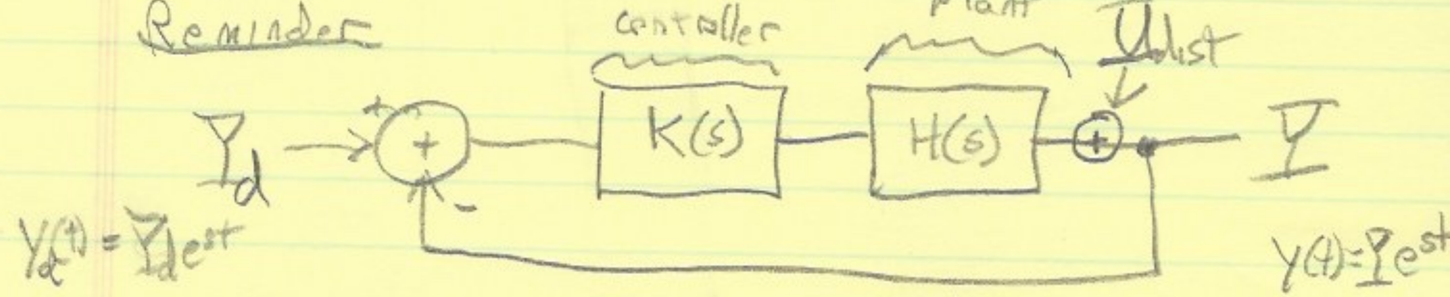
6.3100 10/30/23 1 L

<u>Reminder</u>



$$Y = G(s)\, Y_d = \frac{K(s)H(s)}{1+K(s)H(s)}\, Y_d$$

$$= \frac{n(s)}{d(s)}\, Y_d = \frac{b_0 + b_1 s + b_2 s^2 + \dots}{a_0 + a_1 s + a_2 s^2 \dots}\, Y_d$$

λ_i's ≡ roots($d(s)$) = nat freqs = poles

<u>stability</u> $Re(\lambda_i) < 0 \quad \forall\, i$

<u>In S.S.S.</u> $s = j\omega$

$$y_d(t) = Y_d e^{j\omega t} \qquad y(t) = Y e^{j\omega t}$$

$U_{dist}=0$ { $$Y = G(s)\Big|_{s=j\omega} Y_d = \frac{K(j\omega)H(j\omega)}{1+K(j\omega)H(j\omega)}\, Y_d$$

$Y_d = 0$ { $$Y = G_{dist}(s)\Big|_{s=j\omega} U_d = \frac{1}{1+K(j\omega)H(j\omega)}\, U_d$$

<u>for all ω for which</u> $K(j\omega)H(j\omega) >> 1$

$G(j\omega) \simeq 1$ Good Tracking

$G_{dist}(j\omega) \simeq 0$ Good disturb Rej for some ω_o

If $K(j\omega_o)\,H(j\omega_o)$ is near -1

Then $G(j\omega_o) \to \infty$ $\quad G_{dist}(j\omega_o) \to \infty$

Both are bad!

Two Conditions for $K(j\omega_o)\,H(j\omega_o) = -1$

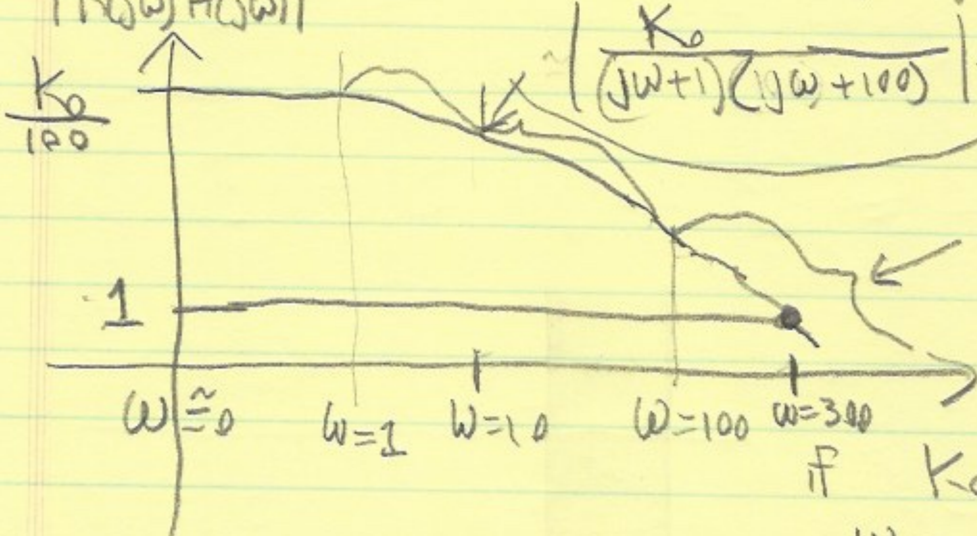
Magnitude Condition $\quad |K(j\omega_{unity})\,H(j\omega_{unity})| = 1$

$\angle$ Condition $\quad \angle K(j\omega_{unity})\,H(j\omega_{unity}) = -180$

or $+180$, -360, $+360$, etc.

