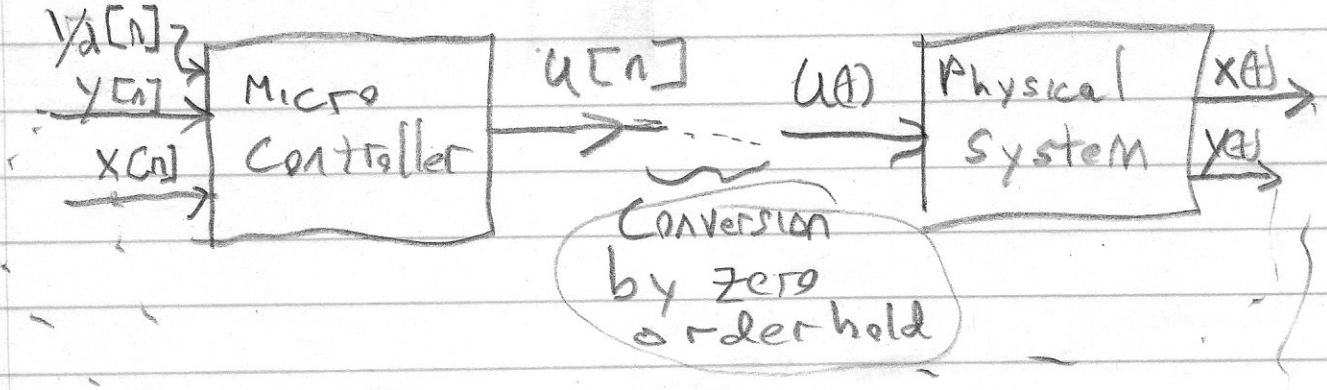
6.3100/2 11/27/23 —> 11/29/23

Mixed Discrete – Continuous System

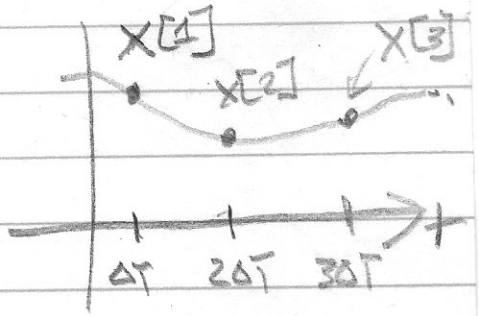


Conversion by sampling

ignoring finite # bits effect!

Sampling Analog → Digital Conversion every ΔT seconds

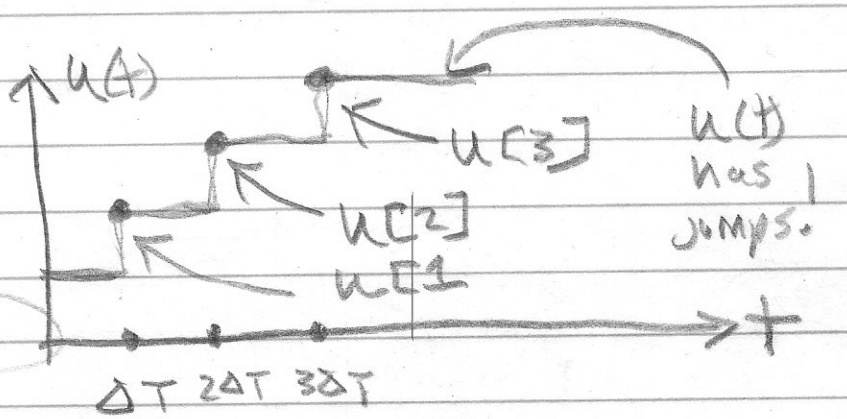
$$x[n] = x(n\Delta T)$$
$$y[n] = y(n\Delta T)$$
$$y_d[n] = y_d(n\Delta T)$$



Zero Order Hold

$$u(t) = u\left[\left[\frac{t}{\Delta T}\right]\right]$$

$[x]$ = Greatest integer $\leq x$



Quick Reminder about c2b

Solution to constant input case:

	Scalar Case	Vector Case
Eqn	$\frac{d}{dt}x(t) = ax(t) + bu_0$ $x(t)\big\|_{t=0} = x(0)$ u_0: constant	$\frac{d}{dt}x = Ax + Bu_0$ $x(t)\big\|_{t=0} = x(0)$
Soln	$x(t) = e^{at}x(0) + (e^{at}-1)\frac{b}{a}u_0$	$x(t) = e^{At}x(0) + (e^{At}-I)A^{-1}Bu_0$
Proof	$x(t)\big\|_{t=0} = e^{a\cdot 0}x(0) + (1-1)\frac{b}{a}u_0 = x(0)$ ✓ Matches I.C.	Since $e^{At}\big\|_{t=0} = I$ $x(t)\big\|_{t=0} = x(0)$ ✓
	$\frac{d}{dt}x(t) = ae^{at}x(0) + ae^{at}\frac{b}{a}u_0$ $= a\left(e^{at}x(0) + (e^{at}-1)\frac{b}{a}u_0\right) + bu_0$ $= a\,x(t) + bu_0$ ✓ Matches Diff. Eqn.	Since $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$ $\frac{d}{dt}x(t) = Ae^{At}x(0) + Ae^{At}A^{-1}Bu_0$ $= A\left(e^{At}x(0) + (e^{At}-I)A^{-1}Bu_0\right) + Bu_0$ ✓

$e^{At}\vec{x} = \left(I + At + \frac{1}{2}(At)^2 + \ldots\right)\vec{x}$

$e^{at}x = \left(1 + at + \frac{1}{2}(at)^2 + \frac{1}{6}(at)^3 + \ldots\right)x$

1B

Repeating From Previous Page

Does $x(t) = e^{at}x(0) + \left(e^{at}-1\right)\frac{b}{a}u_0$

Solve $\frac{d}{dt}x(t) = ax(t) + bu_0$?

If $a < 0$, look at steady state

$\lim_{t\to\infty} x(t) = x(\infty) = x_{ss}$ ← steady state

$\frac{d}{dt}x_{ss} = ax_{ss} + bu_0$ ($\frac{d}{dt}x_{ss} = 0$)

$\Rightarrow x_{ss} = -\frac{b}{a}u_0$

$x(t) = e^{at}x(0) + \left(e^{at}-1\right)\frac{b}{a}u_0$

$\Rightarrow \lim_{t\to\infty}\; 0\cdot x(0) + \left(0-1\right)\frac{b}{a}u_0$

$\Rightarrow x_{ss} = -\frac{b}{a}u_0$ ✓ ✓

Does $\vec{x}(t) = e^{At}\vec{x}(0) + \left(e^{At}-I\right)A^{-1}Bu_0$

Solve $\dot{\vec{x}}(t) = A\vec{x}(t) + Bu_0$?

If $\text{real}(\text{eig}(A)) < 0$ look at steady state

$\lim_{t\to\infty} \vec{x}(t) = \vec{x}(\infty) = \vec{x}_{ss}$

$\frac{d}{dt}\vec{x}_{ss} = A\vec{x}_s + Bu_0$ ($\frac{d}{dt}\vec{x}_{ss} = \vec{0}$)

