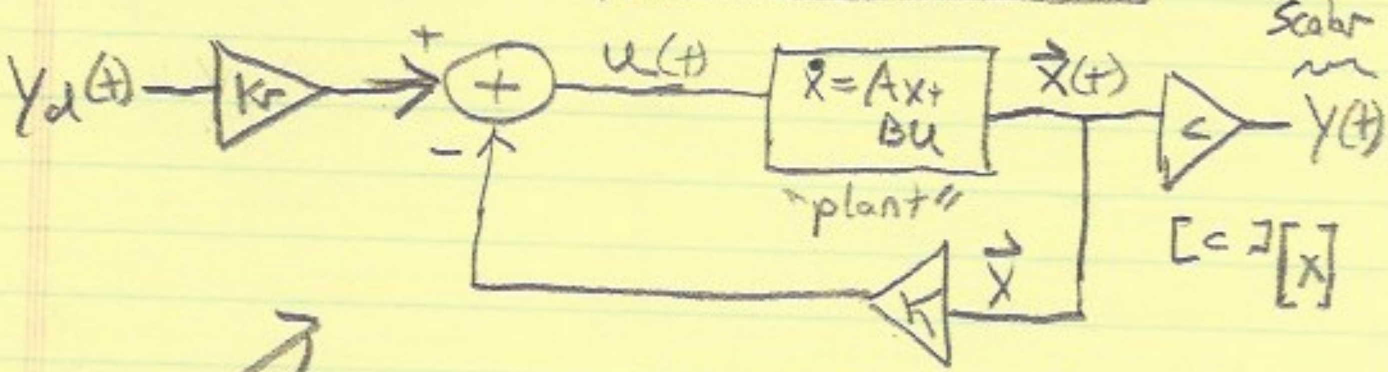
6.310 11/28/23

Today Observers in C.T. (D.T. after Holiday)

C.T. State Space Reminder



$$\underset{\text{scalar}}{Kx} = [K]\begin{bmatrix}x\end{bmatrix}$$

Diagram

Equation → $\begin{cases} \dot{\vec{x}} = A\vec{x} + Bu \quad u = K_r y_d - K\vec{x} \\ \underline{\text{or}} \\ \dot{\vec{x}} = (A - BK)\vec{x} + BK_r y \end{cases}$

state

output

AND

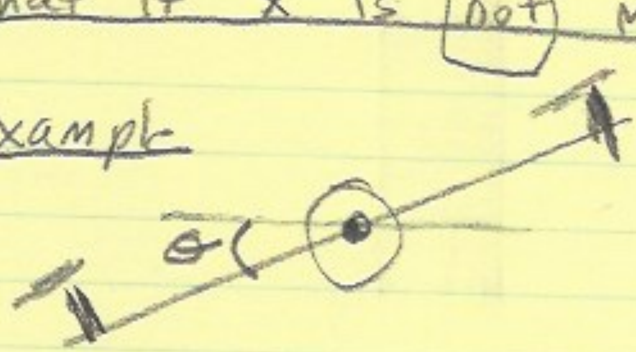
$$y(t) = C\vec{x}(t)$$

Ex ~~y~~: Arm-angle, umbrella position

2

What if x is $\boxed{\text{not}}$ measurable (or some of x isn't)

Example



propeller arm
-driven

We can measure θ

We can approx $\omega \doteq \dfrac{\theta(t) - \theta(t-\Delta T)}{\Delta T}$

We can $\boxed{\text{not}}$ measure:

$\underbrace{\text{Differential}}$ Propeller speed (proportional to Torque)

Left prop speed − Right prop speed

Set up an Estimator

original system: $\dot{x} = Ax + Bu$

Estimation system (computed using simulation): $\dot{\hat{x}} = A\hat{x} + Bu$

estimate state

We generate u so we know it. Same for both systems

Use estimated (e.g. simulated) state for control

$$u(t) = \underbrace{K_r\, y_d(t)}_{\text{Known}} - K\hat{x}$$

estimated state

"Obvious Problems"

1) is $x(t)\big|_{t=0} = \hat{x}(t)\big|_{t=0}$?

(Do initial conds match)

2) Must simulate estimation system as part of control

Examine Initial Condition Issue

Define $err(t) = x(t) - \hat{x}(t)$

(1) $\frac{d}{dt}x(t) = A x(t) + B u(t)$

(2) $\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + B u(t)$

Subtracting (2) from (1) (Assuming A, B model match exactly)

$$\frac{d}{dt}(\underbrace{x(t) - \hat{x}(t)}_{err(t)}) = A(\underbrace{x(t) - \hat{x}(t)}_{err(t)}) + \cancel{Bu(t)} - \cancel{Bu(t)}$$

$$\frac{d}{dt}err(t) = A\, err(t)$$

Recall $e^{At} \to 0$ iff $\mathrm{Re}(\text{eigenvals}(A)) < 0$

nat freqs

So

if $err(t)\big|_{t=0} = x(0) - \hat{x}(0) \neq 0$

and $\mathrm{real}(\mathrm{eig}(A)) < 0$ strictly stable

then $err(t) \to 0$

So initial condition error decays for a strictly stable system!

"Less Obvious Problems"

1) Modeling Errors will cause mismatches
2) Physical system disturbances will not be in estimation (simulation), so **No** disturbance rejection

Fix with Feedback

We always measure $y(t) = Cx(t)$
if single output system
then $y(t)$ is a scalar
but $x(t)$, $\hat{x}(t)$ are vectors!

