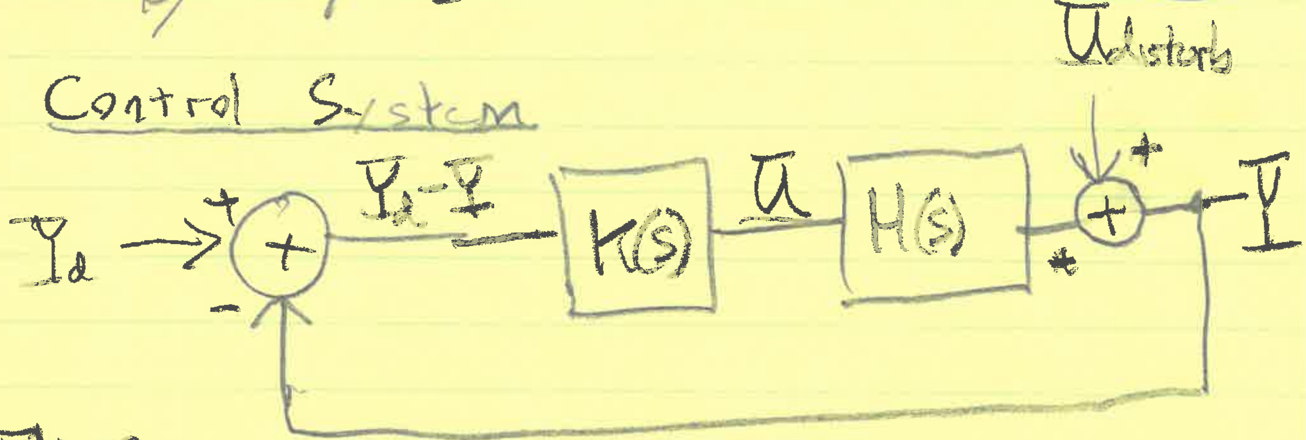
1L

10/25/23

Control System



$U_d = 0$

$$Y = \frac{K(s)H(s)}{1+K(s)H(s)} Y_d$$

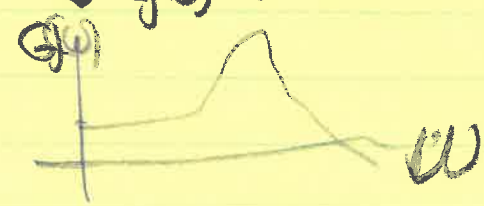
In S.S.S.

$$Y = \underbrace{\frac{K(j\omega)H(j\omega)}{1+K(j\omega)H(j\omega)}}_{G(j\omega)} Y_d$$

Good $K(j\omega)H(j\omega) >> 1 \Rightarrow G(j\omega) = 1 \quad Y = Y_d$

Bad $K(j\omega)H(j\omega) = -1 \quad G(j\omega) \to \infty$

Not as Bad $K(j\omega)H(j\omega)$ near -1



$Y_d = 0$

In S.S.S.

$$Y = \underbrace{\frac{1}{1+K(j\omega)H(j\omega)}}_{G_{dist}(j\omega)} U_{dist}$$

$K(j\omega)H(j\omega) >> 1 \Rightarrow G_{dist}(j\omega) \cong 0$

Bad $K(j\omega)H(j\omega) = -1 \quad G_{dist}(j\omega) \to \infty$

Not as bad $K(j\omega)H(j\omega)$ near -1

for $K(j\omega)H(j\omega)$ to equal -1 $\quad \angle K(j\omega)H(j\omega) = -180$ or $+180$ or

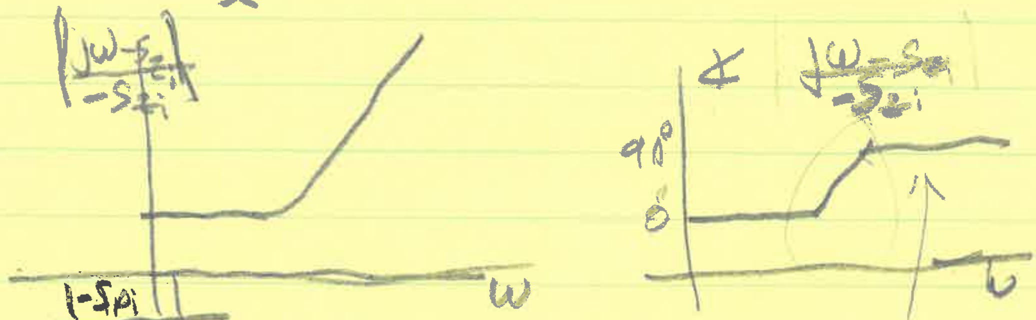
When $|K(j\omega)H(j\omega)| = 1$

ZL

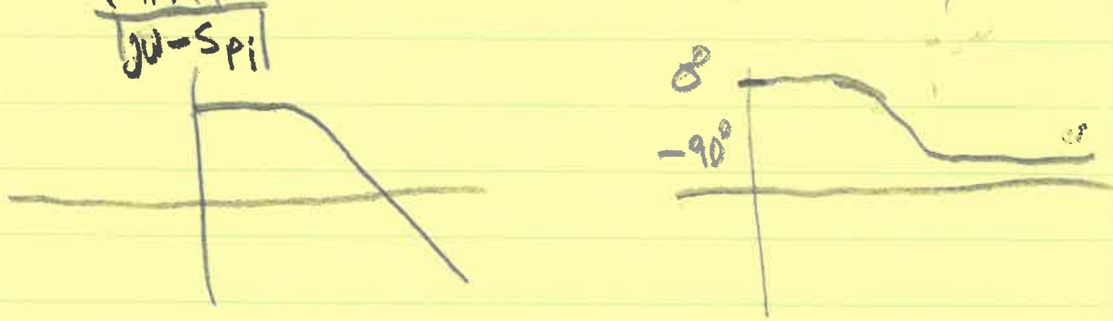
$$K(s)H(s) = \left.\frac{(s-s_{z_1})(s-s_{z_2})\cdots(s-s_{z_{\tilde{L}}})}{(s-s_{p_1})(s-s_{p_2})\cdots(s-s_{p_L})}\right|_{s=j\omega}$$

$$\angle K(j\omega)H(j\omega) = \sum_{l=1}^{\tilde{L}} \angle (j\omega - s_{z_i}) - \angle \sum_{l=1}^{L} (j\omega - s_{p_i})$$

Zero



Pole



Pop Quiz

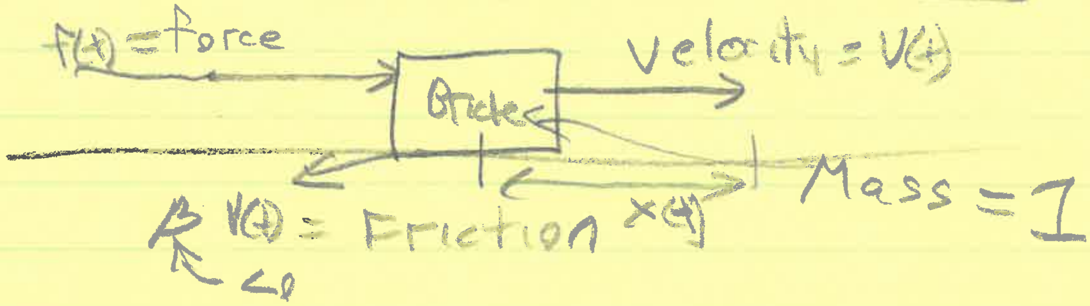
if there is 1 zeros & 2 poles

$$\text{as } \omega \to \infty \quad \angle K(j\omega)H(j\omega) \to -90^\circ$$

$$|K(j\omega)H(j\omega)| \to 0$$

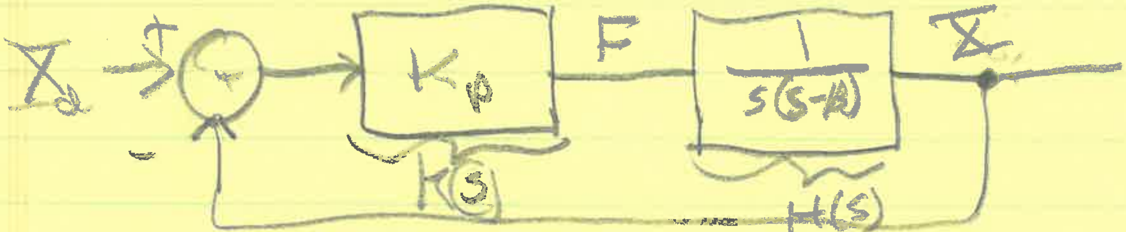
1R

Force control of Brick Position

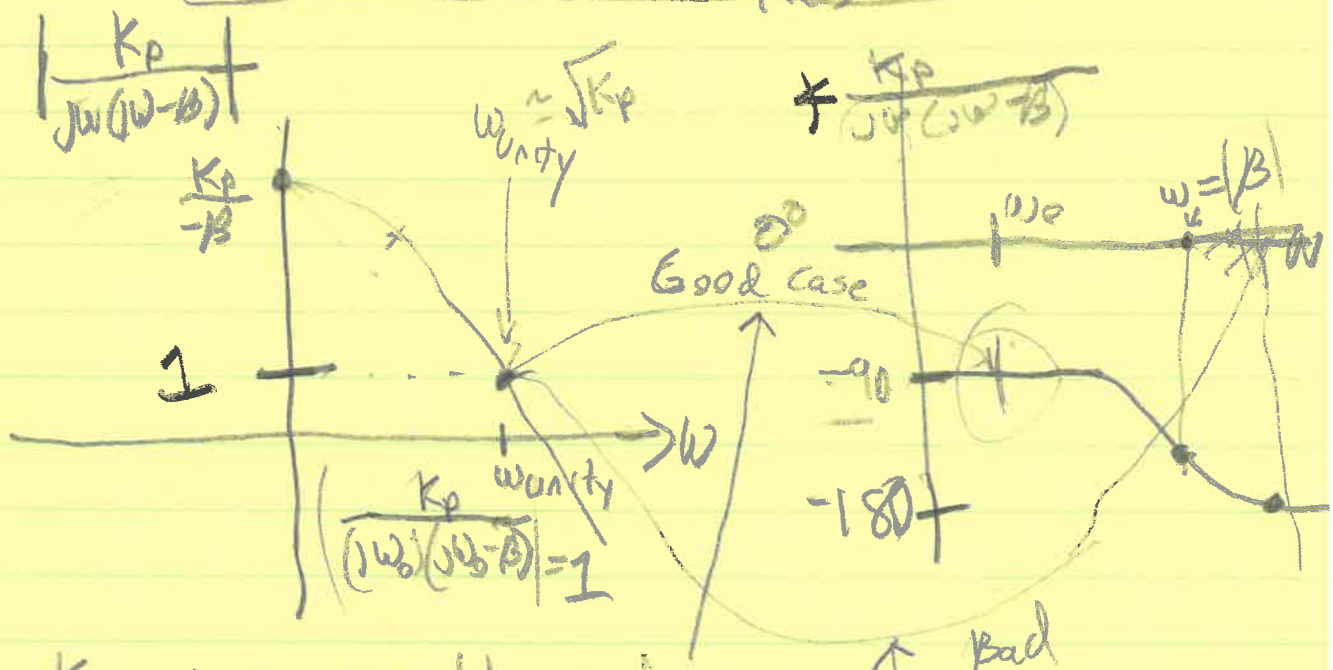


$$\frac{d}{dt} x(t) = V(t) \qquad X = \frac{1}{s} V = \frac{1}{s}\frac{1}{s-\beta} F$$

