\[
Y = G(s) \overline{Y_d} = \frac{K(s)H(s)}{1 + K(s)H(s)} \overline{Y_d}
\]

\[
\overline{Y_d} = \frac{N(s)}{D(s)} \overline{P_d} = \frac{b_0 + b_1 s + b_2 s^2 + \ldots}{a_0 + a_1 s + a_2 s^2 + \ldots} \overline{Y_d}
\]

\[
\chi \text{'s} \equiv \text{roots}(s) = \text{nat. freqs.} = \text{poles}
\]

\[
\text{Stability } \Re(c(\chi)) < 0 + i
\]

In SSS, \( s = jw \)

\[
Y_d(t) = P_d e^{jw t}
\]

\[
Y(t) = G(s) \overline{Y_d} = \frac{jw K(jw) H(jw)}{1 + jw K(jw) H(jw)} \overline{Y_d}
\]

\[
Y_2(t) = 0 \Rightarrow Y = G_{dist}(s) \overline{Y_d} = \frac{1}{1 + jw K(jw) H(jw)} \overline{U_d}
\]

For all \( w \) for which \( K(jw) H(jw) \gg 1 \)

\[
G(jw) \approx 1 \quad \text{Good Tracking}
\]

\[
G_{dist}(jw) \approx 0 \quad \text{Good disturb Rej}
\]
If \( K(j\omega)H(j\omega) \) is near -1

Then \( G(j\omega) \to \infty \) \( G_{\text{dist}}(j\omega) \to \infty \)

Both are bad!

Two Conditions for \( K(j\omega)H(j\omega) = -1 \)

Magnitude Condition \( |K(j\omega_{\text{unity}})H(j\omega_{\text{unity}})| = 1 \)

\( \neq \) Condition \( |K(j\omega_{\text{unity}})H(j\omega_{\text{unity}})| = -1 \)

Example \( \frac{K(s)H(s)}{1 + K(s)H(s)} \)

\[
\left| \frac{1}{(s+1)(s+100)} \right| = \left| \frac{K_0}{s+100} \right|
\]

\[
\frac{K_0}{(j\omega+1)(j\omega+100)} \Rightarrow |\frac{K_0}{j\omega+100}| = |\frac{K_0}{-\omega^2 + j\omega + 100}|
\]

If \( K_0 = 9800 \)

\( \omega_{\text{unity}} \approx 300 \)

\[
\left| \frac{98000}{(300)^2} \right| = 1
\]
Add a zero

\[ K(j\omega) = \frac{K_0}{300} \left( j\omega - \frac{3}{2} \right) \]

\[ \frac{3}{2} \approx -300 \]

\[ \frac{K_0}{300} \left( j\omega + 300 \right) \]

\[ \frac{K_0}{300} \left( \frac{j\omega + 300}{(300\omega + 100)} \right) \]

\[ w = 300 \]

\[ = +45^\circ \]

\[ +90^\circ \quad \Rightarrow -K(j\omega) \cdot H(j\omega) \]

\[ \phi \]

\[ -120^\circ \]

only need phase at \( w \) unity

\[ w = 0.1 \quad w = 1 \quad w = 10 \quad w = 100 \quad w = 1000 \]

\[ |K(j\omega)|, |K_0 H(j\omega)| \]

\[ K_0 \]

\[ \frac{K_0}{100} \]

\[ 1 \]
Lead

Set a maximum on how much $K$ increases from $w = 0$ to $w = \infty$

$$K(s) = K_0 \frac{S^2}{S^2 + S - 5P}$$

$$K(\omega) = K_0 \frac{\omega}{\omega^2 + S - 5P}$$

$$Sp = 100 S^2$$

$K_{lead}$
6.3100: Dynamic System Modeling and Control Design

Gain Margin, Phase Margin, and Root Locus

March 22, 2023
Design a controller based solely on the frequency response of the plant.

\[ Y = G(s)X \]

\[ |H(j\omega)| \quad [\text{dB}] \]

\[ \angle (H(j\omega)) \quad [\text{rad.}] \]
Characterizing Performance

Important performance characteristics of a feedback control system

\[ X + K(s)H(s)Y = G(s)X \]

may be specified in the time domain

\[ y(t) \]

and/or frequency domain.

These metrics are easy to compute from a model. Other metrics work well for the frequency domain approach.
Characterizing Performance

Stability criteria based on the frequency response of the plant.

A pole is a zero of the denominator of the closed-loop system function:

\[ G(s) = K \frac{(s - z_1)(s - z_2)(s - z_3) \cdots}{(s - p_1)(s - p_2)(s - p_3) \cdots} \]

If there is a pole at \( j\omega_0 \), then \( |G(j\omega_0)| \to \infty \).