Example $\quad K(s)\,H(s)\Big|_{s=j\omega} = K_o \dfrac{1}{(s+1)\,(s+100)^2}\Big|_{s=j\omega}$

$\lambda_{1,2} = -1, -10$



$\simeq \left|\dfrac{K_o}{(j\omega+1)(j\omega+100)}\right| \simeq \left|\dfrac{K_o}{(\omega)(100)}\right|$

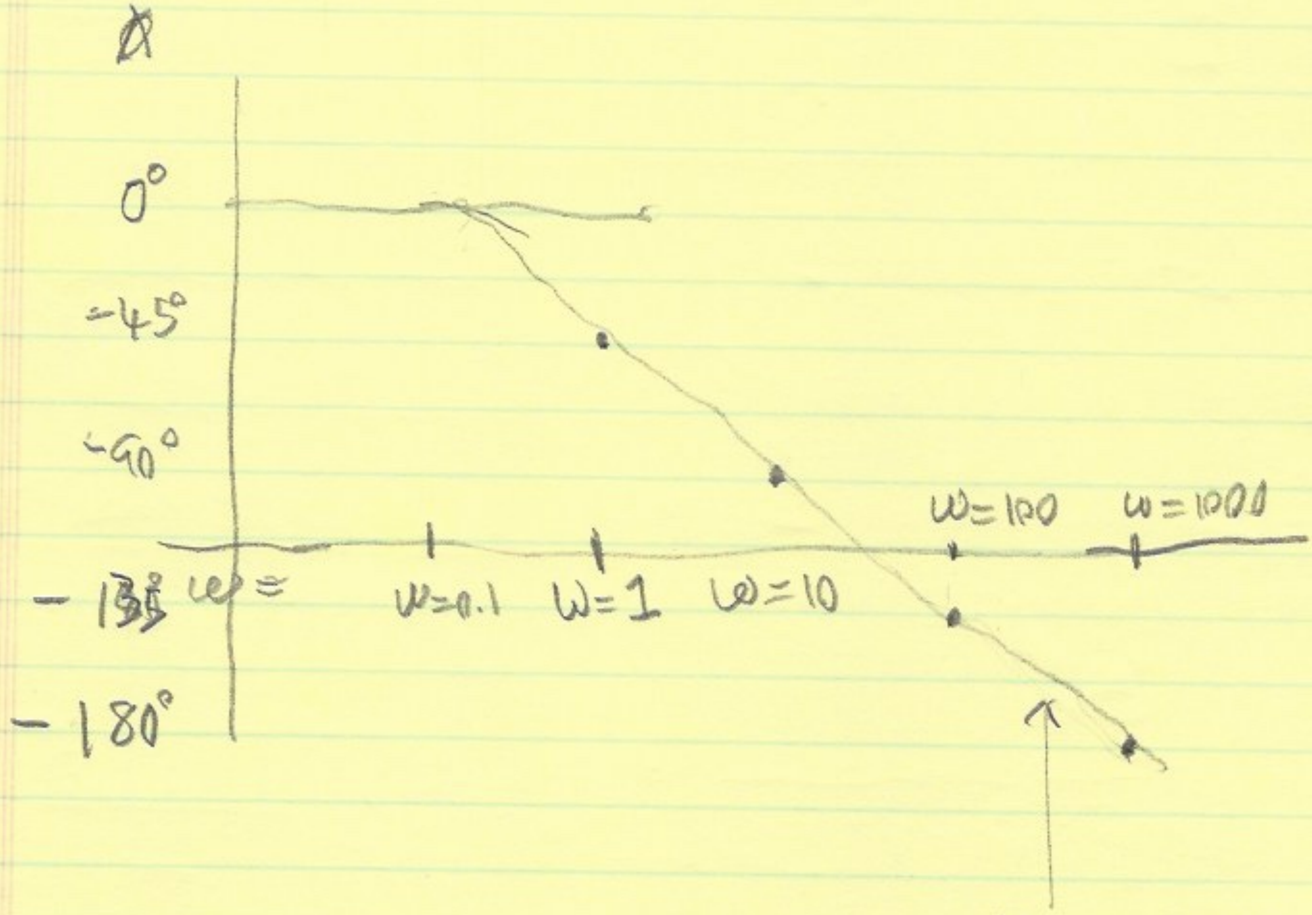
$\simeq \left|\dfrac{K_o}{-\omega^2}\right|$

if $K_o = 90{,}000$

$\omega_{unity} \simeq 300$

$\left|\dfrac{10{,}000}{(300)^2}\right| \simeq 1$

$$\measuredangle \frac{K_0}{(j\omega+1)(j\omega+100)} = -\measuredangle(j\omega+1) + -\measuredangle(j\omega+100)$$