$$\frac{d}{dt} V(t) = \beta V(t) + F(t) = K_p \left(x_d(t) - x(t) \right)$$



$\beta < 0$



K_p very small ($\ll \beta^2$) good case

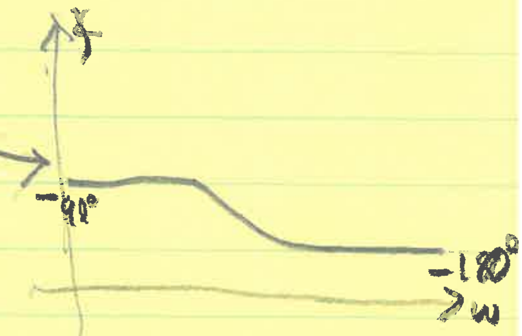
K_p very big ($> \beta^2$) bad case

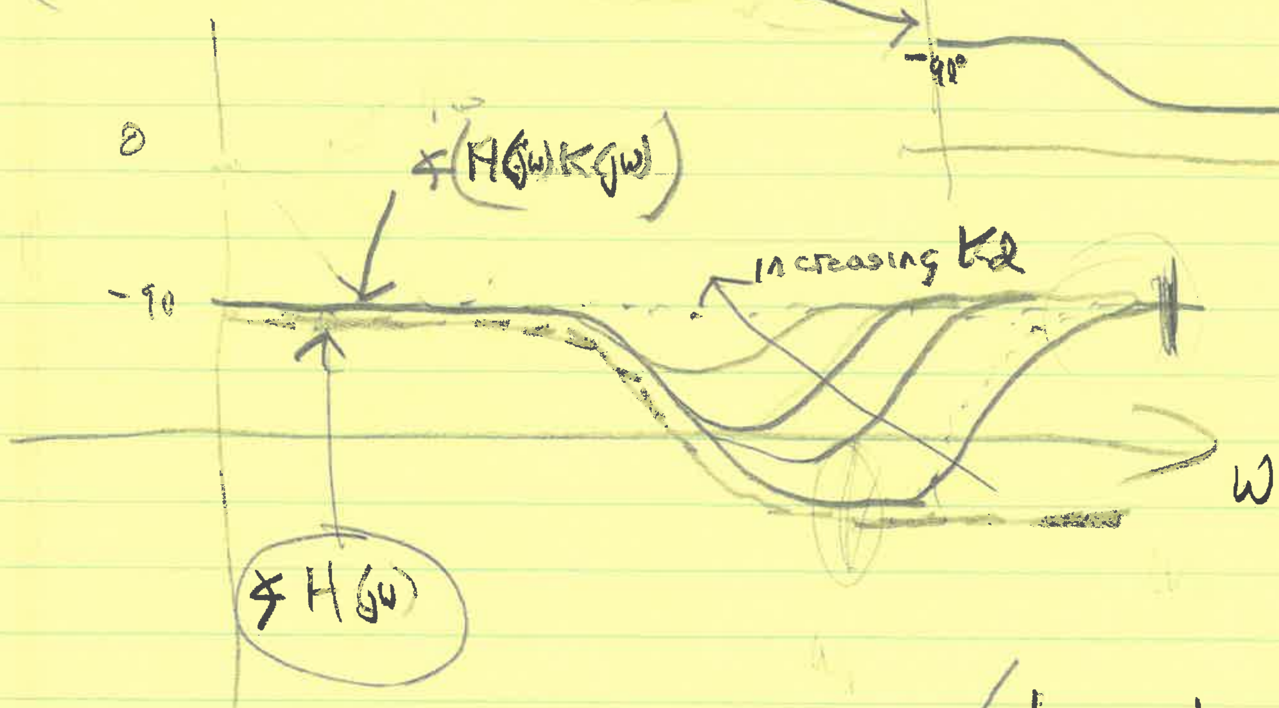
If $|B|$ is very small $\Rightarrow$ K_p will have to be small
(Low Friction) (Low Quality)

What if $K(s) = K_p + K_d s$

Angle Plot

$$\measuredangle(K(s)H(s)) = \measuredangle K(j\omega) + \measuredangle H(j\omega)$$





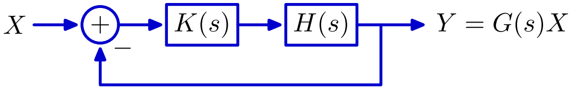
6.3100: Dynamic System Modeling and Control Design

Gain Margins, Phase Margins, and Lead Compensation

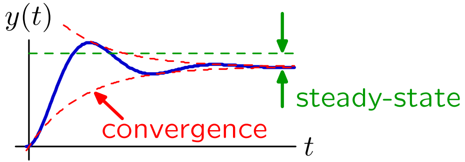
March 20, 2023

Controller Design: Big Picture in Review

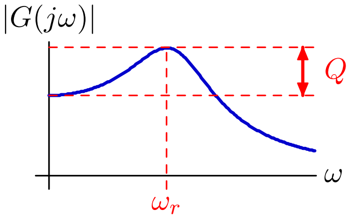
Goal: Given a hardware system $H(s)$ (the plant), design a controller $K(s)$ to achieve some set of performance goals.



The goals may be specified in the time domain

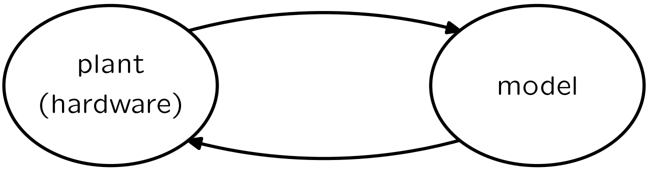


and/or frequency domain.



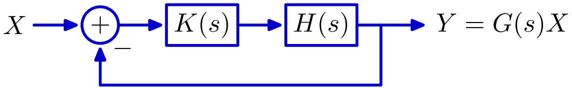
Controller Design: Model-Based Approach

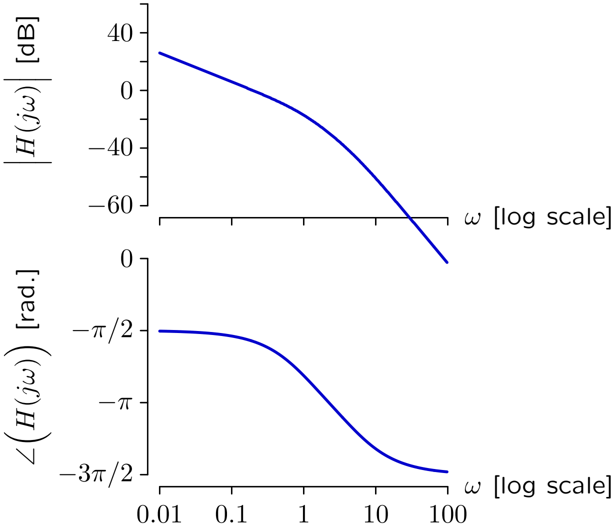
Measure → Model → Optimize → Repeat



Controller Design: Frequency Response Approach

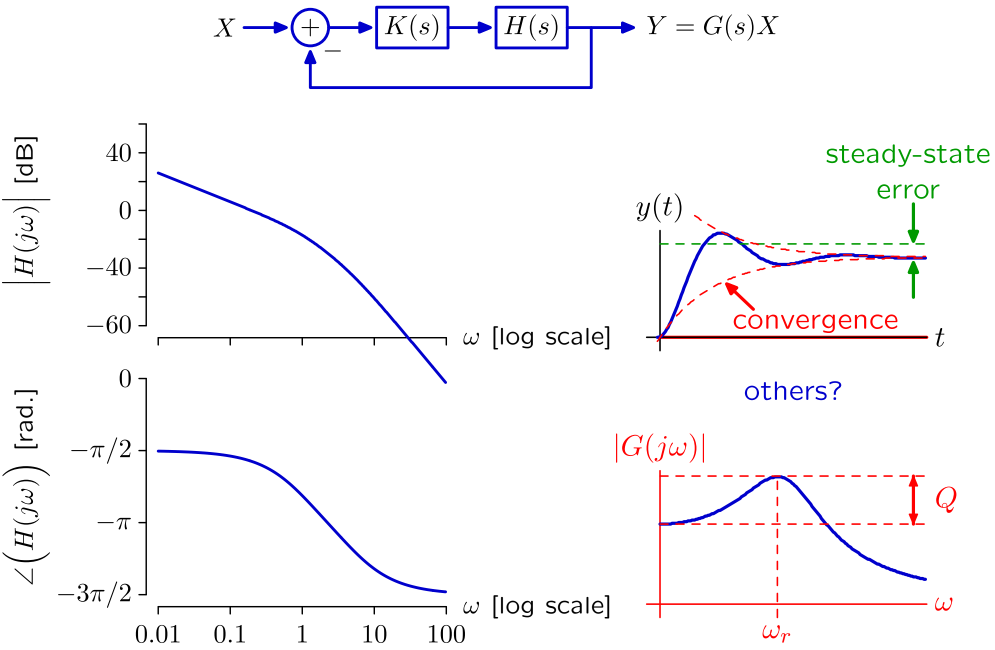
Design a controller based **solely** on the frequency response of the plant.