From Black’s equation,

\[ G(j\omega_0) = \frac{K_p H(j\omega_0)}{1 + K_p H(j\omega_0)} \]

\[ |G(j\omega_0)| \to \infty \text{ if } K_p H(j\omega_0) = -1 \]

- \( \left| K_p H(j\omega_0) \right| = 1 \) and
- \( \angle(K_p H(j\omega_0)) = -\pi \ (\pm k2\pi) \).

\} open loop
Controller Design: Frequency Response Approach

Specify performance parameters based on the plant’s frequency response.

\[ X + K(s)H(s) - X Y = G(s)X \]

- Gain margin
- Phase margin

\[ K_p \left| H(j\omega) \right| \] [dB] vs. \( \omega \) [log scale]

\[ \angle H(j\omega) \] [rad.] vs. \( \omega \) [log scale]
Check Yourself

Use the following stability criteria

- $|K_p H(j\omega_0)| = 1$ and
- $\angle(K_p H(j\omega_0)) = -\pi \ (\pm k 2\pi)$

\[ \begin{align*}
K_p H(j\omega_0) &= -1 \\
\end{align*} \]

...to determine values of $K_p$ (if any) for which the following system is marginally stable.

\[ \begin{align*}
X &\rightarrow + \rightarrow K_p \rightarrow \frac{1}{s} \rightarrow \frac{1}{s} \rightarrow Y
\end{align*} \]
Check Yourself

Use the following stability criteria

\[
\begin{align*}
& \left| K_p H(j\omega_0) \right| = 1 \text{ and } \\
& \angle(K_p H(j\omega_0)) = -\pi \left( \pm k2\pi \right) \} \quad K_p H(j\omega_0) = -1
\end{align*}
\]

to determine values of \( K_p \) (if any) for which the following system is marginally stable.

\[ X \xrightarrow{+} K_p \xrightarrow{} 4 \left( \frac{s+1}{s+4} \right) \xrightarrow{} \frac{1}{s} \xrightarrow{} \frac{1}{s} \xrightarrow{} Y \]
Controller Design: Frequency Response Approach

Gain and phase margin characterize stability of the closed-loop system \((G = Y/X)\) in terms of properties of the open-loop system \((H(s))\).

These same stability criteria
\[ K_p H(j\omega_0) = -1 \]
- \[ \left| K_p H(j\omega_0) \right| = 1 \] and
- \[ \angle(K_p H(j\omega_0)) = -\pi (\pm k\pi) \],
underlie other important design tools → root locus.
A **root locus** shows points in the $s$-plane that are poles of the closed loop system function \( G(s) = \frac{Y}{X} \) for values of \( K_p > 0 \).

![Block diagram of a control system with a root locus](image)

Example: Points of the following root locus indicate the closed-loop poles that result for different values of \( K_p \).

A root locus is easy to calculate from a model. Given the poles and zeros of \( H(s) \), we can use Black’s equation to find the poles of \( G(s) = \frac{K_pH(s)}{1+K_pH(s)} \) and then find the roots of the denominator numerically.
A root locus shows points in the $s$-plane that are poles of the closed loop system function ($G(s) = \frac{Y}{X}$) for values of $K_p > 0$.

Example: Points of the following root locus indicate the closed-loop poles that result for different values of $K_p$.

A more intuitive (and often informative) method is to solve the stability criteria from last time using vectors.
Vector Analysis

The **frequency response** of a system composed of adders, gains, and integrators

$$
\cos(\omega_0 t) \xrightarrow{H(j\omega_0)} |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))
$$

can be determined from **vectors** associated with the system’s poles/zeros.

$$
H(j\omega_0) = K \frac{(j\omega_0 - z_0)(j\omega_0 - z_1)(j\omega_0 - z_2) \cdots}{(j\omega_0 - p_0)(j\omega_0 - p_1)(j\omega_0 - p_2) \cdots}
$$

Useful for constructing frequency responses in general and Bode plots in specific. Combine with stability criteria to generate a root locus.

- $|K_p H(j\omega_0)| = 1$ and
- $\angle (K_p H(j\omega_0)) = -\pi \ (\pm k2\pi)$

$$
\begin{align*}
\begin{cases}
|K_p H(j\omega_0)| = 1 \\
\angle (K_p H(j\omega_0)) = -\pi \ (\pm k2\pi)
\end{cases} \Rightarrow K_p H(j\omega_0) = -1
\end{align*}
$$
Root Locus

The shape of the root locus follows from a few simple rules.

\[ G(s) = \frac{K_p H(s)}{1 + K_p H(s)} \]

**Starting Rule:** Each root locus branch starts at an open-loop pole.

For small values of \( K_p \), the denominator of \( G(s) \rightarrow 1 \) and

\[ G(s) \rightarrow K_p H(s) \]

The closed-loop poles of \( G(s) \) are equal to the open-loop poles of \( H(s) \).