$\Rightarrow A\vec{x}_{ss} = -Bu_0$

$\Rightarrow \vec{x}_{ss} = -A^{-1}Bu_0$

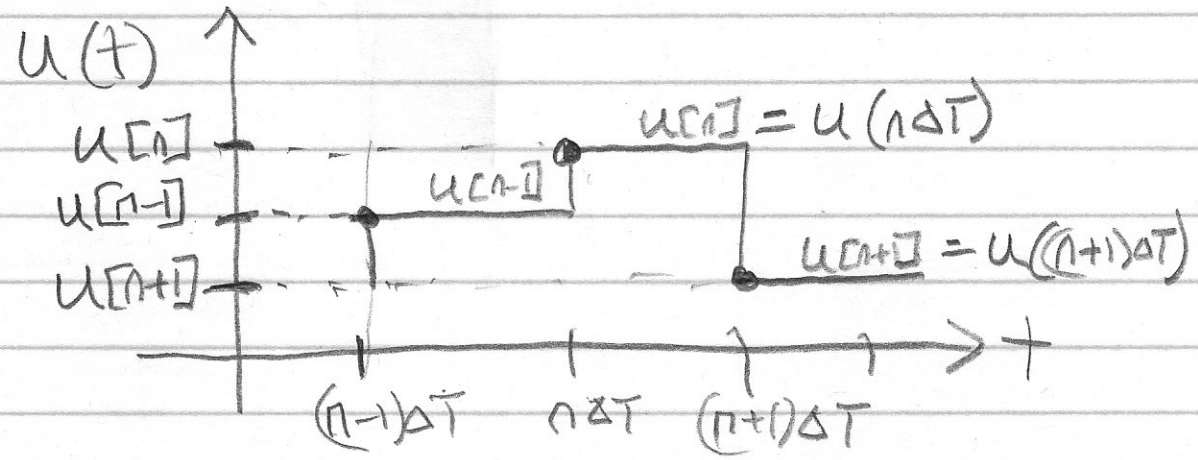
$\vec{x}(t) = e^{At}\vec{x}(0) + \left(e^{At}-I\right)A^{-1}Bu_0$

($e^{At} = 0$, $e^{At}-I = 0$)

$= 0\cdot\vec{x}(0) + (-I)A^{-1}Bu_0$ ✓

$= -A^{-1}Bu_0$ ✓

Zero-Order Hold



Let $X(0) = X((n-1)\Delta T) = X[n-1]$, $u_0 = u((n-1)\Delta T) = u[n-1]$

in Constant input solution

Scalar Case

$$X(n\Delta T) = e^{a\Delta T} X((n-1)\Delta T) + \left(e^{a\Delta T} - 1\right)\frac{b}{a} u((n-1)\Delta T)$$

$$X[n] = e^{a\Delta T} X[n-1] + \left(e^{a\Delta T} - 1\right)\frac{b}{a} u[n-1]$$

A_d

B_d

Vector Case

$$X(n\Delta T) = e^{A\Delta T} X((n-1)\Delta T) + \left(e^{A\Delta T} - I\right)A^{-1}B\, u((n-1)\Delta T)$$