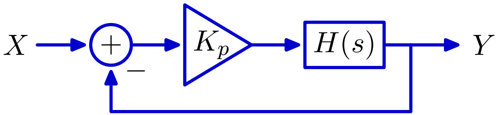
Controller Design: Frequency Response Approach

Is it possible to **characterize performance** using just frequency response?



Controller Design: Frequency Response Approach

Design a controller based **solely** on the frequency response of the plant.



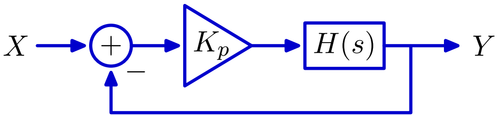
Q: Under what conditions will the closed-loop system be **stable/unstable**?

A: **Stable** if all closed-loop poles are in the left half plane.
Unstable if any closed-loop pole is in the right half plane.
Oscillatory if the right-most pole is on the $j\omega$ axis.

Can we infer stability from the **open-loop** frequency response of the plant?

Controller Design: Frequency Response Approach

Marginal stability occurs when there is a **closed-loop pole** on the $j\omega$ axis.



A pole is a zero of the denominator of the (closed-loop) system function:

$$G(s) = K\frac{(s-z_1)(s-z_2)(s-z_3)\cdots}{(s-p_1)(s-p_2)(s-p_3)\cdots}$$

If there is a pole at $j\omega_0$, then $|G(j\omega_0)| \to \infty$.

From Black's equation,

$$G(j\omega_0) = \frac{K_pH(j\omega_0)}{1+K_pH(j\omega_0)}$$

$|G(j\omega_0)| \to \infty$ if $K_pH(j\omega_0) = -1$:

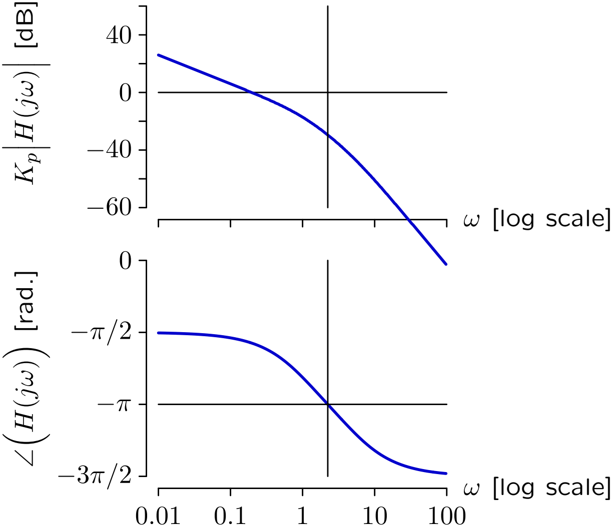
- $\Big|K_pH(j\omega_0)\Big| = 1$ and
- $\angle(K_pH(j\omega_0) = -\pi$ $(\pm k2\pi)$.

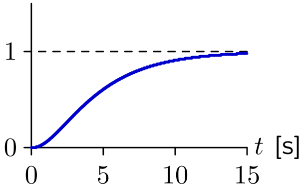
Stability of the closed-loop system can be determined directly from $H(j\omega)$.

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 1$



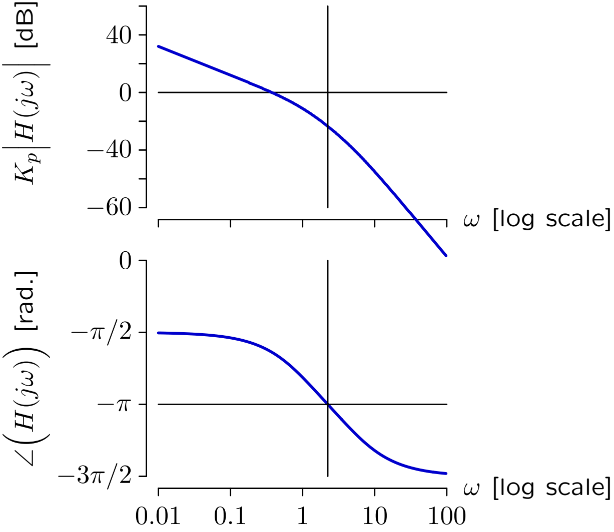


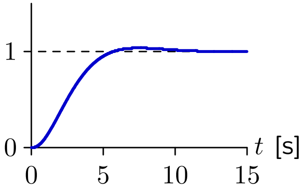
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 2$



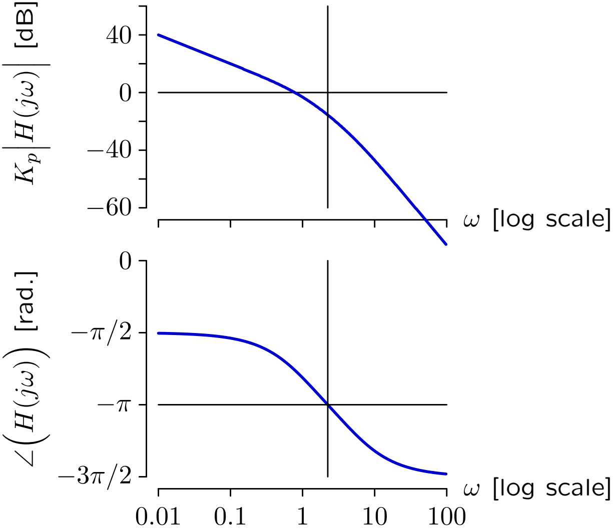


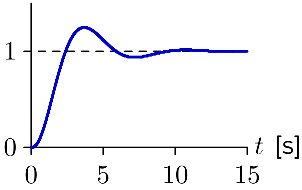
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 5$



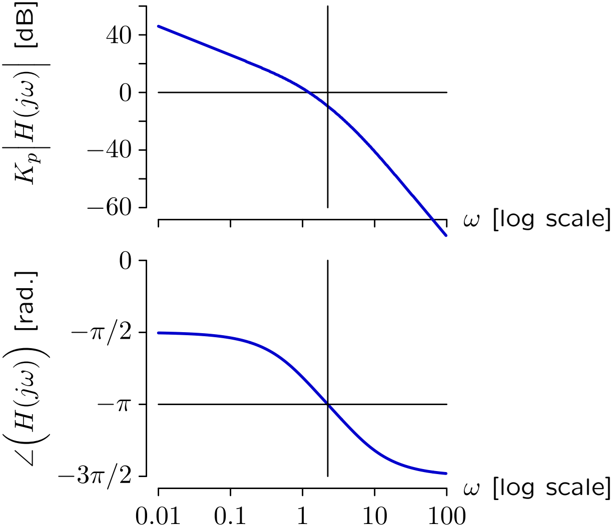


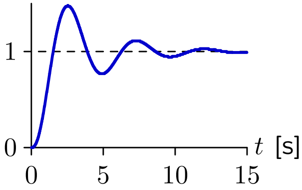
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 10$



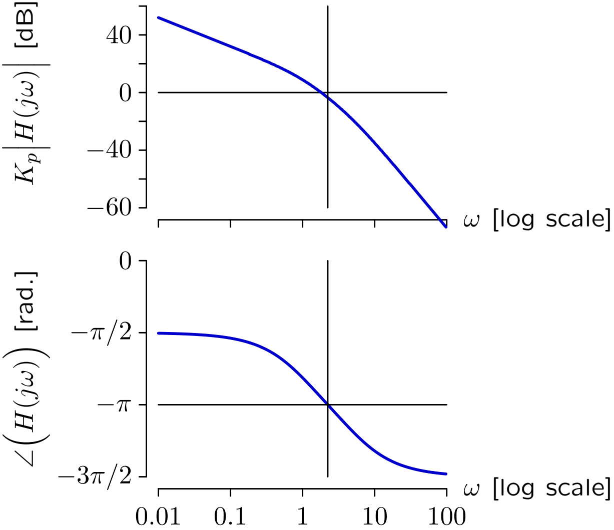


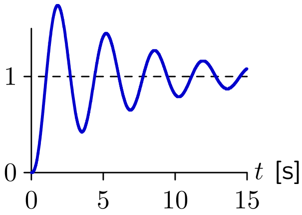
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 20$



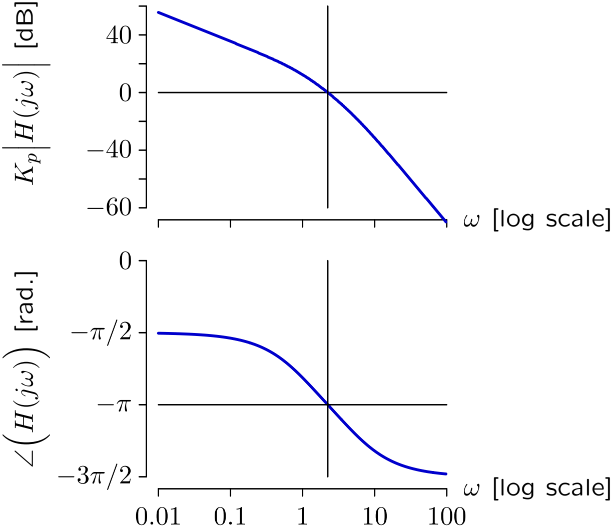


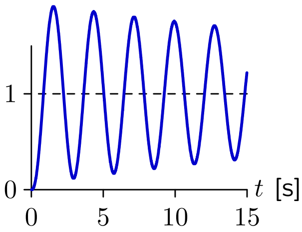
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 30$



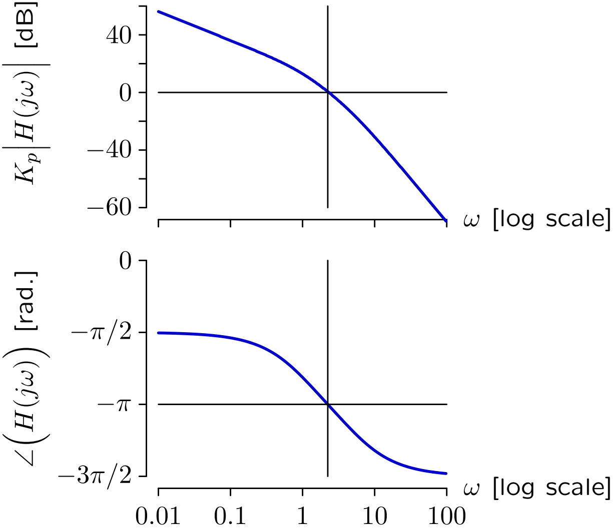


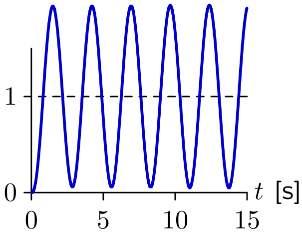
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 32$



