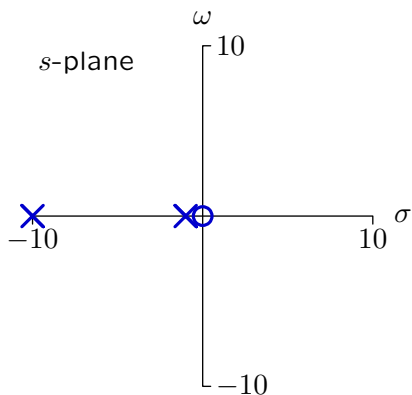
The log of the magnitude is a sum of logs.

$$|H(s_0)| = \left| K \frac{\prod_{q=1}^Q (s_0 - z_q)}{\prod_{p=1}^P (s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^Q |s_0 - z_q|}{\prod_{p=1}^P |s_0 - p_p|}$$

$$\log |H(j\omega)| = \log |K| + \sum_{q=1}^Q \log |j\omega - z_q| - \sum_{p=1}^P \log |j\omega - p_p|$$

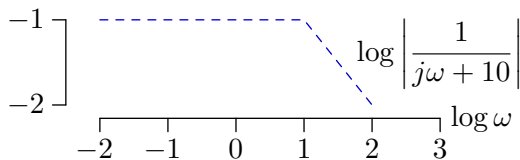
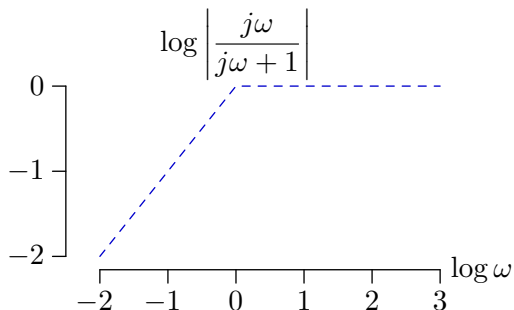
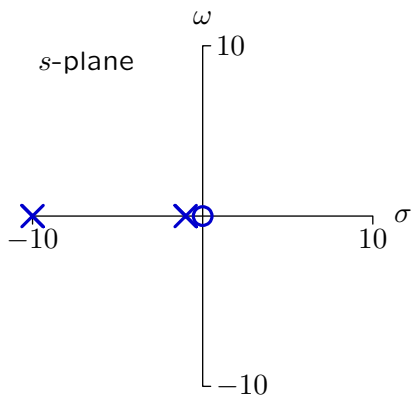
## Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$



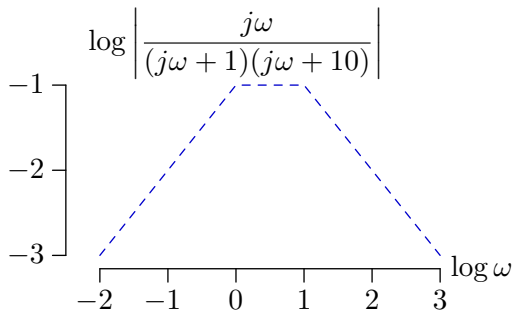
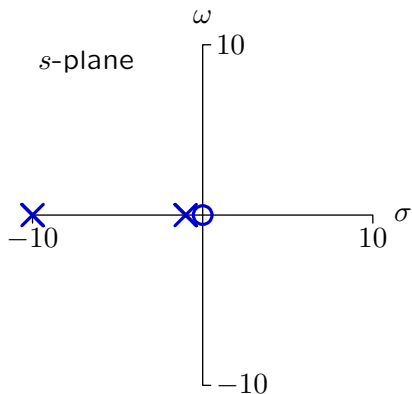
## Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$



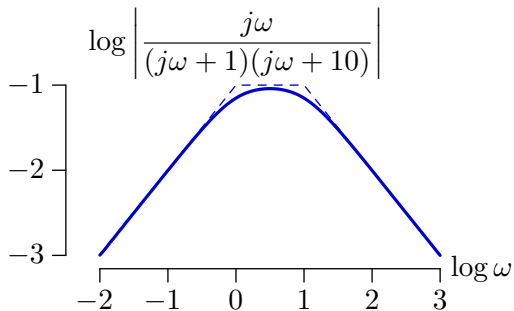
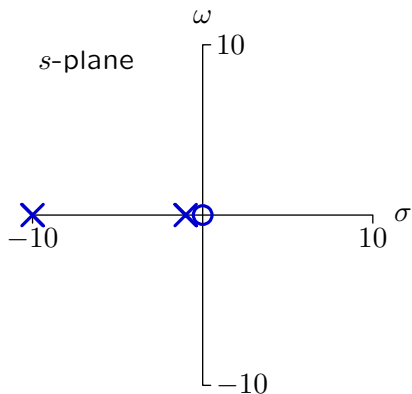
## Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$



## Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$

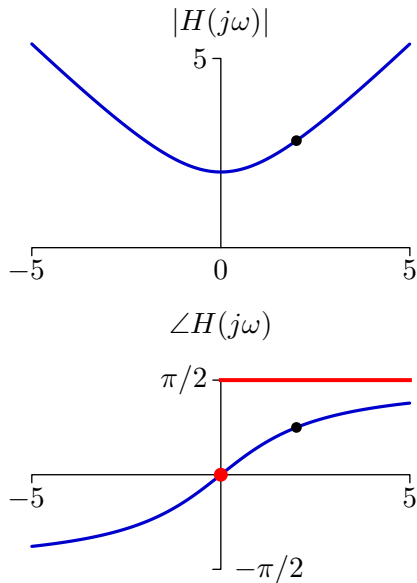
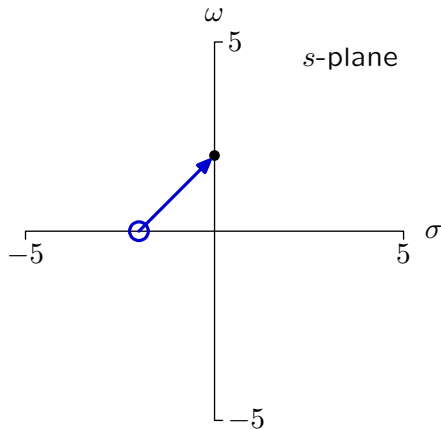




## Asymptotic Behavior: Single Zero

The angle response is simple at low and high frequencies.