$\omega_{unity} = 300$

$\measuredangle K(j\omega)H(j\omega)\big|_{\omega=\omega_u} \simeq -160°$

Add a zero

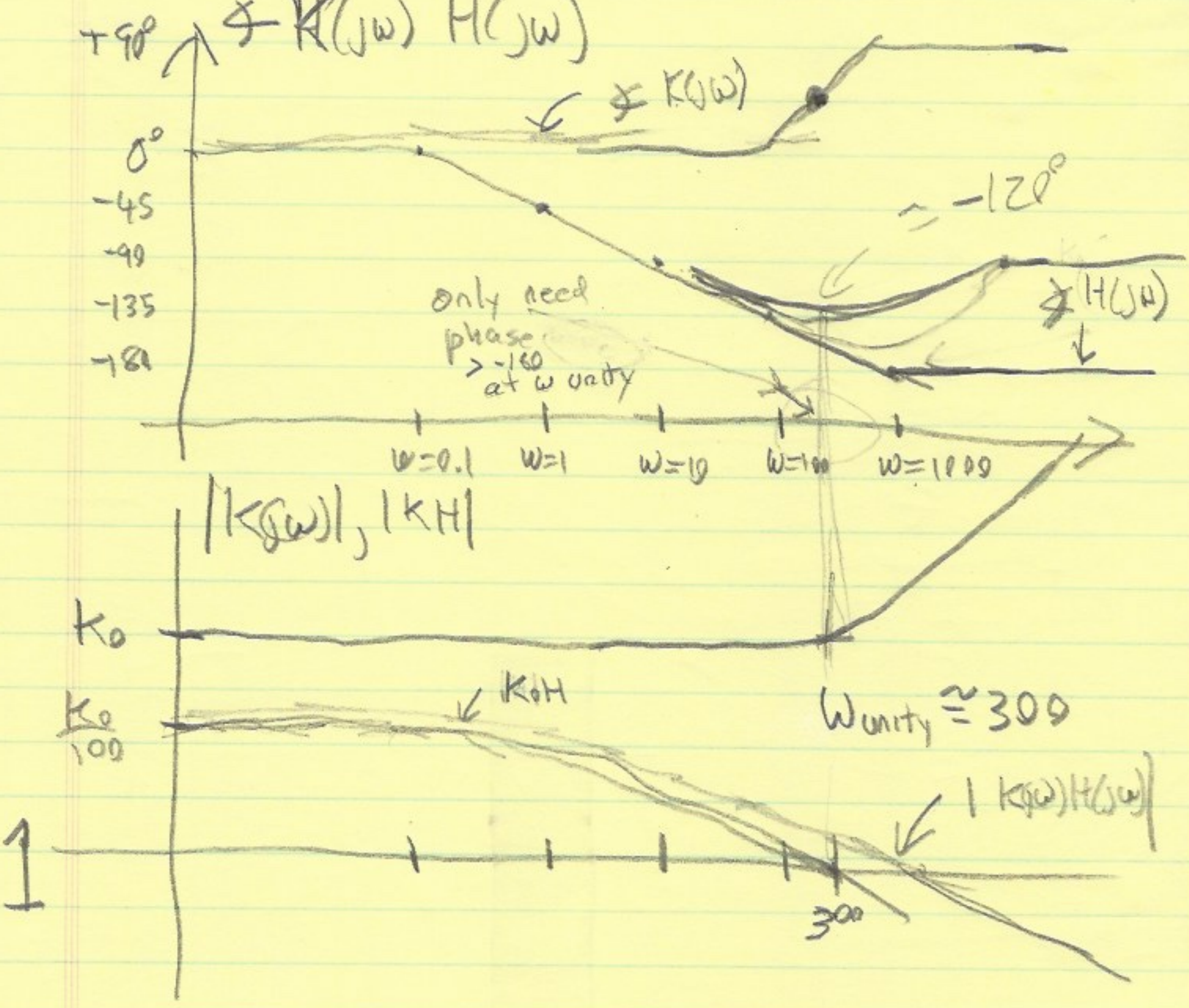
$s_z = ?$

$$K(j\omega) = \frac{K_0}{-s_z}(j\omega - s_z)$$

$s_z = -300$?

$$\angle \frac{K_0}{300}(j\omega + 300)\Big|_{\omega=300} = +45^\circ$$

$$\frac{K_0}{300}\left|\frac{j\omega + 300}{(300j+1)(300j+100)}\right| \quad (\omega = 300)$$