$$X[n] = e^{A\Delta T} X[n-1] + \left(e^{A\Delta T} - I\right)A^{-1}B\, u[n-1]$$

$[x] \equiv$ largest integer $< x$

Plant + Controller

$u(t) = u\left[\left[\frac{t}{\Delta T}\right]\right]$

$u(t)$ → $\dot{x} = Ax + Bu$, $y = Cx$ → $x(t)$ → C → $y(t)$

Zero order Hold

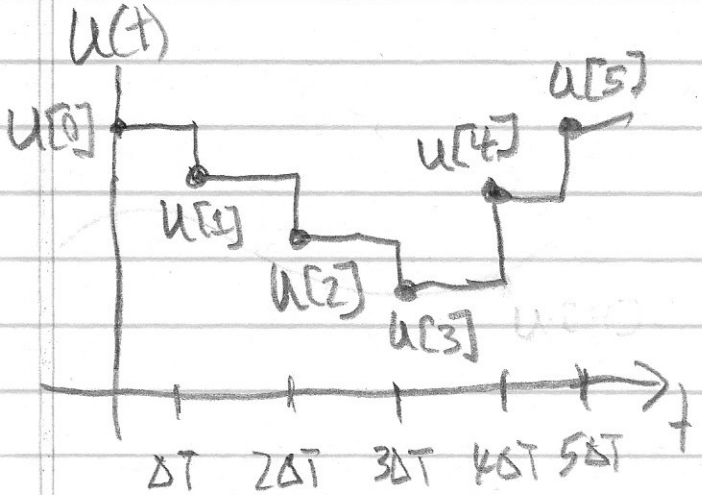
$= x(n\Delta T)$

$x[n]$

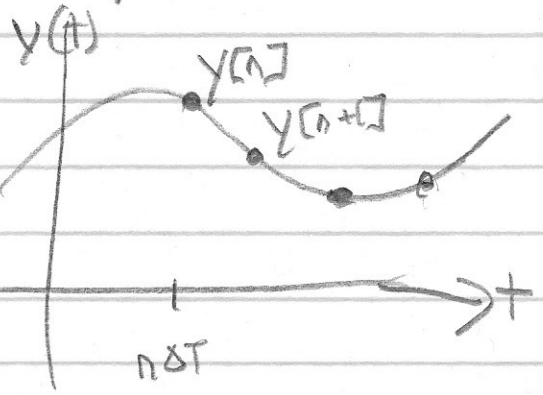
$u[n]$ → Controller → Sampler

$y[n]$

Replace with DT System



$= y(n\Delta T)$



Discrete-Time State-Space Model

$$x[n] = A_d x[n-1] + B_d u[n-1]$$

$$y[n] = C x[n]$$

← CT & DT C's are equal!

$e^{A\Delta T}$

$(e^{A\Delta T} - I)A^{-1}B$

State Feedback Control

$$u[n] = K_{r_d} y_d[n] - K_d x[n]$$

Closed-Loop System

$$x[n] = (A_d - B_d K_d) x[n-1] + B_d K_{r_d} y_d[n-1]$$

If $y_d = 0$

$$x[n] = (A_d - B_d K_d)^n x[0]$$

Want $x[n] \to 0$ as fast as possible!

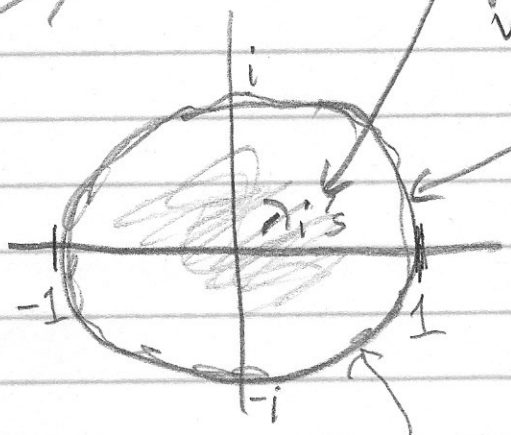
Picking Kd

Pole placement

$$\min_{K_d} \left(\max_i \left| \lambda_i (A_d - B_d K_d) \right| \right)$$

can be ineffective!