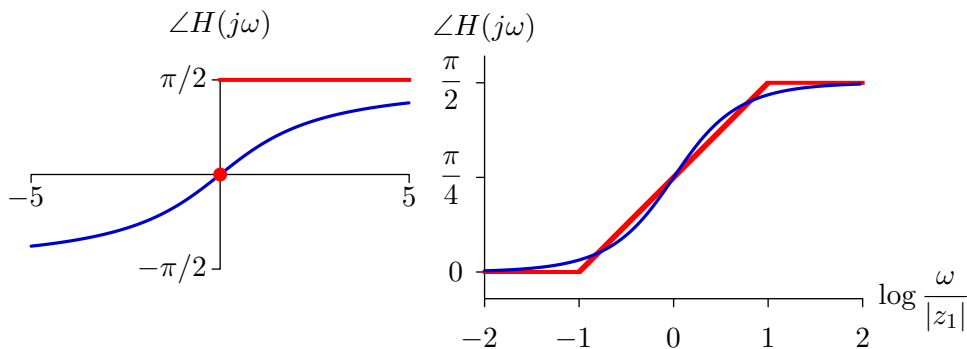
$$H(s) = s - z_1$$



## Asymptotic Behavior: Single Zero

Three straight lines provide a good approximation versus  $\log \omega$ .

$$H(s) = s - z_1$$



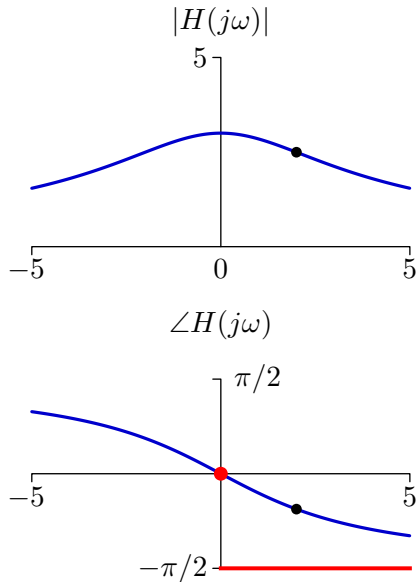
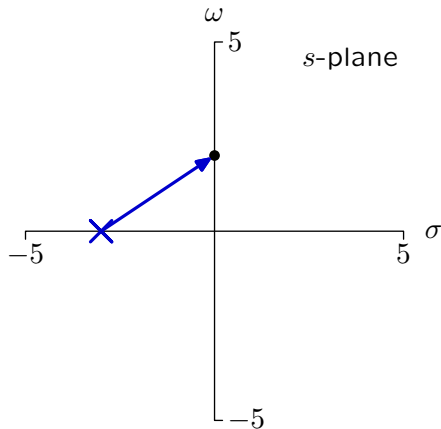
$$\lim_{\omega \rightarrow 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \rightarrow \infty} \angle H(j\omega) = \pi/2$$

## Asymptotic Behavior: Single Pole

The angle response is simple at low and high frequencies.

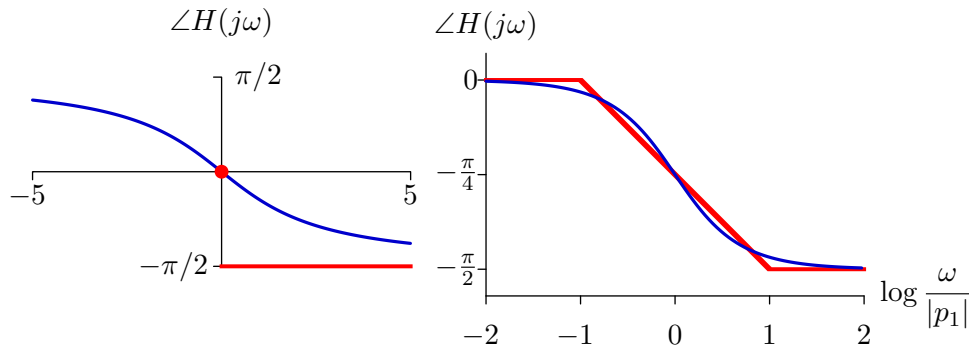
$$H(s) = \frac{9}{s - p_1}$$



## Asymptotic Behavior: Single Pole

Three straight lines provide a good approximation versus  $\log \omega$ .

$$H(s) = \frac{9}{s - p_1}$$



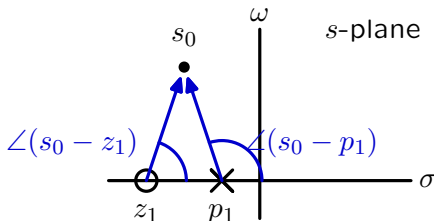
$$\lim_{\omega \rightarrow 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \rightarrow \infty} \angle H(j\omega) = -\pi/2$$

## Bode Plot

The angle of a product is the sum of the angles.

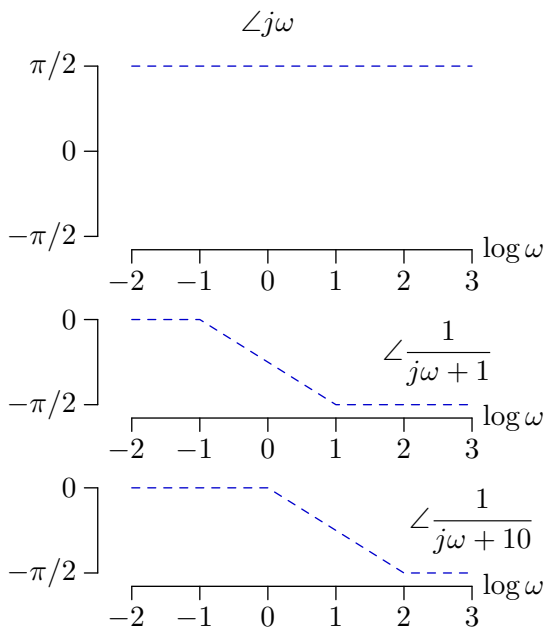
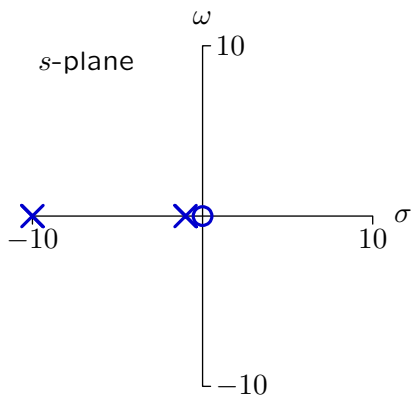
$$\angle H(s_0) = \angle \left( K \frac{\prod_{q=1}^Q (s_0 - z_q)}{\prod_{p=1}^P (s_0 - p_p)} \right) = \angle K + \sum_{q=1}^Q \angle (s_0 - z_q) - \sum_{p=1}^P \angle (s_0 - p_p)$$



The angle of  $K$  can be  $0$  or  $\pi$  for systems described by linear differential equations with constant, real-valued coefficients.