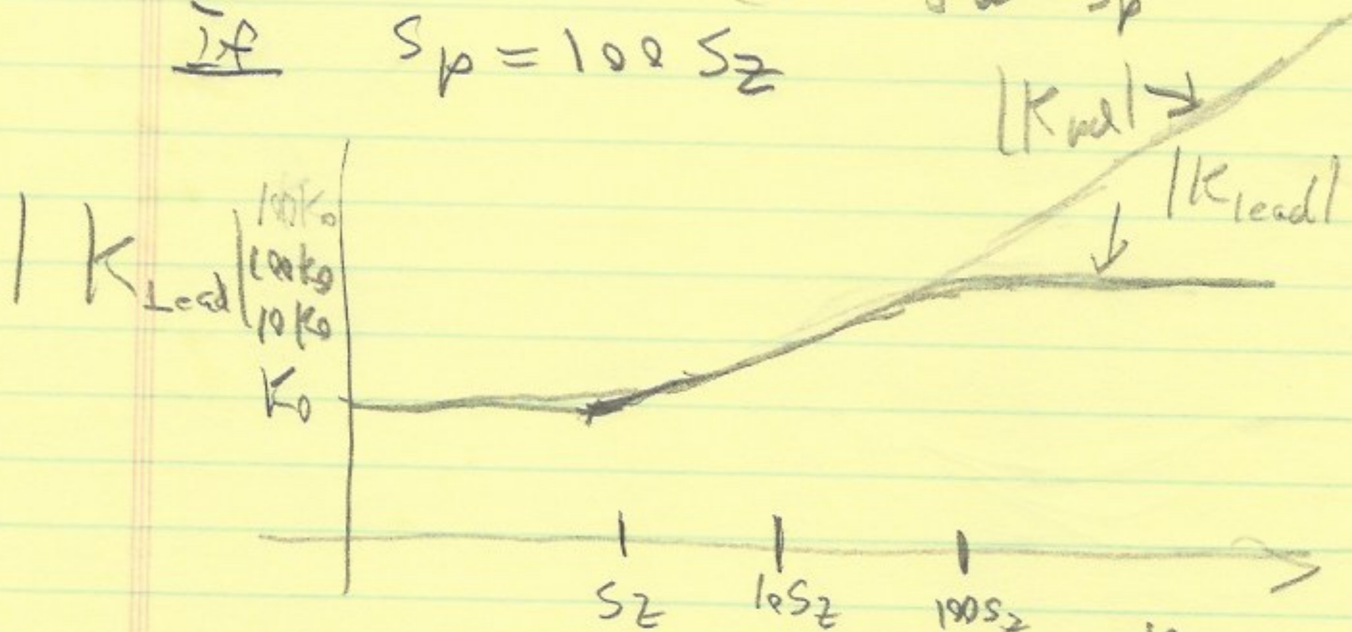
Lead

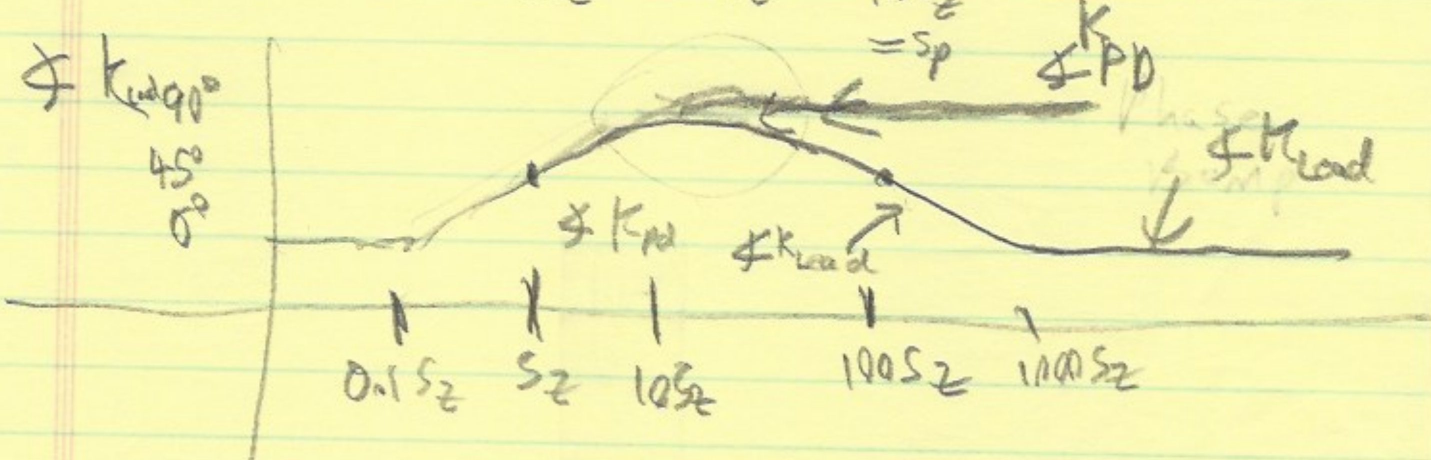
Idea Set a maximum on how much K increases from $\omega = 0$ to $\omega = \infty$

$$K_{Lead}(s) = K_0 \left(\frac{s_p}{s_z}\right)\left(\frac{s - s_z}{s - s_p}\right)$$

$$K_{Lead}(j\omega) = K_0 \left(\frac{s_p}{s_z}\right)\frac{j\omega - s_z}{j\omega - s_p}$$