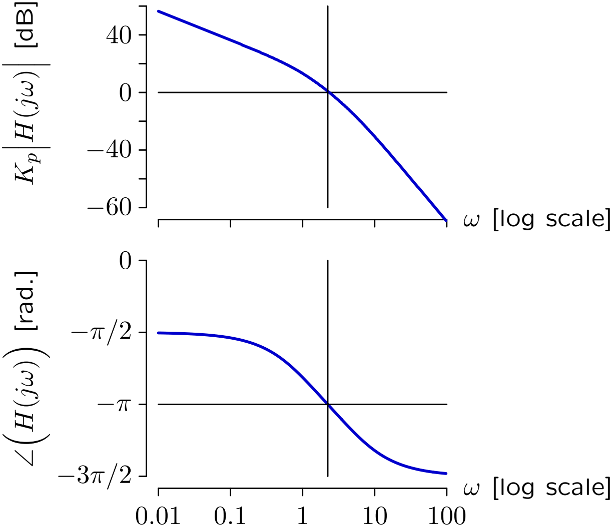


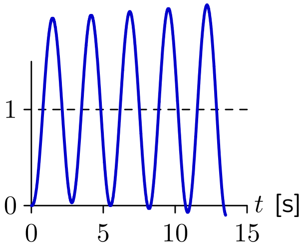
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 33$



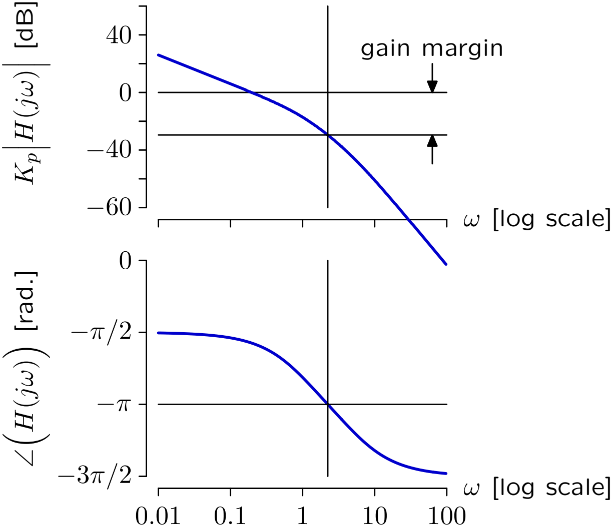


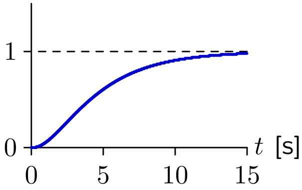
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 1$



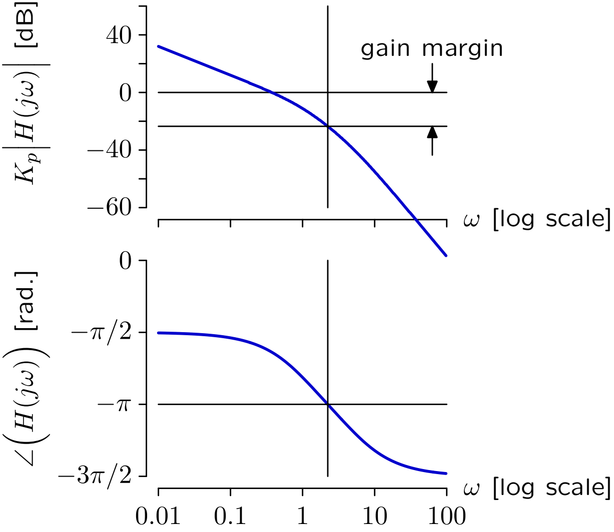


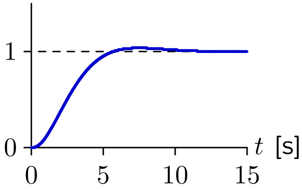
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 2$



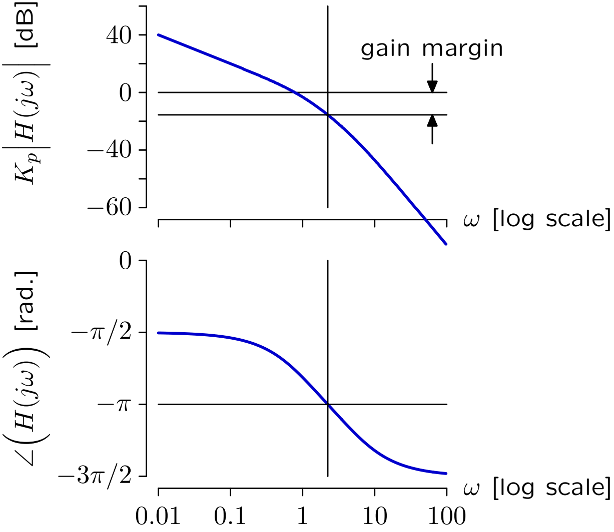


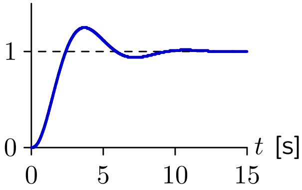
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 5$



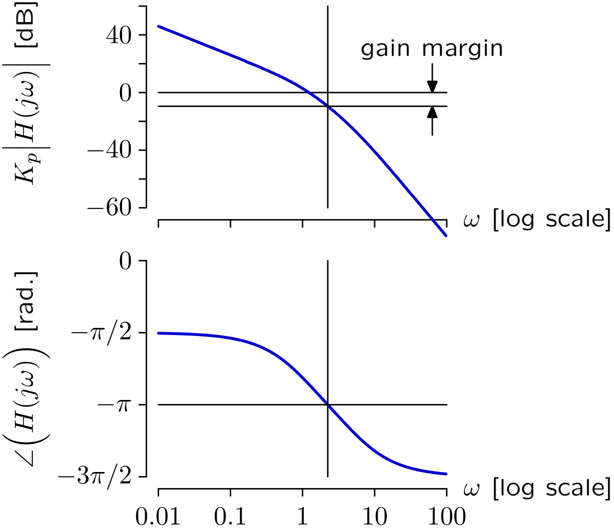


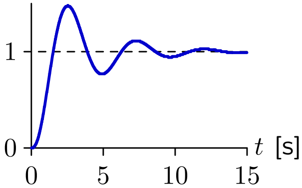
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 10$



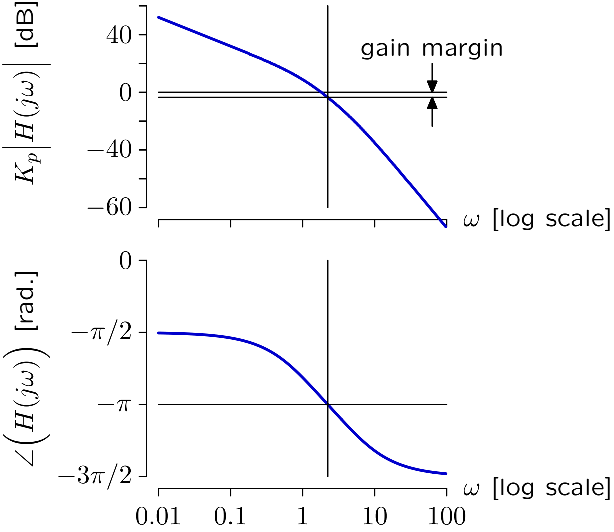


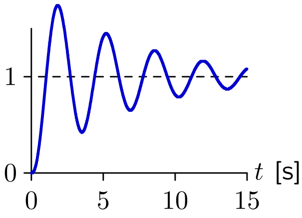
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 20$



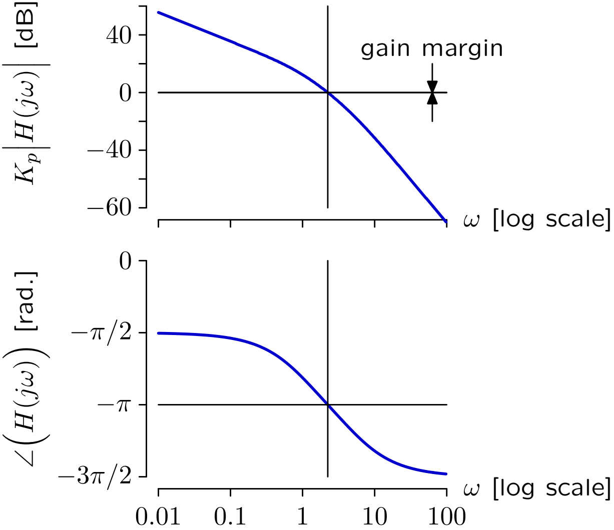


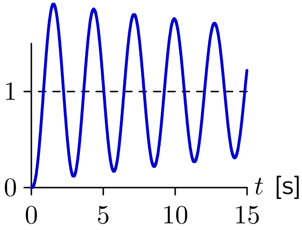
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 30$