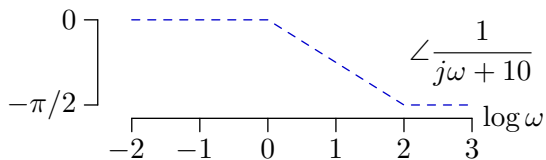
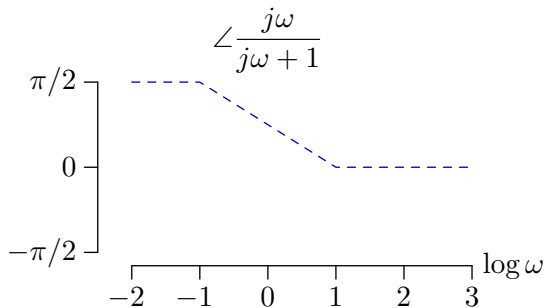
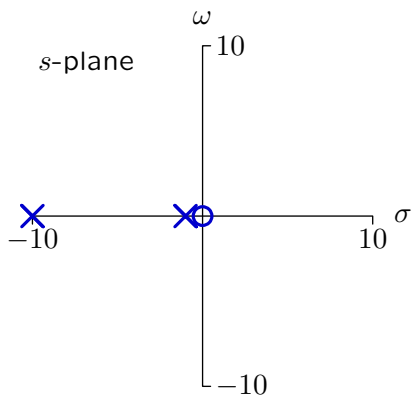
# Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$



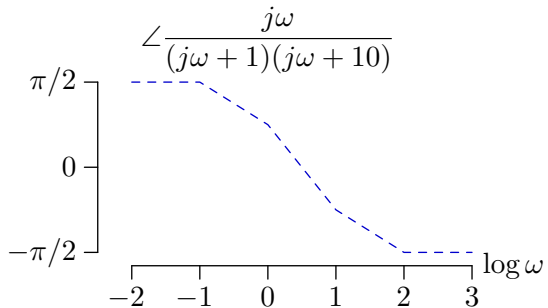
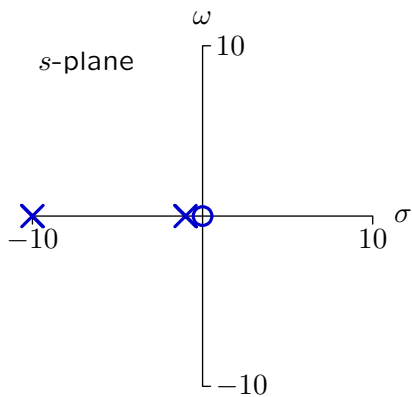
# Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$



## Bode Plot

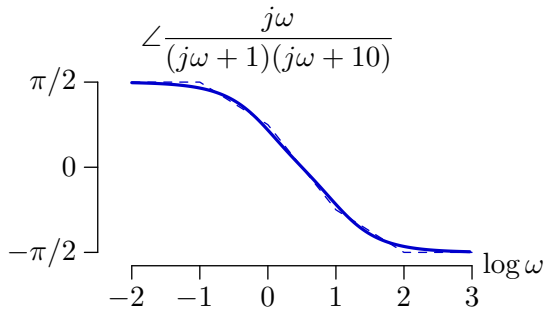
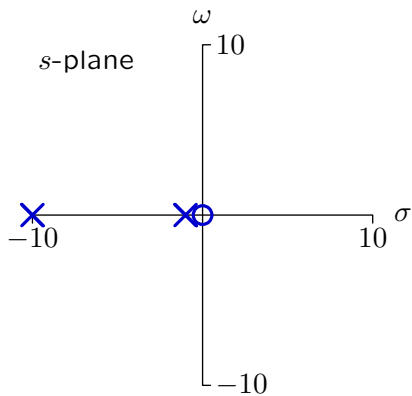
$$H(s) = \frac{s}{(s+1)(s+10)}$$





## Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$



## From Frequency Response to Bode Plot

---

The magnitude of  $H(j\omega)$  is a product of magnitudes.

$$|H(j\omega)| = |K| \frac{\prod_{q=1}^Q |j\omega - z_q|}{\prod_{p=1}^P |j\omega - p_p|}$$

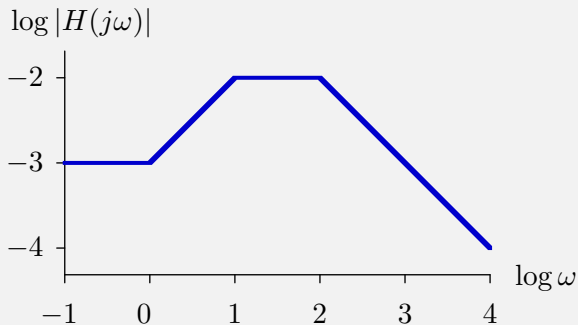
The log of the magnitude is a sum of logs.

$$\log |H(j\omega)| = \log |K| + \sum_{q=1}^Q \log |j\omega - z_q| - \sum_{p=1}^P \log |j\omega - p_p|$$

The angle of  $H(j\omega)$  is a sum of angles.

$$\angle H(j\omega) = \angle K + \sum_{q=1}^Q \angle (j\omega - z_q) - \sum_{p=1}^P \angle (j\omega - p_p)$$

## Check Yourself



Which corresponds to the Bode approximation above?