If $s_p = 100\, s_z$





$$\measuredangle K_{Lead} = \measuredangle(j\omega - s_z) - \measuredangle(j\omega - s_p)$$

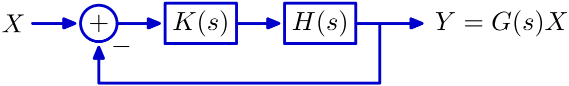
6.3100: Dynamic System Modeling and Control Design

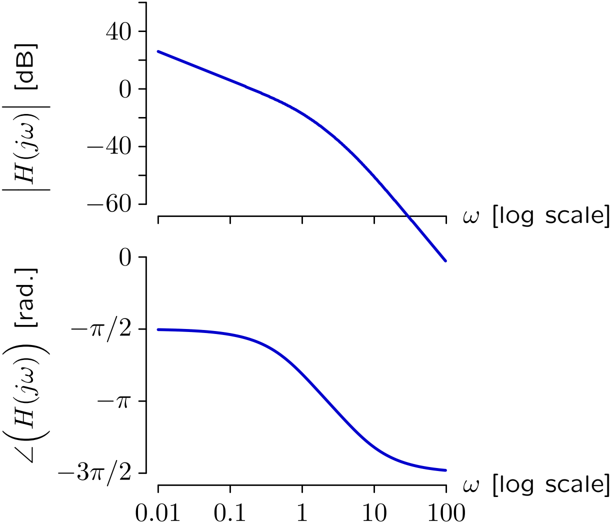
Gain Margin, Phase Margin, and Root Locus

March 22, 2023

Last Time

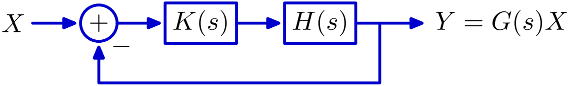
Design a controller based **solely** on the frequency response of the plant.



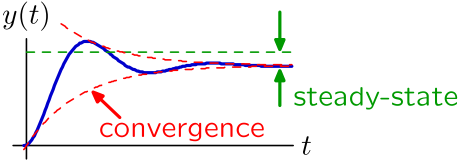


Characterizing Performance

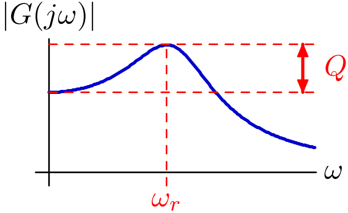
Important performance characteristics of a feedback control system



may be specified in the time domain



and/or frequency domain.

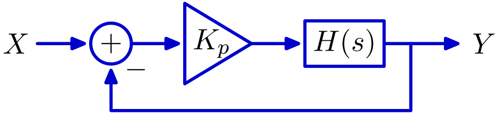


These metrics are easy to compute from a model.

Other metrics work well for the frequency domain approach.

Characterizing Performance

Stability criteria based on the frequency response of the plant.



A pole is a zero of the denominator of the **closed-loop** system function:

$$G(s) = K\frac{(s-z_1)(s-z_2)(s-z_3)\cdots}{(s-p_1)(s-p_2)(s-p_3)\cdots}$$

If there is a pole at $j\omega_0$, then $|G(j\omega_0)| \to \infty$.

From Black's equation,

$$G(j\omega_0) = \frac{K_pH(j\omega_0)}{1+K_pH(j\omega_0)}$$

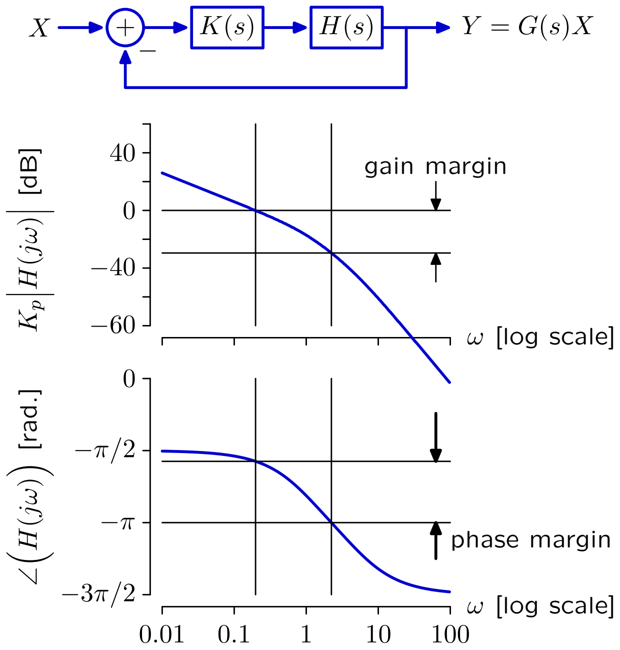
$|G(j\omega_0)| \to \infty$ if $K_pH(j\omega_0) = -1$

- $\Big|K_pH(j\omega_0)\Big| = 1$ and
- $\angle(K_pH(j\omega_0) = -\pi$ $(\pm k2\pi)$.