The following plot shows the open-loop poles and zeros of a plant \( H(s) \):

The associated root locus has 3 branches, one starting from each pole.
Root Locus

Real-Axis Rule: A point on the real axis is in the root locus if the number of poles to the right of the point plus the number of zeros to the right of the point is odd.

If a system contains just adders, gains, and integrators, then poles (and zeros) with nonzero imaginary parts come in conjugate pairs, and do not contribute to the angle of $H(s)$ if $s$ is on the real axis.

A real-valued pole or zero contributes 0 or $\pi$ to the angle of $H(s_0)$ depending on whether $s_0$ is to the right or left of the pole or zero.
Root Locus

**Real-Axis Rule:** A point on the real axis is in the root locus if \# of poles to the right of the point plus \# of zeros to the right of the point is **odd**.

Examples:
**Root Locus**

**Break-Away Rule:** Increasing $K_p$ after two real-valued closed-loop poles collide causes them to split off the real axis.

The left panel below shows two real-valued, closed-loop poles approaching each other. Notice that their angles sum to $\pi$ prior to collision. The right panel below shows that the angles still sum to $\pi$ after the collision.
Root Locus

High-Gain Rule: If the # of poles exceeds the # of zeros by $N > 0$, there will be $N$ high-gain asymptotes with angles at odd multiples of $\pi/N$.

When $|s|$ is large, vectors from the poles and zeros of $H(s)$ to $s$ will be approximately equal. Since the angle from a pole will be equal to the angle from a zero, the angles from pole/zero pairs will cancel, leaving a net number of excess poles ($N$) whose angles must sum to $\pi$.

\[ \theta = \frac{\pi}{N} \]
**Root Locus**

**High-Gain Rule:** If the \# of poles exceeds the \# of zeros by \( N \), there will be \( N \) high-gain asymptotes with angles at \((2n + 1)\pi/N\).

- **Zero Excess Poles** (no asymptotes)
- **One Excess Pole** (one asymptote)
- **Two Excess Poles** (two asymptotes)
- **Three Excess Poles** (three asymptotes)
Root Locus

Mean Rule: If # of poles is at least two greater than the # of zeros, then the average closed-loop pole position is independent of $K_p$.

Example:

$$H(s) = \frac{s+z}{(s+p_1)(s+p_2)(s+p_3)}$$

$$G(s) = 1 + \frac{K_p(s+z)}{(s+p_1)(s+p_2)(s+p_3)}$$

$$= \frac{s+z}{(s+p_1)(s+p_2)(s+p_3) + K_p(s+z)}$$

$$= \frac{s+z}{s^3 + (p_1+p_2+p_3)s^2 + (p_1p_2+p_1p_3+p_2p_3)s + (p_1p_2p_3) + K_p(s+z)}$$

$$= \frac{s+z}{s^3 + (q_1+q_2+q_3)s^2 + (q_1q_2+q_1q_3+q_2q_3)s + (q_1q_2q_3)}$$

The sum of the closed-loop poles ($q_i$) does not depend on $K_p$. 
Root Locus

Ending Rule: Each root locus branch ends at an open-loop zero or $\infty$.

If $K_p$ is large, then $|H(s)|$ must be small so that the magnitude criterion

$$|K_pH(s)| = 1$$

is satisfied.

$|H(s)|$ will be zero
(a) if $s$ is equal to an open-loop zero, or
(b) if $H(s)$ has more poles than zeros and $s \to \infty$.

Rule 2b follows from the factored representation for $H(s)$:

$$H(s) = K \frac{(s - z_1)(s - z_2)(s - z_3) \cdots (s - z_{n_z})}{(s - p_1)(s - p_2)(s - p_3) \cdots (s - p_{n_p})}$$

If the number of poles ($n_p$) equals the number of zeros ($n_z$), then

$$\lim_{|s| \to \infty} = K$$

But if $n_p > n_z$, as $|s| \to \infty$, $H(s)$ will approach $K/s^{(n_p-n_z)} \to 0$. 
Summary

Today we focused on applications of stability criteria based on gain and phase margins.

Combining these criteria with the vector method we used to evaluate frequency responses provides an intuitive and often informative way to think about root locus plots.