1.  $\frac{1}{(s+1)(s+10)(s+100)}$

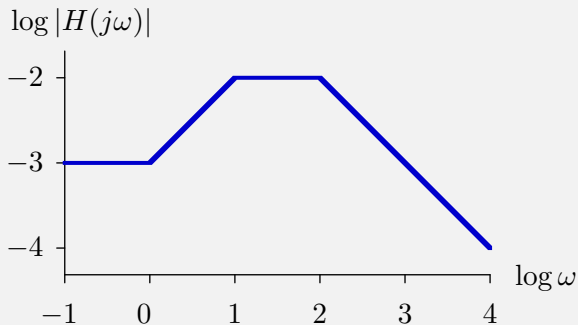
2.  $\frac{s+1}{(s+10)(s+100)}$

3.  $\frac{(s+10)(s+100)}{s+1}$

4.  $\frac{s+100}{(s+1)(s+10)}$

5. none of the above

## Check Yourself



Which corresponds to the Bode approximation above? 2

1. 
$$\frac{1}{(s+1)(s+10)(s+100)}$$

3. 
$$\frac{(s+10)(s+100)}{s+1}$$

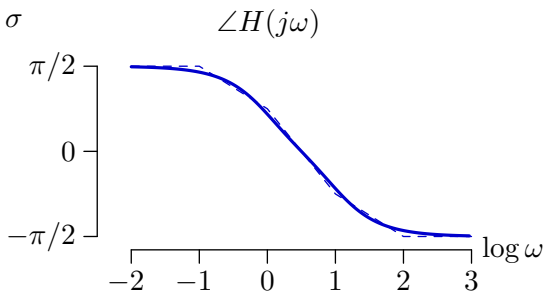
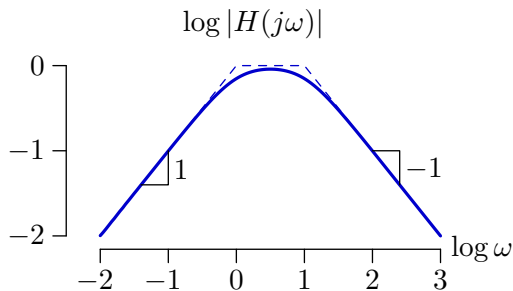
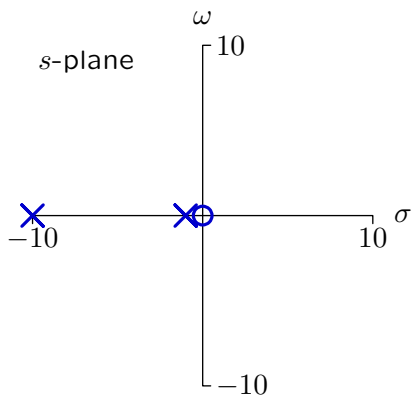
2. 
$$\frac{s+1}{(s+10)(s+100)}$$

4. 
$$\frac{s+100}{(s+1)(s+10)}$$

5. none of the above

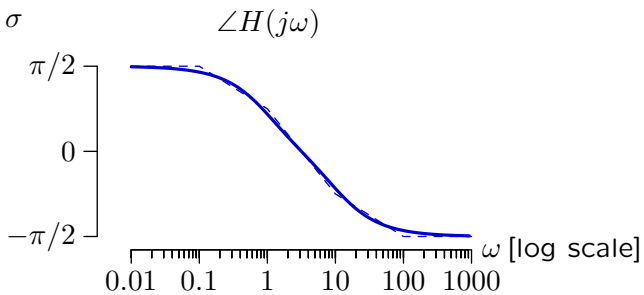
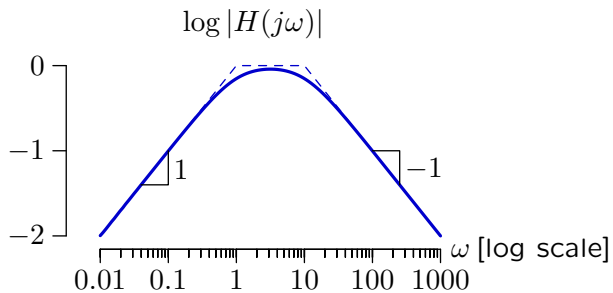
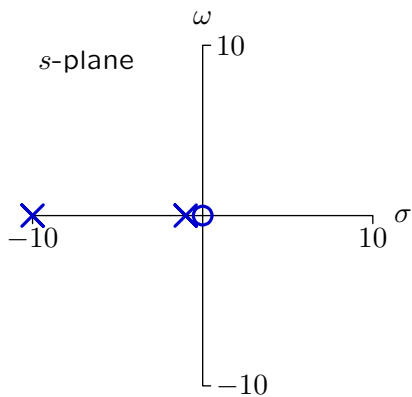
## Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$



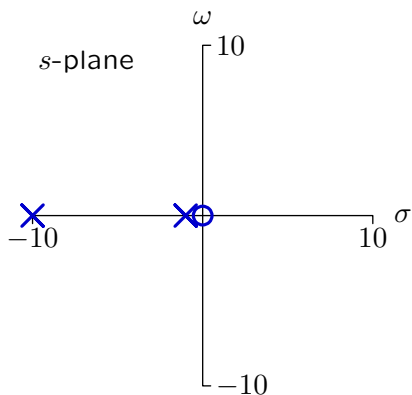
## Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$

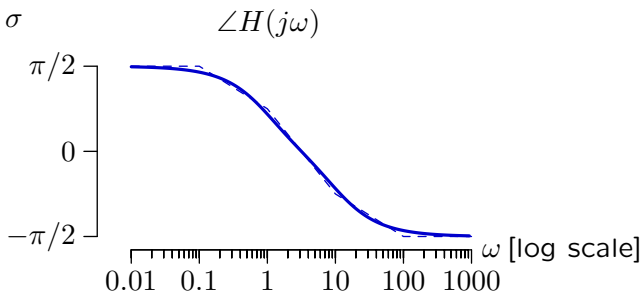
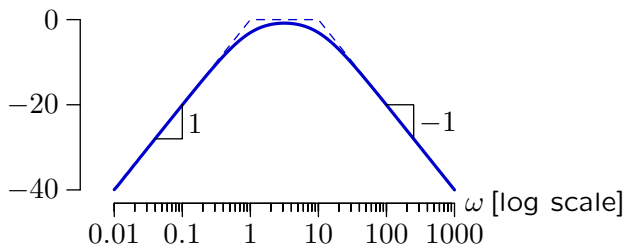


## Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$

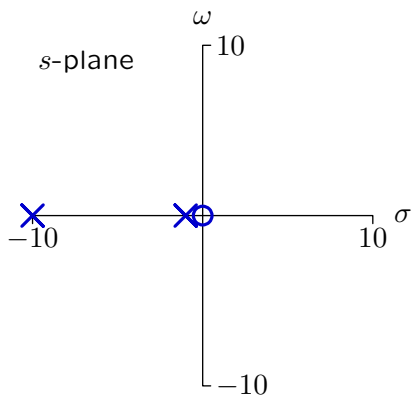


$$|H(j\omega)|[\text{dB}] = 20 \log_{10} |H(j\omega)|$$

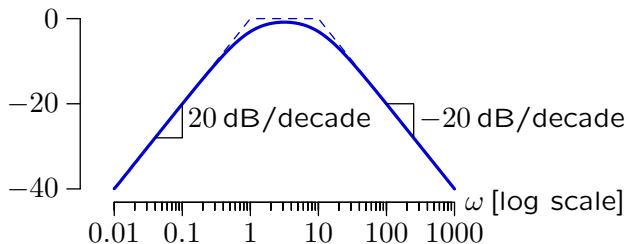


## Bode Plot: dB

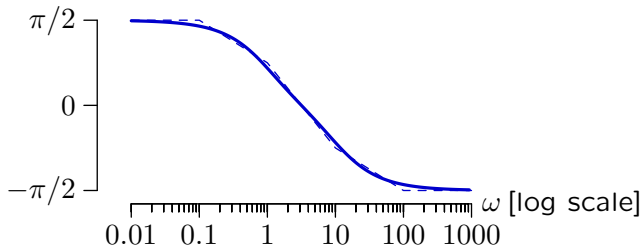
$$H(s) = \frac{10s}{(s+1)(s+10)}$$



$$|H(j\omega)|[\text{dB}] = 20 \log_{10} |H(j\omega)|$$



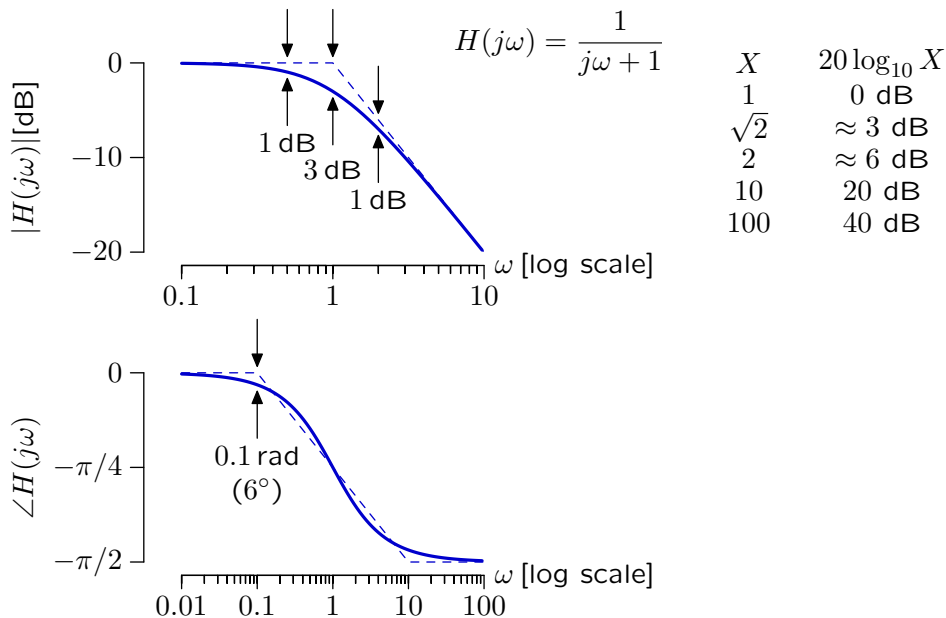
$$\angle H(j\omega)$$





## Bode Plot: Accuracy

The straight-line approximations are surprisingly accurate.

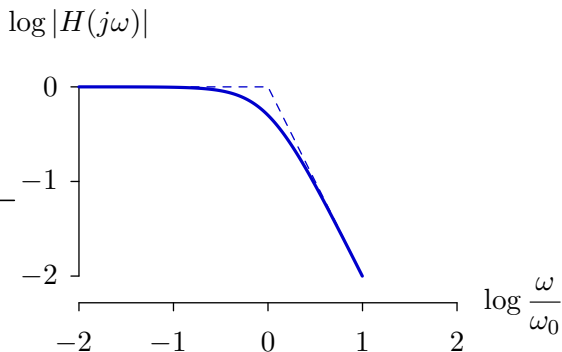
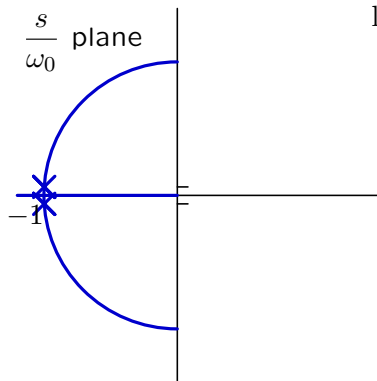


## Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 0.501$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

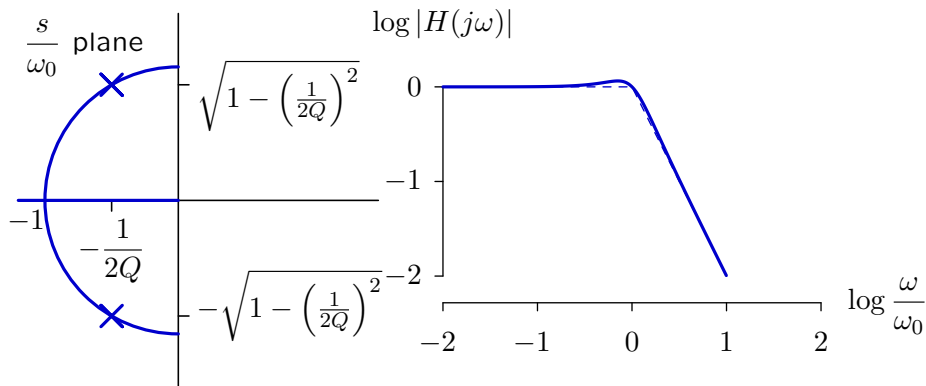


## Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 1$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

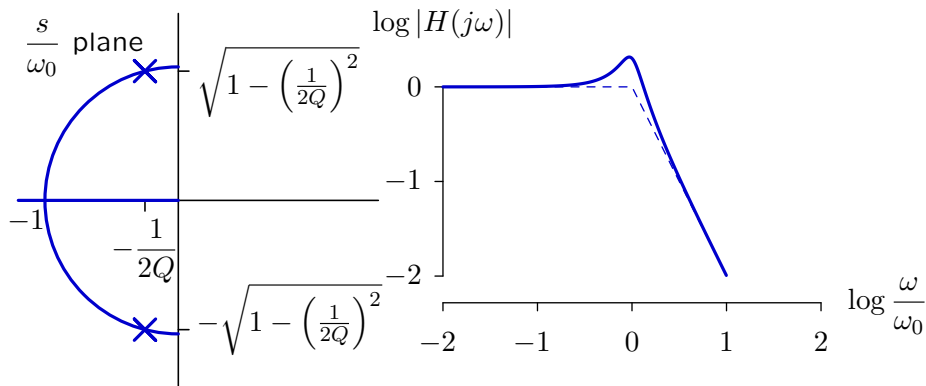


## Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 2$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

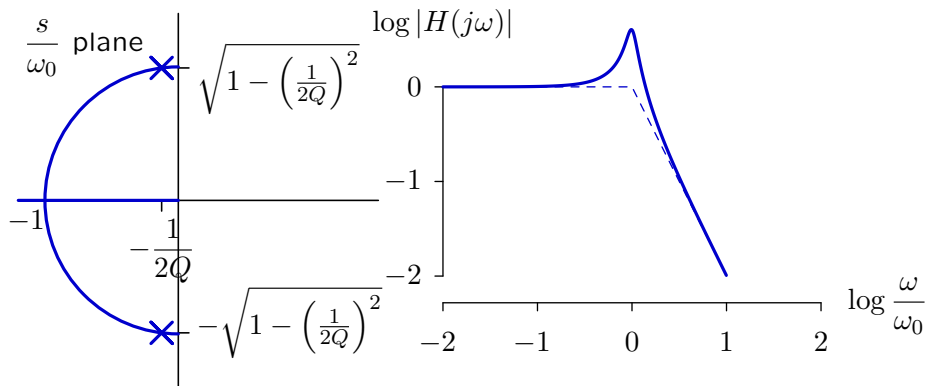


## Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 4$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