} **open loop**

Controller Design: Frequency Response Approach

Specify performance parameters based on the plant's frequency response.



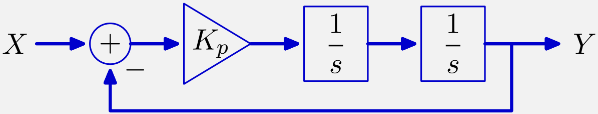
Check Yourself

Use the following stability criteria

- $\left|K_pH(j\omega_0)\right| = 1$ and
- $\angle(K_pH(j\omega_0) = -\pi\ (\pm k2\pi)$

$\Big\}\ K_pH(j\omega_0) = -1$

to determine values of K_p (if any) for which the following system is marginally stable.



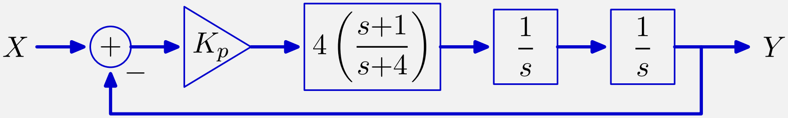
Check Yourself

Use the following stability criteria

- $\left|K_pH(j\omega_0)\right| = 1$ and
- $\angle(K_pH(j\omega_0) = -\pi \ (\pm k2\pi)$

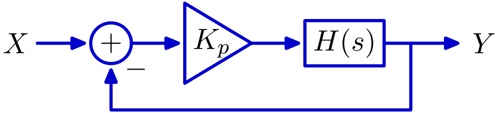
$$\left.\begin{array}{l}\left|K_pH(j\omega_0)\right| = 1 \text{ and} \\ \angle(K_pH(j\omega_0) = -\pi \ (\pm k2\pi)\end{array}\right\} K_pH(j\omega_0) = -1$$

to determine values of K_p (if any) for which the following system is marginally stable.



Controller Design: Frequency Response Approach

Gain and phase margin characterize stability of the closed-loop system ($G = Y/X$) in terms of properties of the open-loop system ($H(s)$).



These same stability criteria

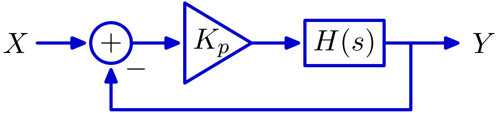
$K_p H(j\omega_0) = -1$

- $\left| K_p H(j\omega_0) \right| = 1$ and
- $\angle(K_p H(j\omega_0) = -\pi \ (\pm k2\pi)$.

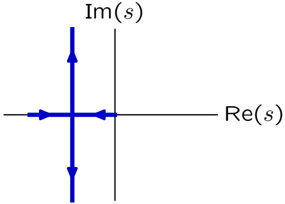
underlie other important design tools $\rightarrow$ root locus.

Root Locus

A **root locus** shows points in the s-plane that are poles of the closed loop system function ($G(s) = \frac{Y}{X}$) for values of $K_p > 0$.



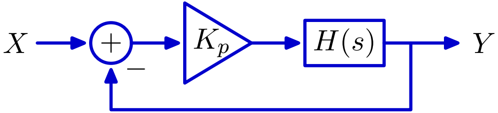
Example: Points of the following root locus indicate the closed-loop poles that result for different values of K_p.



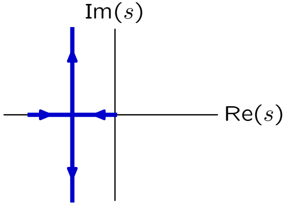
A root locus is easy to calculate from a model. Given the poles and zeros of $H(s)$, we can use Black's equation to find the poles of $G(s) = \frac{K_pH(s)}{1+K_pH(s)}$ and then find the roots of the denominator numerically.

Root Locus

A **root locus** shows points in the s-plane that are poles of the closed loop system function ($G(s) = \frac{Y}{X}$) for values of $K_p > 0$.



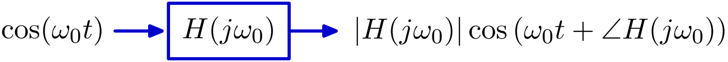
Example: Points of the following root locus indicate the closed-loop poles that result for different values of K_p.



A more intuitive (and often informative) method is to solve the stability criteria from last time using vectors.