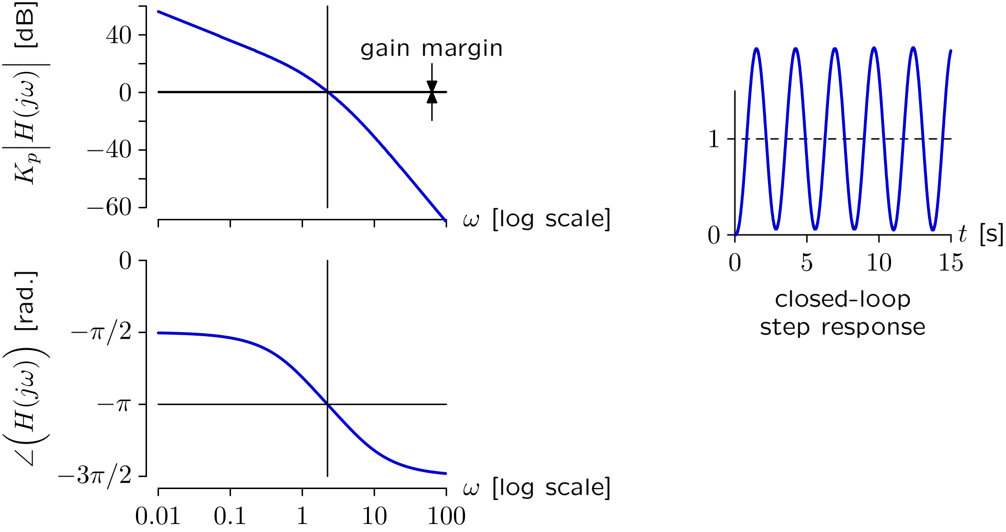


closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 32$

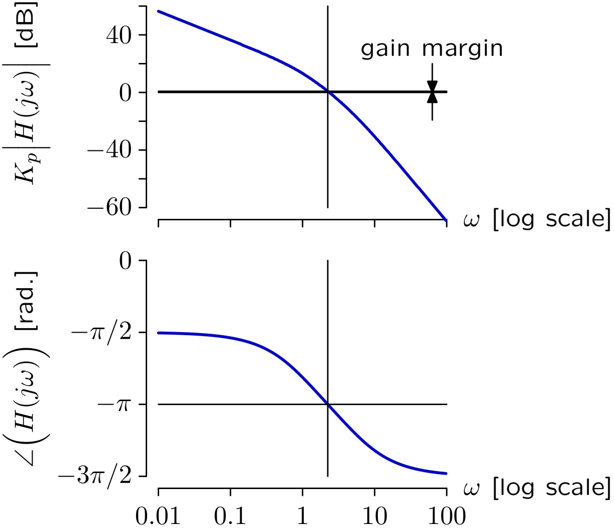


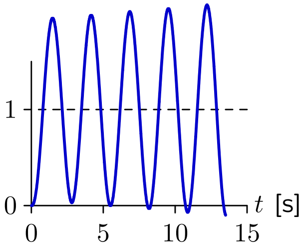
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 33$



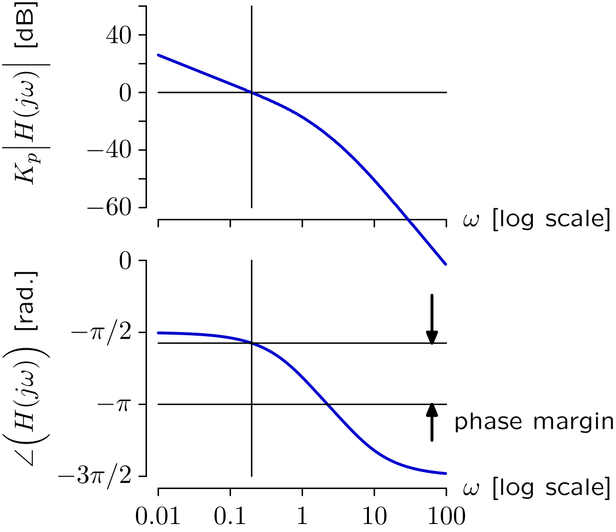


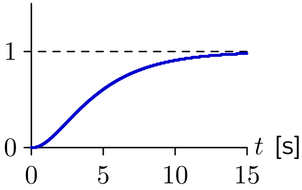
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 1$



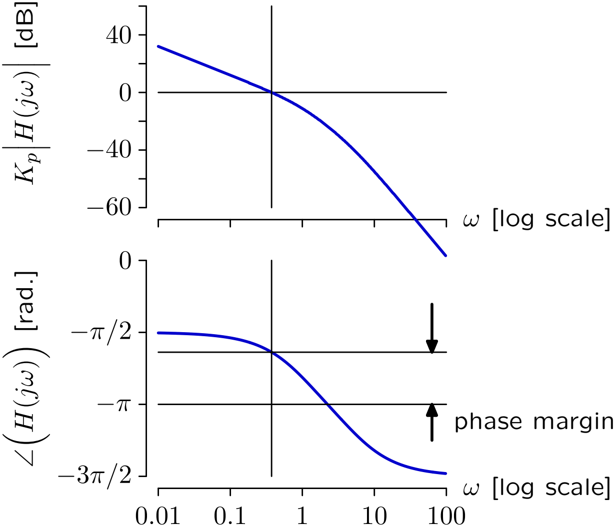


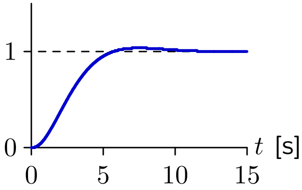
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 2$



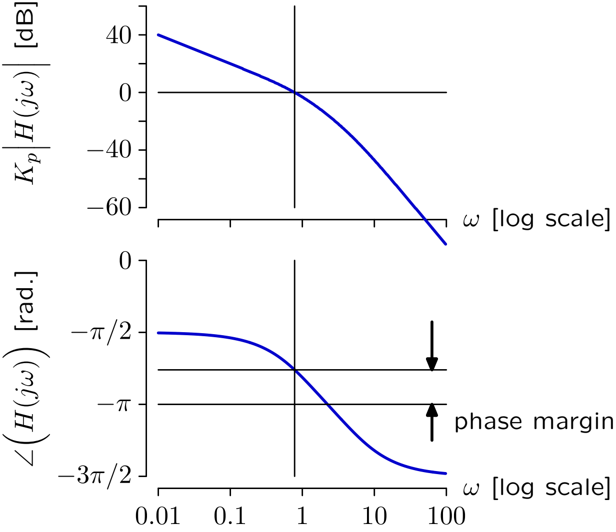


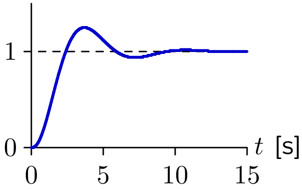
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 5$



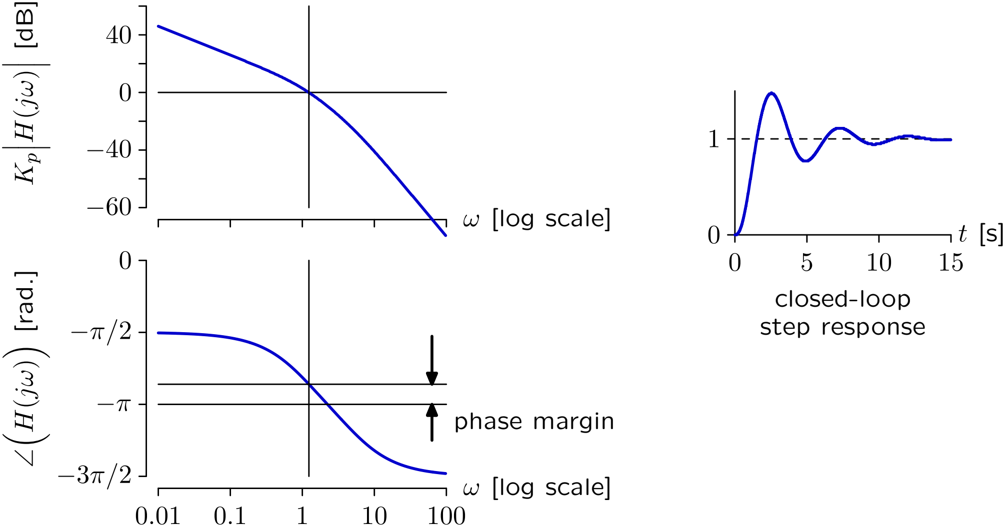


closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 10$

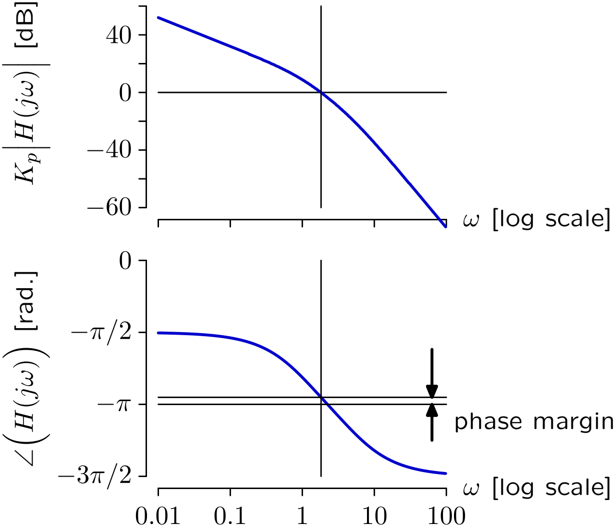


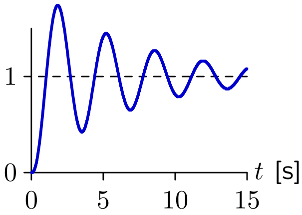
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 20$



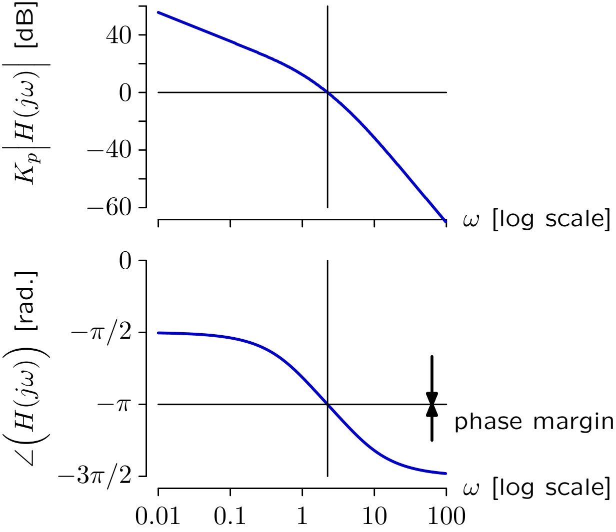


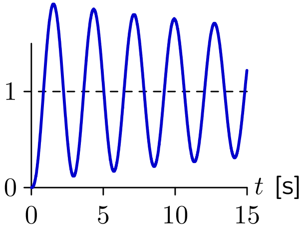
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 30$