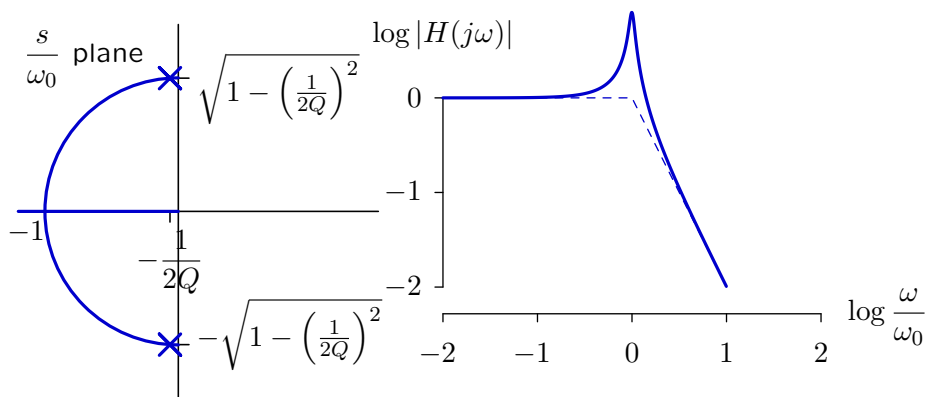


## Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 8$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

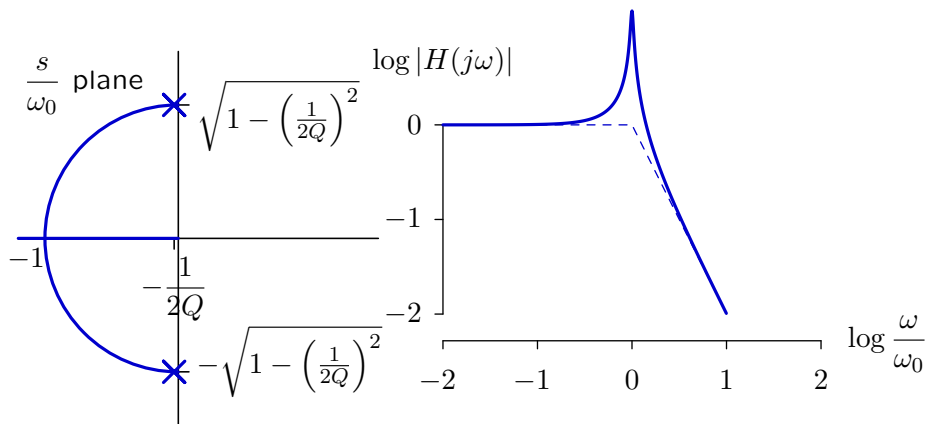


## Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 16$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

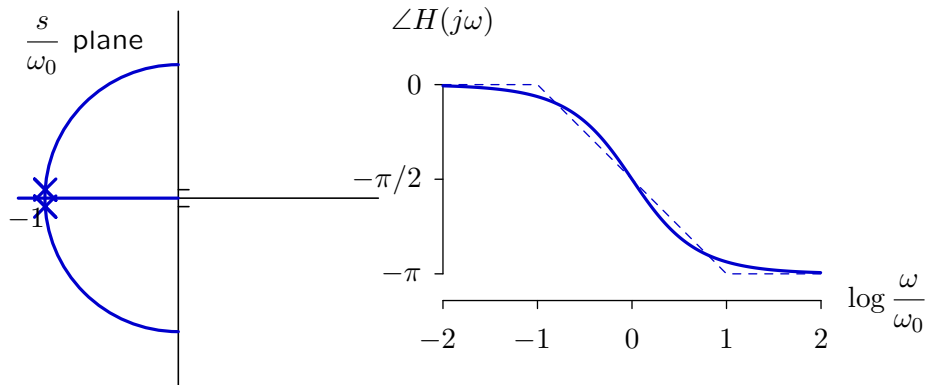


## Frequency Response of a High-Q System

As  $Q$  increases, the phase changes more abruptly with  $\omega$ .

$$Q = 0.501$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



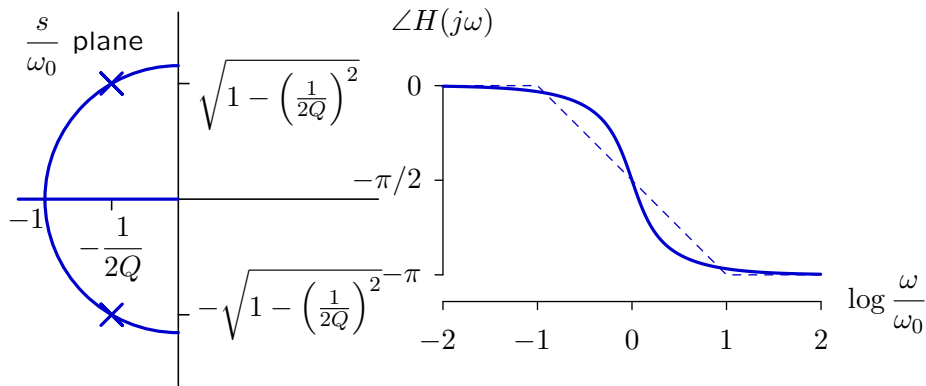


## Frequency Response of a High-Q System

As  $Q$  increases, the phase changes more abruptly with  $\omega$ .

$$Q = 1$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

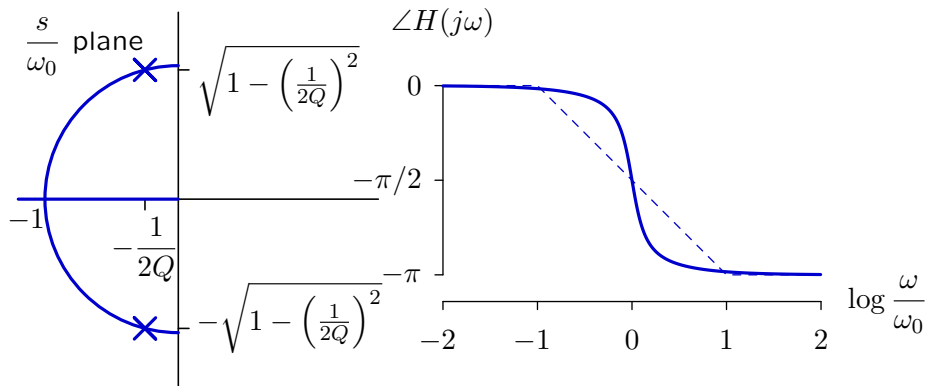


## Frequency Response of a High-Q System

As  $Q$  increases, the phase changes more abruptly with  $\omega$ .