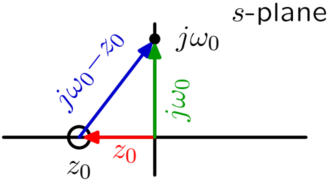
Vector Analysis

The **frequency response** of a system composed of adders, gains, and integrators



can be determined from **vectors** associated with the system's poles/zeros.

$$H(j\omega_0) = K \frac{(j\omega_0 - z_0)(j\omega_0 - z_1)(j\omega_0 - z_2)\cdots}{(j\omega_0 - p_0)(j\omega_0 - p_1)(j\omega_0 - p_2)\cdots}$$



Useful for constructing frequency responses in general and Bode plots in specific. Combine with stability criteria to generate a root locus.

- $\left|K_pH(j\omega_0)\right| = 1$ and
- $\angle(K_pH(j\omega_0) = -\pi\ (\pm k2\pi)$

$\Big\}\ K_pH(j\omega_0) = -1$

Root Locus

The shape of the root locus follows from a few simple rules.

$$G(s) = \frac{K_p H(s)}{1 + K_p H(s)}$$

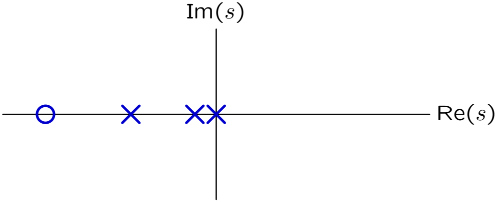
Starting Rule: Each root locus branch starts at an **open-loop pole**.

For small values of K_p, the denominator of $G(s) \to 1$ and

$$G(s) \to K_p H(s)$$

The closed-loop poles of $G(s)$ are equal to the open-loop poles of $H(s)$.

The following plot shows the open-loop poles and zeros of a plant $H(s)$:

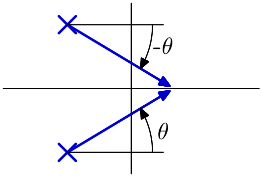


The associated root locus has 3 branches, one starting from each pole.

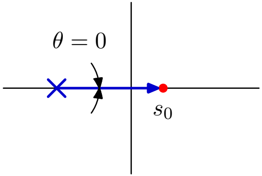
Root Locus

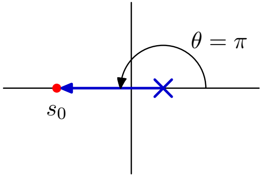
Real-Axis Rule: A point on the real axis is in the root locus if # of poles to the right of the point plus # of zeros to the right of the point is **odd**.

If a system contains just adders, gains, and integrators, then poles (and zeros) with nonzero imaginary parts come in conjugate pairs, and do not contribute to the angle of $H(s)$ if s is on the real axis.



A real-valued pole or zero contributes 0 or π to the angle of $H(s_0)$ depending on whether s_0 is to the right or left of the pole or zero.

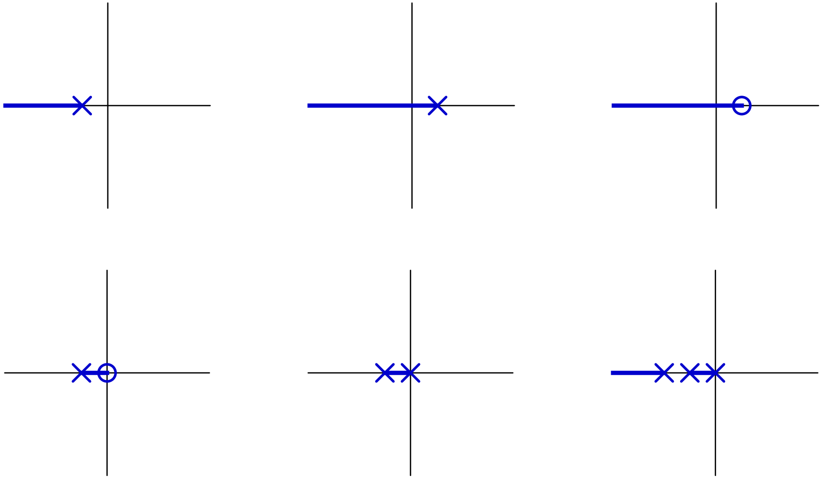




Root Locus

Real-Axis Rule: A point on the real axis is in the root locus if # of poles to the right of the point plus # of zeros to the right of the point is **odd**.

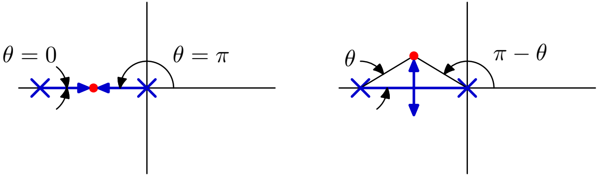
Examples:



Root Locus

Break-Away Rule: Increasing K_p after two real-valued closed-loop poles collide causes them to split off the real axis.