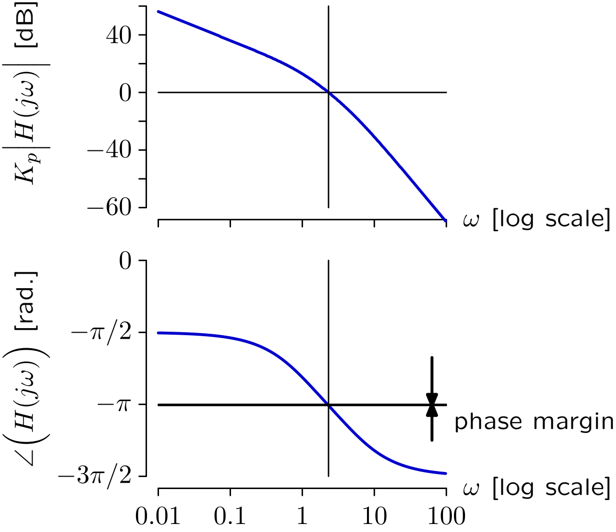


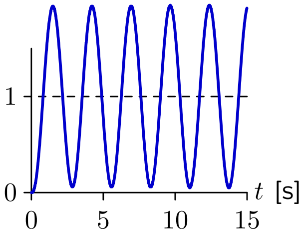
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 32$



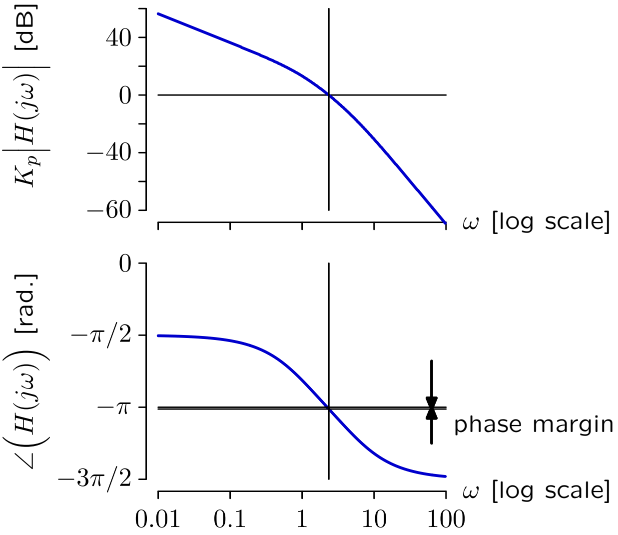


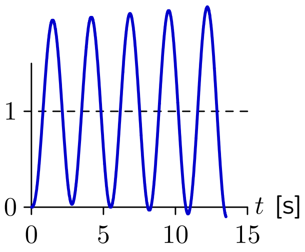
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 33$



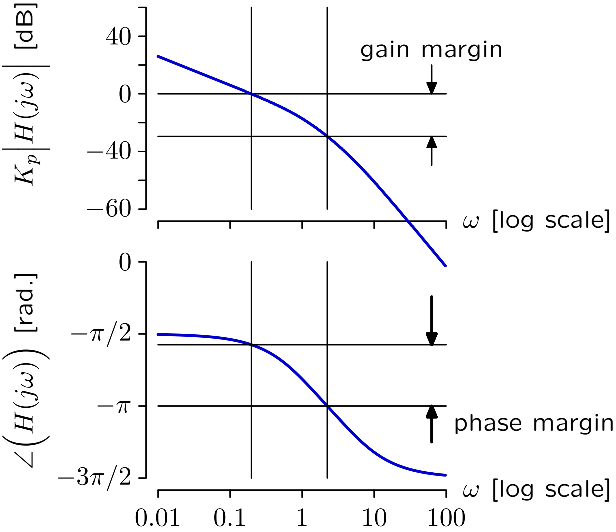


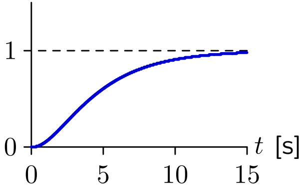
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 1$



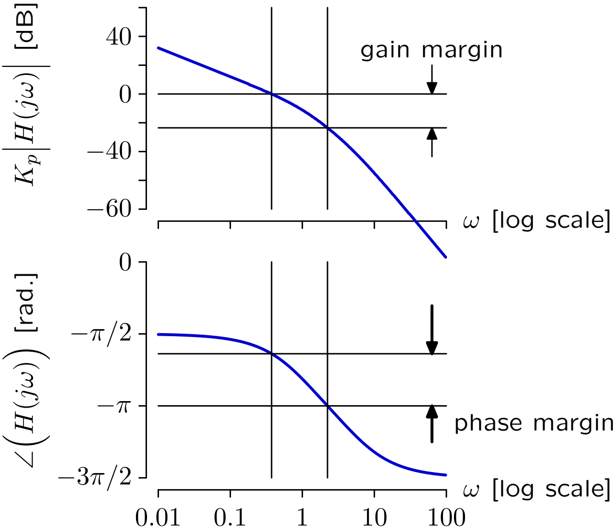


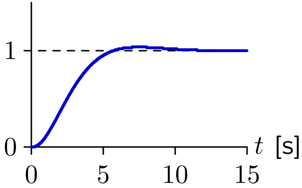
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 2$



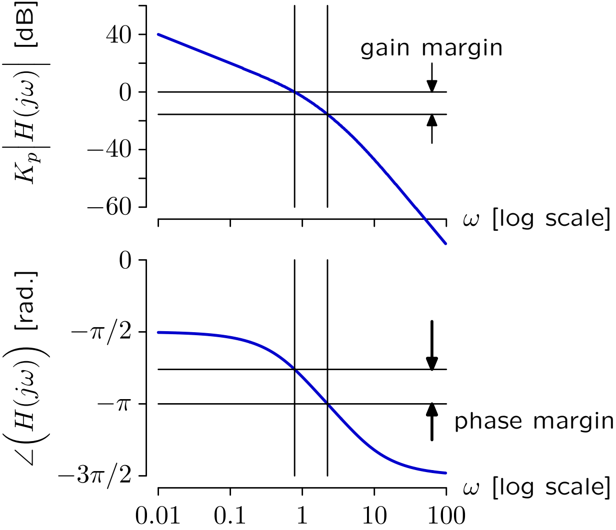


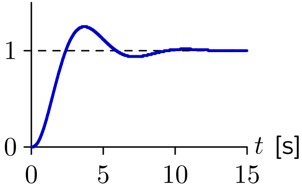
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 5$



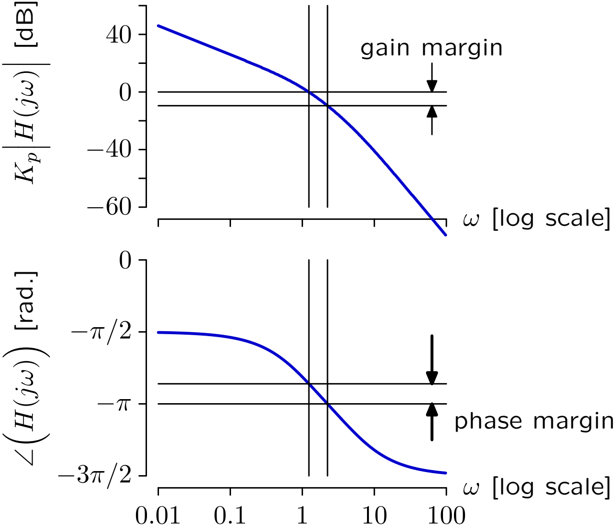


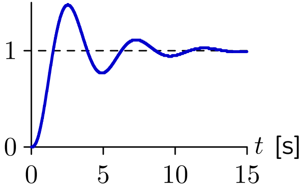
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 10$



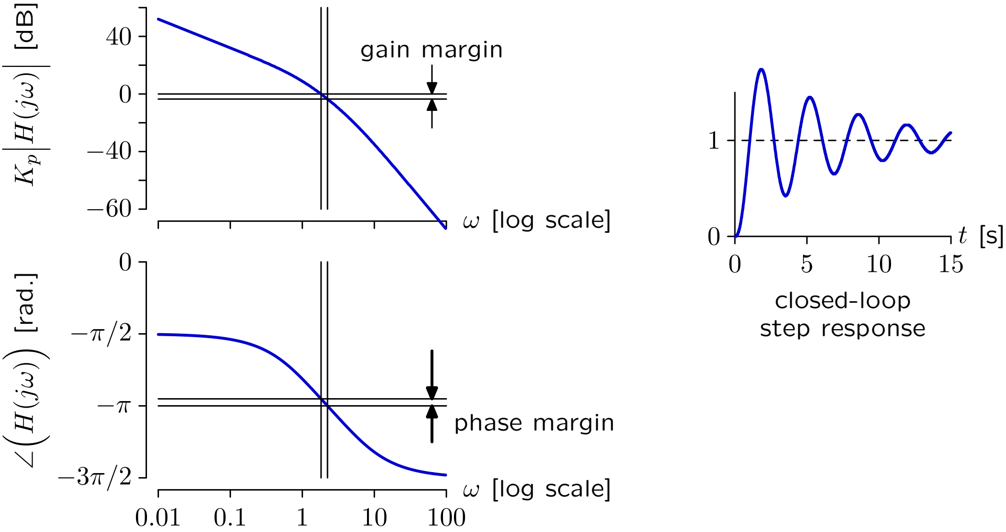


closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

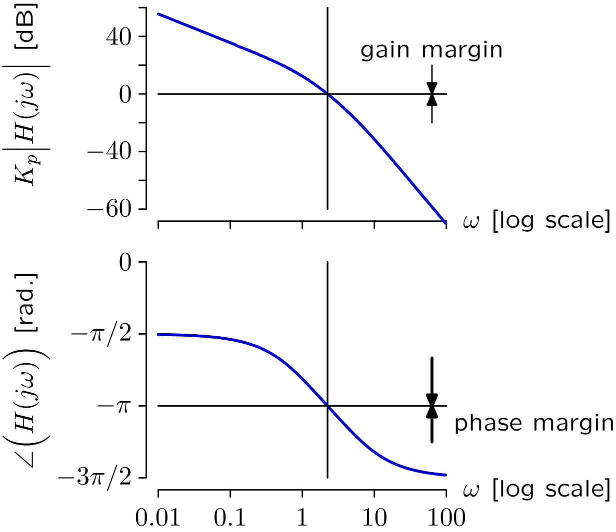
$K_p = 20$

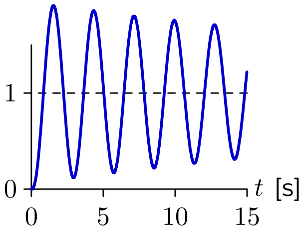


Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 30$



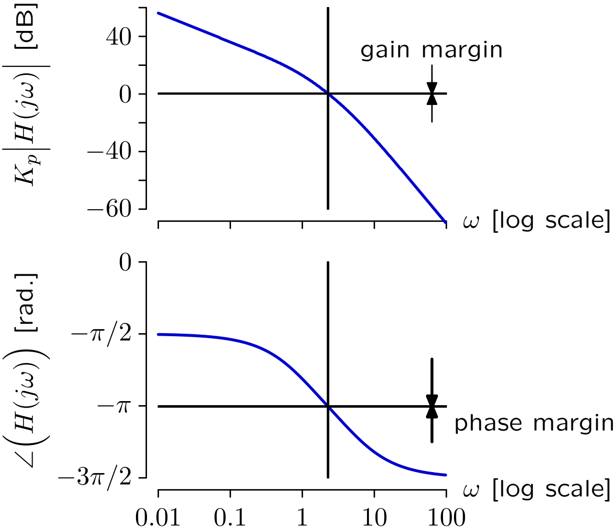


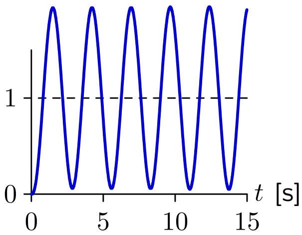
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 32$