$$Q = 2$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

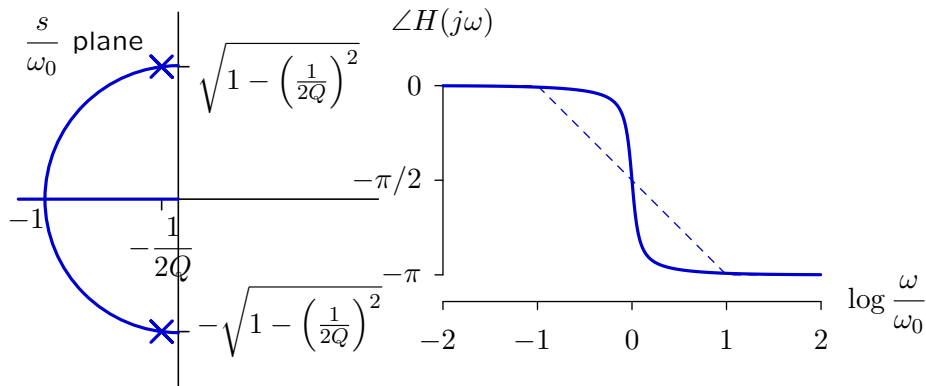


## Frequency Response of a High-Q System

As  $Q$  increases, the phase changes more abruptly with  $\omega$ .

$$Q = 4$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

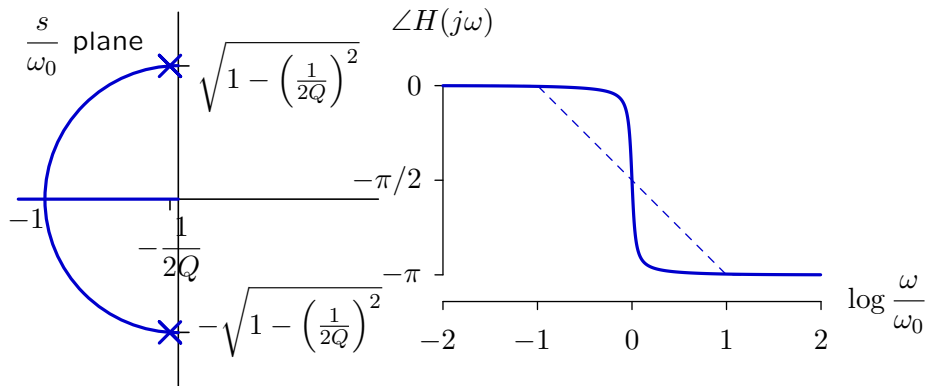


## Frequency Response of a High-Q System

As  $Q$  increases, the phase changes more abruptly with  $\omega$ .

$$Q = 8$$

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



## Check Yourself

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Determine Bode diagrams (magnitude and phase) for the following system function:

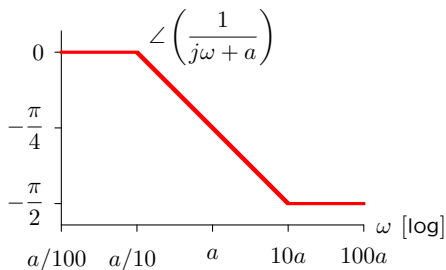
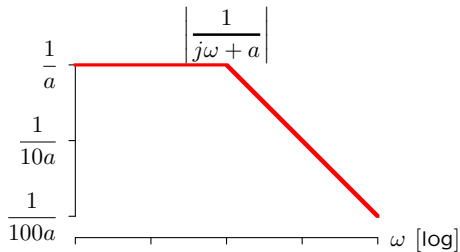
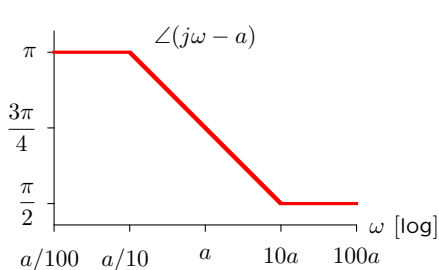
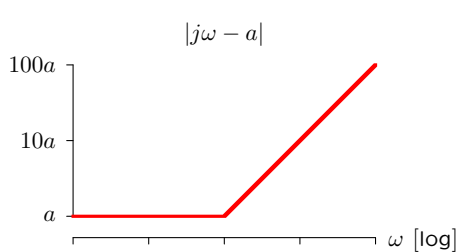
$$H(s) = \frac{s - a}{s + a}$$

where  $a$  is a positive, real number.

## Check Yourself

Determine Bode diagrams (magnitude and phase) for the following system function:

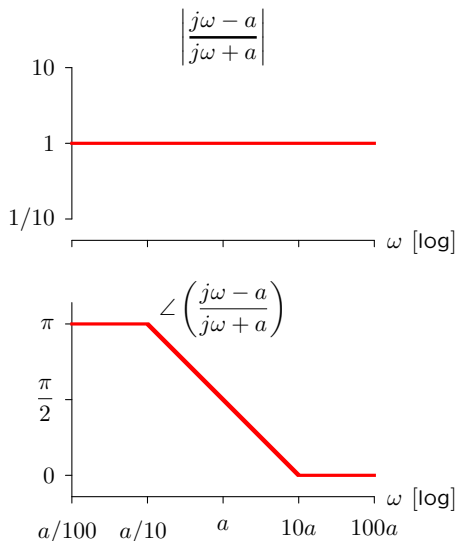
$$H(s) = \frac{s - a}{s + a}$$



## Check Yourself

Determine Bode diagrams (magnitude and phase) for the following system function:

$$H(s) = \frac{s - a}{s + a}$$



## Parameter Estimation

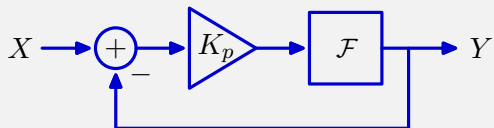
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Application of frequency response in **parameter estimation**.



## Check Yourself

A proportional control scheme is used in the following system:



When  $K_p = 1$ , the closed-loop system function is given by

$$H(s) = \frac{Y}{X} = \frac{s + \alpha}{s^2 + s + \alpha}$$

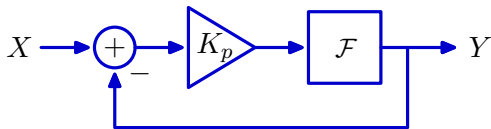
Determine the system function  $\mathcal{F}$  for the plant.