How do we use a scalar measurement to correct $\begin{bmatrix} v \\ e \\ c \end{bmatrix}$

Correct $\hat{x}(t)$ using $y(t) = C\vec{x}(t)$

(1) physical system $\dot{x} = Ax + Bu$

(2) Estimation system $\dot{\hat{x}} = A\hat{x} + \underbrace{Bu}_{\text{We know}} + [L](y - \hat{y})$

$y = Cx \qquad \hat{y} = C\hat{x}$

Recall estimation error

$$err(t) \equiv x(t) - \hat{x}(t)$$

(1) − (2)

$$\dot{\vec{err}}(t) = A\vec{err}(t) - L(y(t) - \hat{y}(t))$$

$$= A\vec{err}(t) - L(Cx(t) - C\hat{x}(t))$$

$$= A\vec{err}(t) - LC(\vec{err}(t))$$

$$= (A - LC)\vec{err}(t)$$

Pick L so that $\vec{err}(t) \to 0$ fast

Note

$$\underbrace{(A - B_{n \times 1} K_{1 \times n})} \quad \xleftrightarrow{\text{Transpose}} \quad (A - L_{n \times 1} C_{1 \times n})$$

$K = \text{place}(A, B, \text{poles})$

$K = \text{lqr}(A, B, \text{weights})$

$L = \text{place}(A^T, C^T, \text{poles})^T$

$= \text{lqr}(A^T, C^T, \text{weights})^T$

LQR $L = \text{lqr}(A^T, C^T, Q, R)^T$?

what's being minimized?

Pole Placement $L = \text{place}(A^T, C^T, \text{poles})^T$

$$\text{eig}(A - LC) = \text{eig}((A-LC)^T)$$
$$= \text{eig}(A^T - C^T L^T)$$

so place works the usual way

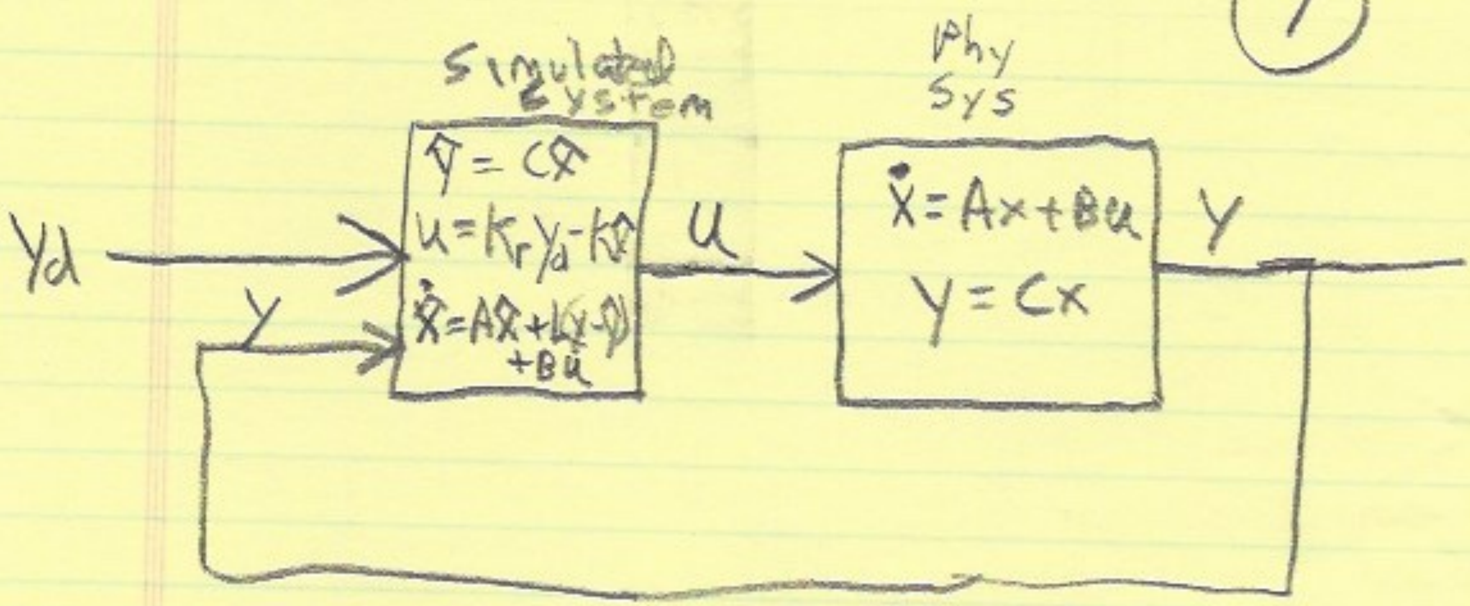
Estimator given L & K

$$u = K_r y_d - K\hat{x}$$

Est $\dot{\hat{x}} = (A - BK)\hat{x} + K_r y_d + L(y - \overset{C\hat{x}}{\hat{y}})$

Phys $\dot{\hat{x}} = (A - BK - LC)\hat{x} + K_r y_d + L y$

Hmmm..



Analyzing

(1) Phys $\quad \dot{x} = Ax + Bu = Ax + B(K_r y_d - K\hat{x})$

(2) Est $\quad \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$

$err(t) \equiv x(t) - \hat{x}(t) \qquad \hat{x}(t) = x(t) - err(t)$

(1)-(2) $\quad \dot{err}(t) = (A - LC)\, err(t)$

$$\dot{x} = Ax + BK_r y_d - BK\,\overbrace{(x(t) - err(t))}^{\hat{x}(t)}$$

$$= (A - BK)x + BK\,err(t) + BK_r y_d$$

$$\begin{bmatrix} \dot{x} \\ \dot{err} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ err \end{bmatrix} + \begin{bmatrix} BK_r \\ 0 \end{bmatrix} y_d$$

Decouples control from estimator!