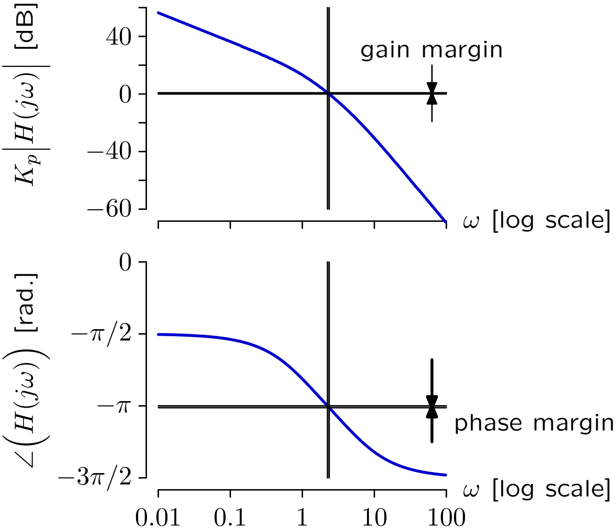


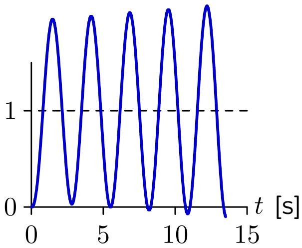
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 33$

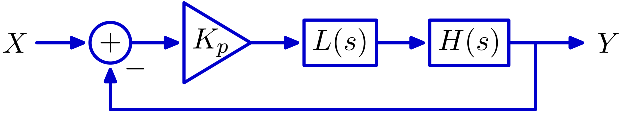




closed-loop
step response

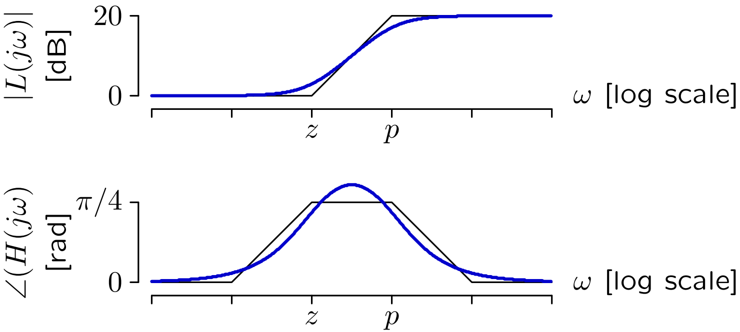
Lead Compensation

Stability can be enhanced by increasing the gain and/or phase margin using a **compensator** as shown below.



We can use a **lead** compensator to increase the phase margin.

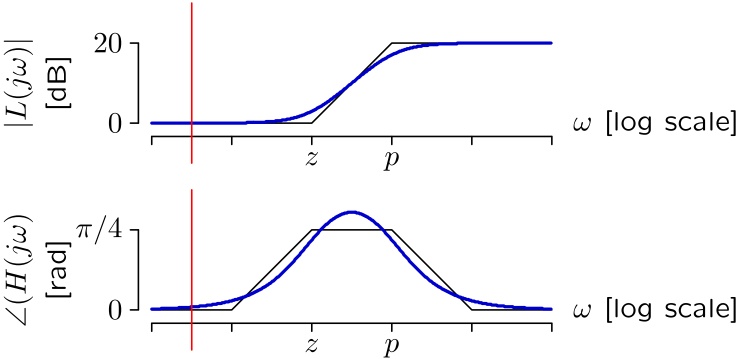
$$L(s) = \left(\frac{p}{z}\right)\left(\frac{s+z}{s+p}\right)$$



Lead Compensation

A lead compensator has no effect on the magnitude or phase at low frequencies.

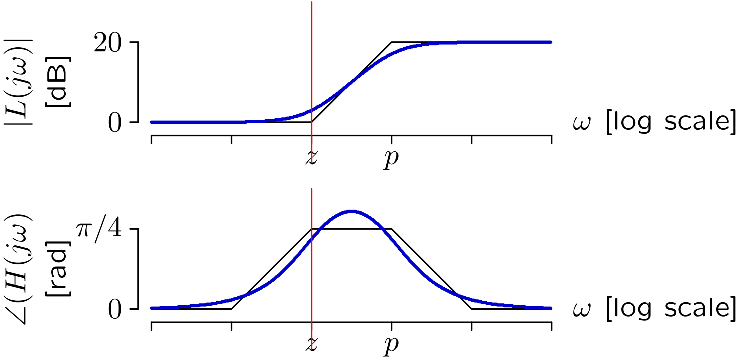
$$L(s) = \left(\frac{p}{z}\right)\left(\frac{s+z}{s+p}\right)$$



Lead Compensation

A lead compensator can significantly increase phase margin (which is good). Unfortunately, it also reduces the gain margin a bit (which is not so good).

$$L(s) = \left(\frac{p}{z}\right)\left(\frac{s+z}{s+p}\right)$$



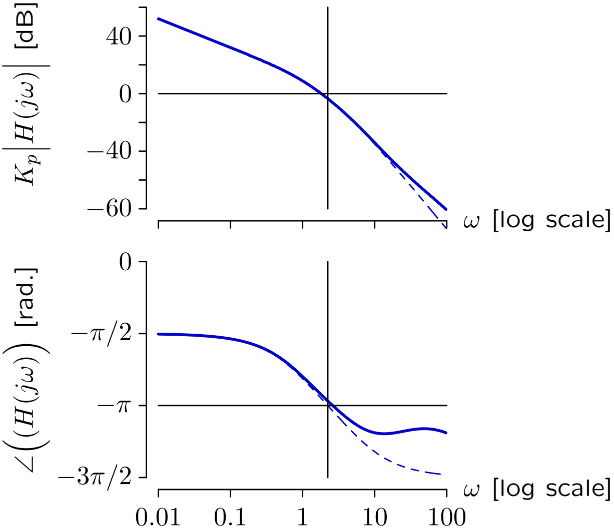
When adjusted appropriately, the increase in phase margin can more than compensate for the slight loss of gain margin.

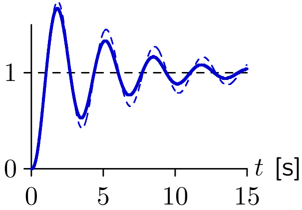
Improving Performance with Lead Compensation

Using a lead compensator with $z = 20$ and $p = 200$ has a very small effect.

$K_p = 20$

$z = 20; \quad p = 200$





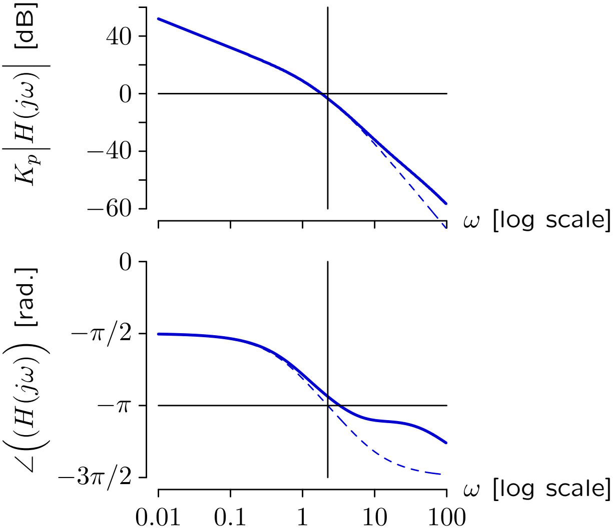
closed-loop
step response

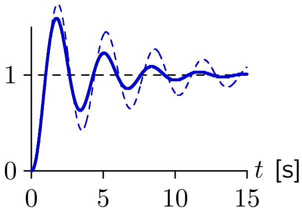
Improving Performance with Lead Compensation

Moving the compensator to a lower frequency increases convergence rate.

$K_p = 20$

$z = 10; \quad p = 100$





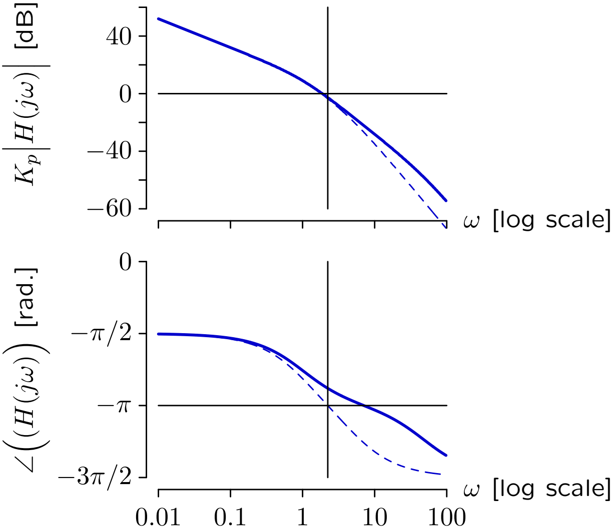
closed-loop
step response

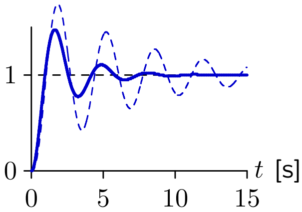
Improving Performance with Lead Compensation

Moving the compensator to a lower frequency increases convergence rate.