1.  $\frac{s + \alpha}{s^2}$
2.  $\frac{s + \alpha}{s^2 + s + \alpha}$
3.  $\frac{s + \alpha}{s^2 + 2s + 2\alpha}$
4.  $\frac{s + \alpha}{s^2 - s - \alpha}$
5. none of the above

## Check Yourself

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A proportional control scheme is used in the following system:



The general relation between  $\mathcal{F}$  and  $\mathcal{H}$  is given by Black's equation:

$$\mathcal{H} = \frac{K_p \mathcal{F}}{1 + K_p \mathcal{F}}$$

As indicated above, Black's equation is normally used to determine  $\mathcal{H}$  from  $\mathcal{F}$ . However, Black's equation can be used to go in the opposite direction.

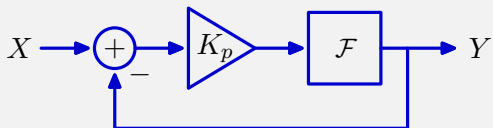
$$\mathcal{H} + K_p \mathcal{F} \mathcal{H} = K_p \mathcal{F}$$

$$\mathcal{F} = \frac{1}{K_p} \left( \frac{\mathcal{H}}{1 - \mathcal{H}} \right) = \frac{\mathcal{H}}{1 - \mathcal{H}} \quad \text{since } K_p = 1$$

$$\text{If } \mathcal{H} = \frac{s + \alpha}{s^2 + s + \alpha} \text{ then } \mathcal{F} = \frac{\left( \frac{s + \alpha}{s^2 + s + \alpha} \right)}{\left( 1 - \frac{s + \alpha}{s^2 + s + \alpha} \right)} = \left( \frac{s + \alpha}{s^2} \right)$$

## Check Yourself

A proportional control scheme is used in the following system:



When  $K_p = 1$ , the closed-loop system function is given by

$$H(s) = \frac{Y}{X} = \frac{s + \alpha}{s^2 + s + \alpha}$$

Determine the system function  $\mathcal{F}$  for the plant. 1

1.  $\frac{s + \alpha}{s^2}$

2.  $\frac{s + \alpha}{s^2 + s + \alpha}$

3.  $\frac{s + \alpha}{s^2 + 2s + 2\alpha}$

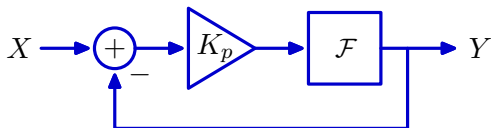
4.  $\frac{s + \alpha}{s^2 - s - \alpha}$

5. none of the above

## Check Yourself

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Notice that the system function for the plant is simpler than that of the closed loop system.



$$F = \frac{s + \alpha}{s^2}$$

$$H = \frac{Y}{X} = \frac{s + \alpha}{s^2 + s + \alpha}$$

This motivates a method to estimate the parameters of a system from the closed-loop frequency response.

## Parameter Estimation

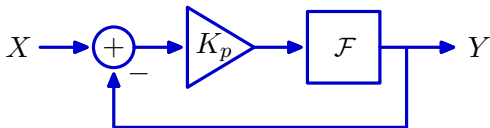
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Assume that we know the form of  $F(s)$  from a model of the system:

$$F(s) = \frac{s + \alpha}{s^2}$$

but we don't know  $\alpha$ .

Estimation procedure: measure the closed-loop frequency response of the system  $\tilde{H}(j\omega)$  when  $K_p = 1$ .



Compute the measured  $\tilde{F}(j\omega)$  from the measured  $\tilde{H}(j\omega)$ :

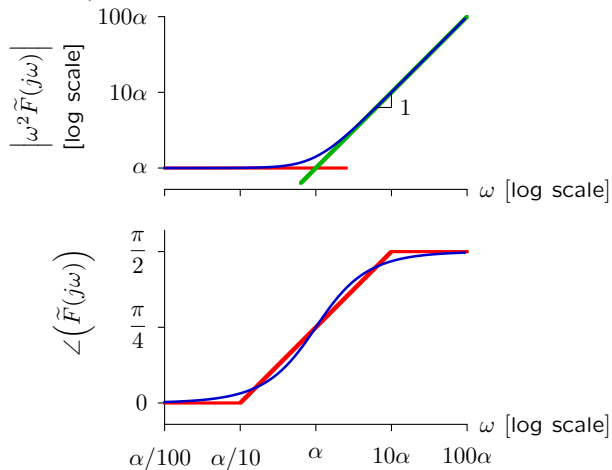
$$\tilde{F}(j\omega) = \frac{1}{K_p} \left( \frac{\tilde{H}(j\omega)}{1 - \tilde{H}(j\omega)} \right) = \frac{\tilde{H}(j\omega)}{1 - \tilde{H}(j\omega)}$$

Compare the resulting frequency response for  $\tilde{F}(j\omega)$  with that of the model:

$$\tilde{F}(j\omega) = \frac{j\omega + \alpha}{(-j\omega)^2} = \frac{j\omega + \alpha}{-\omega^2}$$

## Parameter Estimation

From this analysis, we expect that  $-\omega^2 \tilde{F}(j\omega)$  will have the following form:  $j\omega + \alpha$  (a single zero).

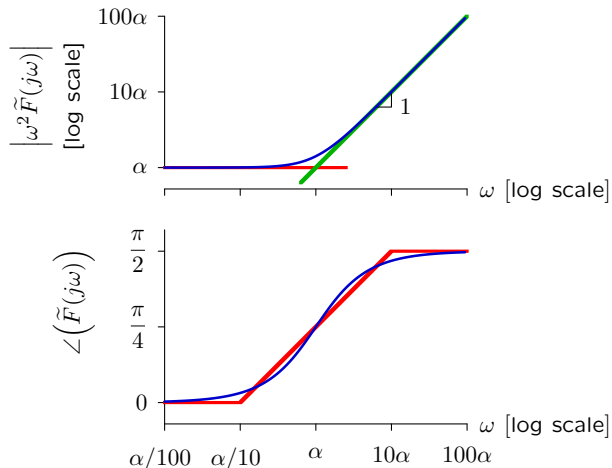


We can estimate the value of the parameter  $\alpha$

- from the low-frequency gain, or
- from the frequency where the two magnitude asymptotes intersect, or
- from the frequency where the phase is  $\pi/4$ .

## Parameter Estimation

From this analysis, we expect that  $-\omega^2 \tilde{F}(j\omega)$  will have the following form:  $j\omega + \alpha$  (a single zero).



We will use a variant of this method to estimate the parameters of the two-propeller arm in this week's postlab.

## Summary

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The frequency response of a system is easily determined using Bode plots.

Each pole and each zero contributes one section to the Bode plot.

The magnitude of the response of the system is given by the sum of the magnitudes for the sections contributed by each pole and zero.

The angle of the response of the system is given by the sum of the angles for the sections contributed by each pole and zero.