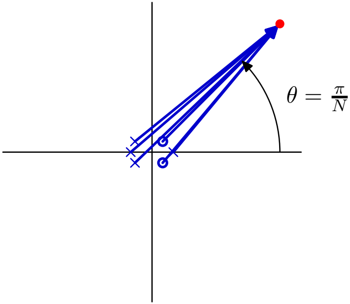
The left panel below shows two real-valued, closed-loop poles approaching each other. Notice that their angles sum to π prior to collision.
The right panel below shows that the angles still sum to π after the collision.



Root Locus

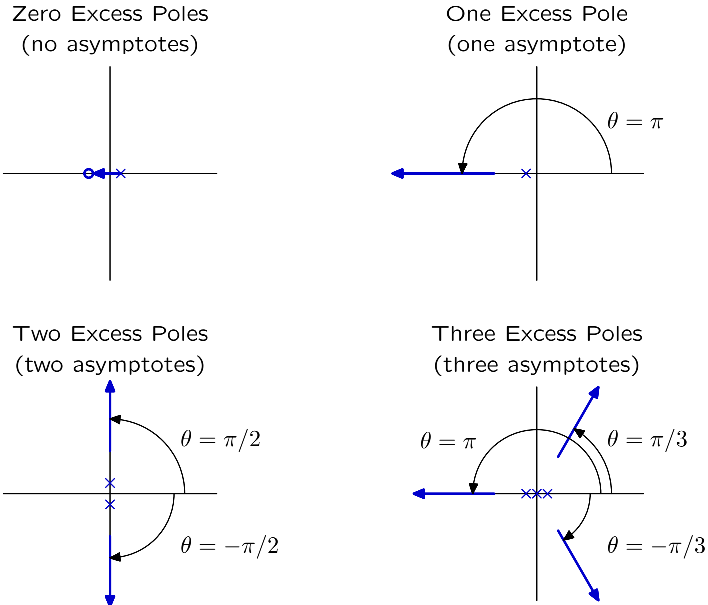
High-Gain Rule: If the # of poles exceeds the # of zeros by $N > 0$, there will be N high-gain asymptotes with angles at odd multiples of π/N.

When $|s|$ is large, vectors from the poles and zeros of $H(s)$ to s will be approximately equal. Since the angle from a pole will be equal to the angle from a zero, the angles from pole/zero pairs will cancel, leaving a net number of excess poles (N) whose angles must sum to π.



Root Locus

High-Gain Rule: If the # of poles exceeds the # of zeros by N, there will be N high-gain asymptotes with angles at $(2n+1)\pi/N$.



Root Locus

Mean Rule: If # of poles is at least two greater than the # of zeros, then the average closed-loop pole position is independent of K_p.

Example:

$$H(s) = \frac{s+z}{(s+p_1)(s+p_2)(s+p_3)}$$

$$\begin{aligned}
G(s) &= \frac{\frac{s+z}{(s+p_1)(s+p_2)(s+p_3)}}{1 + \frac{K_p(s+z)}{(s+p_1)(s+p_2)(s+p_3)}} \\
&= \frac{s+z}{(s+p_1)(s+p_2)(s+p_3) + K_p(s+z)} \\
&= \frac{s+z}{s^3 + (p_1+p_2+p_3)s^2 + (p_1p_2+p_1p_3+p_2p_3)s + (p_1p_2p_3) + K_ps+K_pz} \\
&= \frac{s+z}{s^3 + (q_1+q_2+q_3)s^2 + (q_1q_2+q_1q_3+q_2q_3)s + (q_1q_2q_3)}
\end{aligned}$$

The sum of the closed-loop poles (q_i) does not depend on K_p.

Root Locus

Ending Rule: Each root locus branch **ends at an open-loop zero or ∞.**

If K_p is large, then $|H(s)|$ must be small so that the magnitude criterion

$$|K_p H(s)| = 1$$

is satisfied.

$|H(s)|$ will be zero

(a) if s is equal to an **open-loop zero**, or

(b) if $H(s)$ has more poles than zeros and $s \to \infty$.

Rule 2b follows from the factored representation for $H(s)$:

$$H(s) = K \frac{(s - z_1)(s - z_2)(s - z_3) \cdots (s - z_{n_z})}{(s - p_1)(s - p_2)(s - p_3) \cdots (s - p_{n_p})}$$

If the number of poles (n_p) equals the number of zeros (n_z), then

$$\lim_{|s| \to \infty} = K$$

But if $n_p > n_z$, as $|s| \to \infty$, $H(s)$ will approach $K/s^{(n_p - n_z)} \to 0$.

Summary

Today we focused on applications of stability criteria based on gain and phase margins

Combining these criteria with the vector method we used to evaluate frequency responses provides an intuitive and often informative way to think about **root locus** plots.