$K_p = 20$

$z = 5; \quad p = 50$





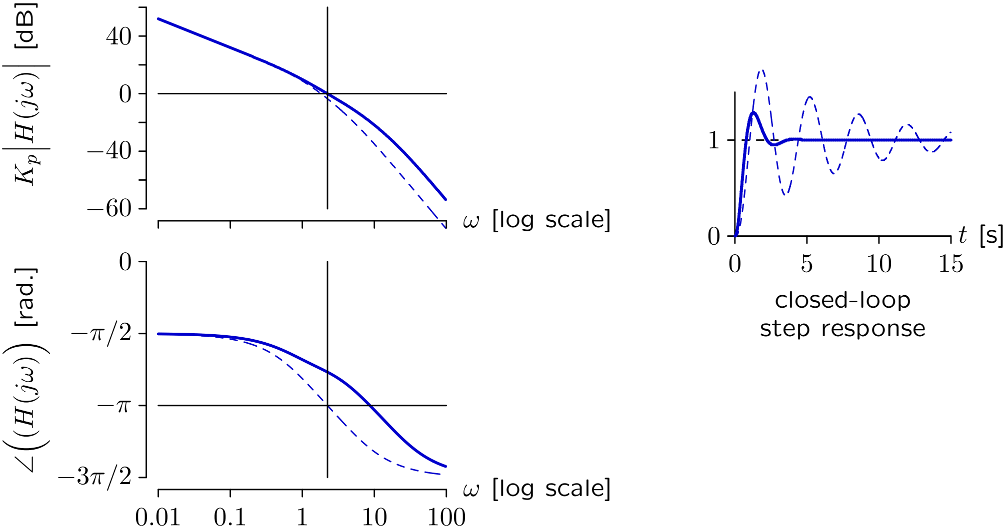
closed-loop
step response

Improving Performance with Lead Compensation

Convergence is dramatically improved when $z = 2$ and $p = 20$.

$K_p = 20$

$z = 2; \quad p = 20$

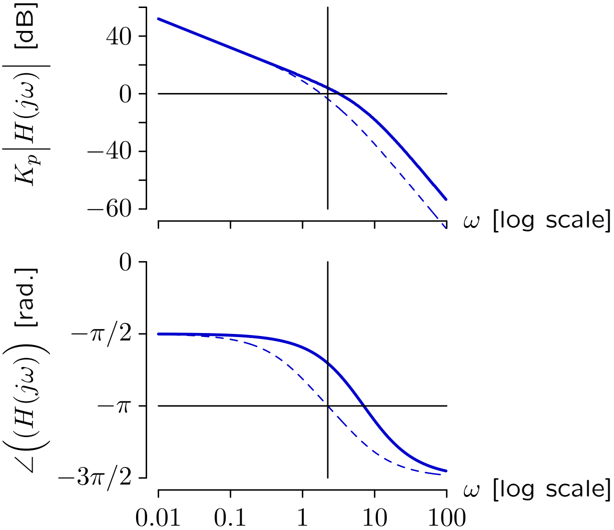


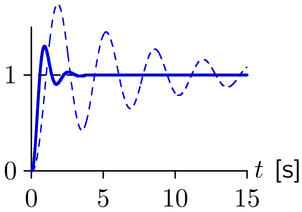
Improving Performance with Lead Compensation

Convergence for $z = 1$ not as good as $z = 2$ – now loosing gain margin.

$K_p = 20$

$z = 1; \quad p = 10$





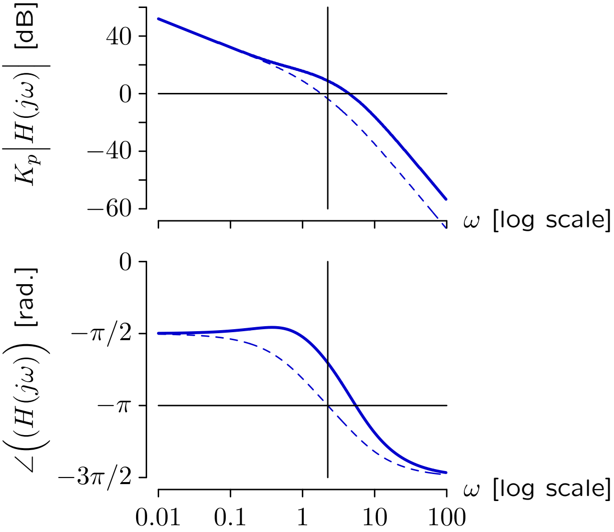
closed-loop
step response

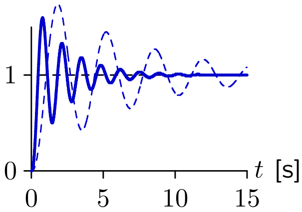
Improving Performance with Lead Compensation

The loss of gain margin is severe when $z = 0.5$.

$K_p = 20$

$z = 0.5; \quad p = 5$





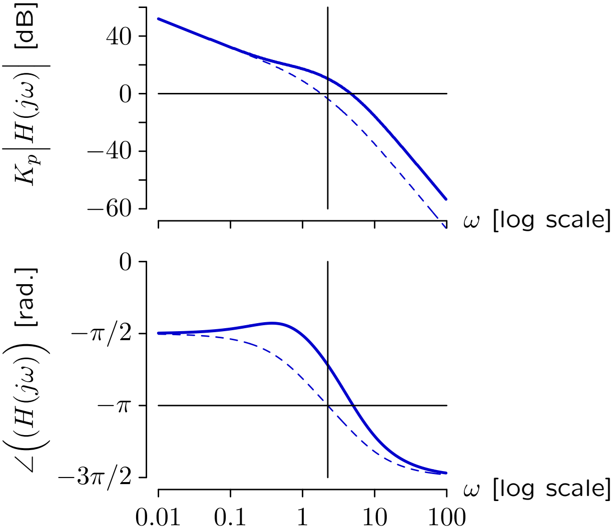
closed-loop
step response

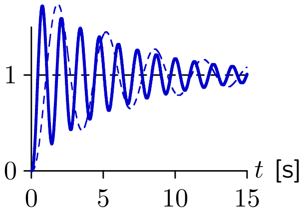
Improving Performance with Lead Compensation

The loss of gain margin is severe when $z = 0.4$.

$K_p = 20$

$z = 0.4; \quad p = 4$





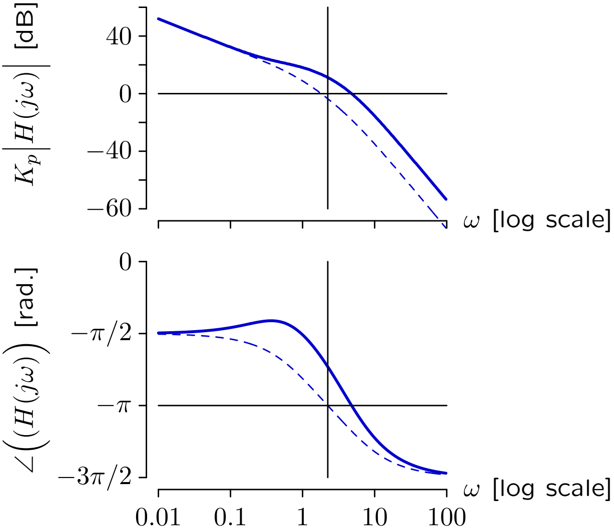
closed-loop
step response

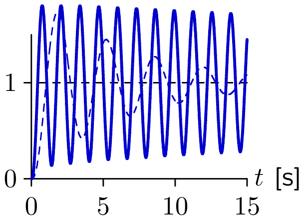
Improving Performance with Lead Compensation

The loss of gain margin is severe when $z = 0.35$.

$K_p = 20$

$z = 0.35; \quad p = 3.5$





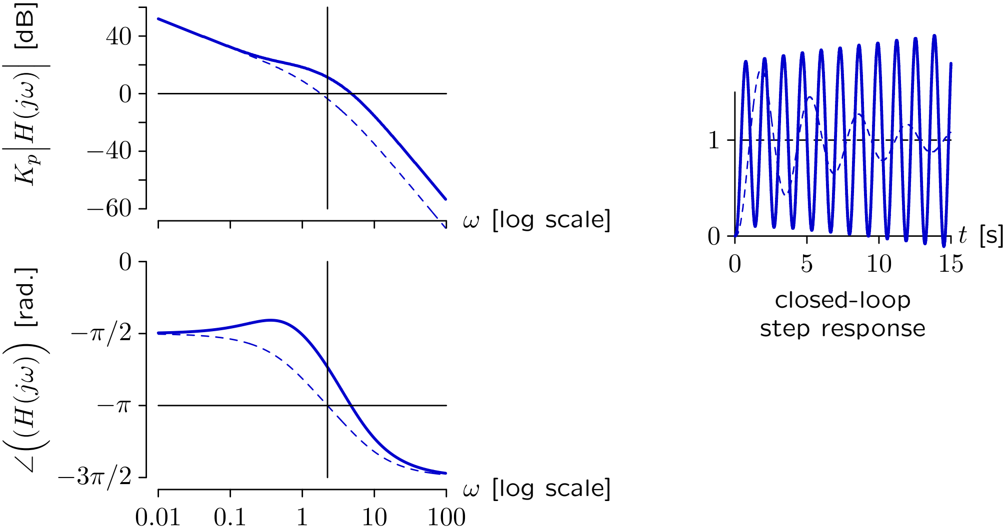
closed-loop step response

Improving Performance with Lead Compensation

The system is unstable when $z = 0.34$.

$K_p = 20$

$z = 0.34; \quad p = 3.4$



Summary

Today we focused on a frequency-response approach to controller design.

Stability criterion: Let ω_0 represent the frequency at which the open-loop phase is $-\pi$. The closed loop system will be stable if the magnitude of the open-loop system at ω_0 is less than 1.

Useful metrics for characterizing relative stability:

- gain margin: ratio of the maximum stable gain to the current gain
- phase margin: additional phase lag needed to make system unstable

Lead compensation can improve performance by increasing phase margin (while also decreasing gain margin slightly).