place λ_i's well inside unit circle



LQR for Discrete-Time

Pick Kd to minimize

$$\sum_{M=0}^{\infty} x^T[M] Q x[M] + u^T[M] R u[M]$$

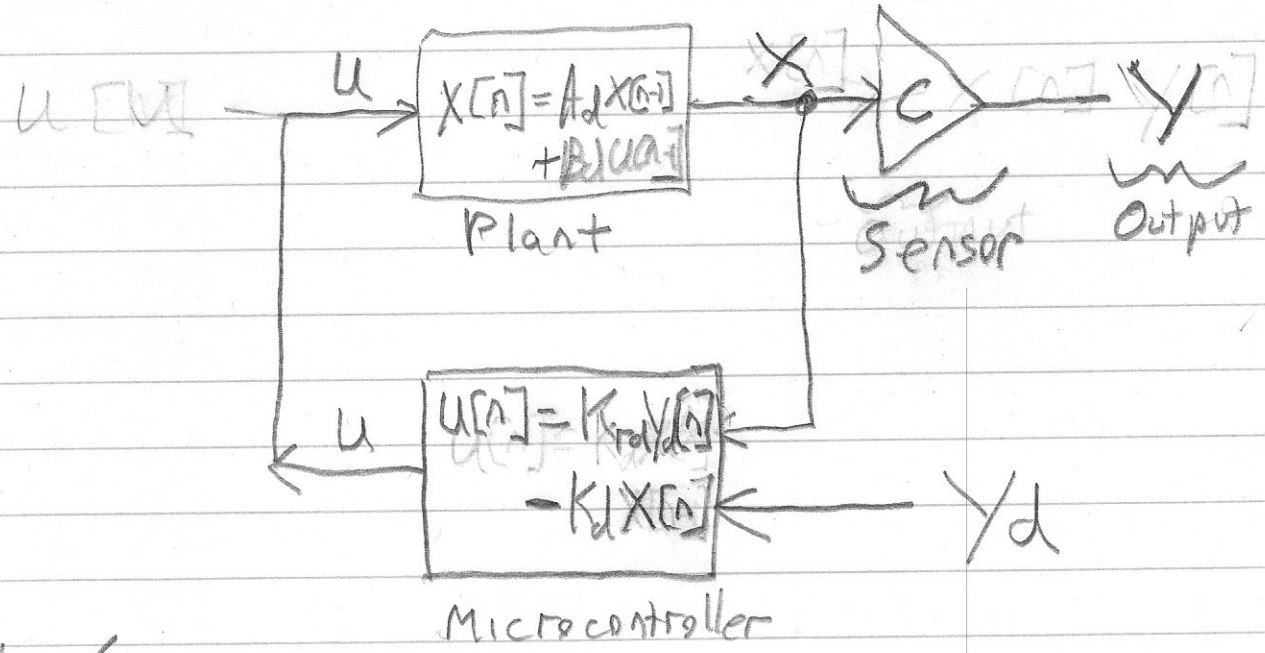
or if Q & R diagonal

$$\sum_{M=0}^{\infty} \left(\sum_{i=1}^{\#\text{states}} Q_{ii} x_i^2[M] \right) + \sum_{j=1}^{\#\text{inputs}} R_{jj} u_j^2[M]$$

What about K_{rd}?

Plant Model:

$$x[n] = A_d x[n-1] + B_d u[n-1]$$
$$y[n] = C x[n]$$



Controller:

$$u[n] = K_{rd} y_d[n] - K_d x[n]$$

K_{rd}: ?

K_d: minimize metric

Closed-Loop

$$x[n] = (A_d - B_d K_d) x[n-1] + B_d K_{rd} y_d[n-1]$$

In Steady State $x[n] \to x_\infty$ $y_d[n] \to y_\infty$

$$x_\infty = (A_d - B_d K_d) x_\infty + B_d K_{rd} y_\infty$$

What is K_{rd} Continued (4B)

In Steady State (Continued)

$$\left(I - (A_d - B_d K_d)\right) X_\infty = B_d K_{rd} Y d_\infty$$

$$\Rightarrow \underbrace{X_\infty}_{\substack{\text{Vector} \\ n \times n}} = \underbrace{\left(I - (A_d - B_d K_d)\right)^{-1}}_{\substack{\text{matrix} \\ n \times n}} \underbrace{B_d K_{rd}}_{\substack{\text{Vector} \\ \uparrow n \times 1}} \underbrace{Yd_\infty}_{\substack{\text{Scalar} \\ 1 \times 1}}$$

$$Y_\infty = C X_\infty = C \left(I - (A_d - B_d K_d)\right)^{-1} B_d K_{rd} Yd_\infty$$

Y_∞: scalar; Yd_∞: Scalar

Should match in steady-state

$$\frac{Y_\infty}{Yd_\infty} = 1 = \underbrace{C\left(I - (A_d - B_d K_d)\right)^{-1} B_d}_{\text{Scalar}} \underbrace{K_{rd}}_{\text{Scalar}}$$

$$\frac{1}{C\left(I - (A_d - B_d K_d)\right)^{-1} B_d} = K_{rd} \; !!$$

Discrete-Time Observers

$$\hat{x}[n] = A_d \hat{x}[n-1] + B_d u[n-1]$$

No hat, we know u

$$+ L_d(y[n-1] - \hat{y}[n-1])$$

correction

$$u[n-1] = K_{r_d} y_d[n-1] - K_d \hat{x}[n-1]$$

Compare Indexing

On a microcontroller

→ Every ΔT interval

1) Use predicted state, create control

$$u \Leftarrow K_{r_d} y_d - K_d \hat{x}$$

2) Measure observable and compare

$$\text{correction} \Leftarrow L_d(y - \hat{y})$$

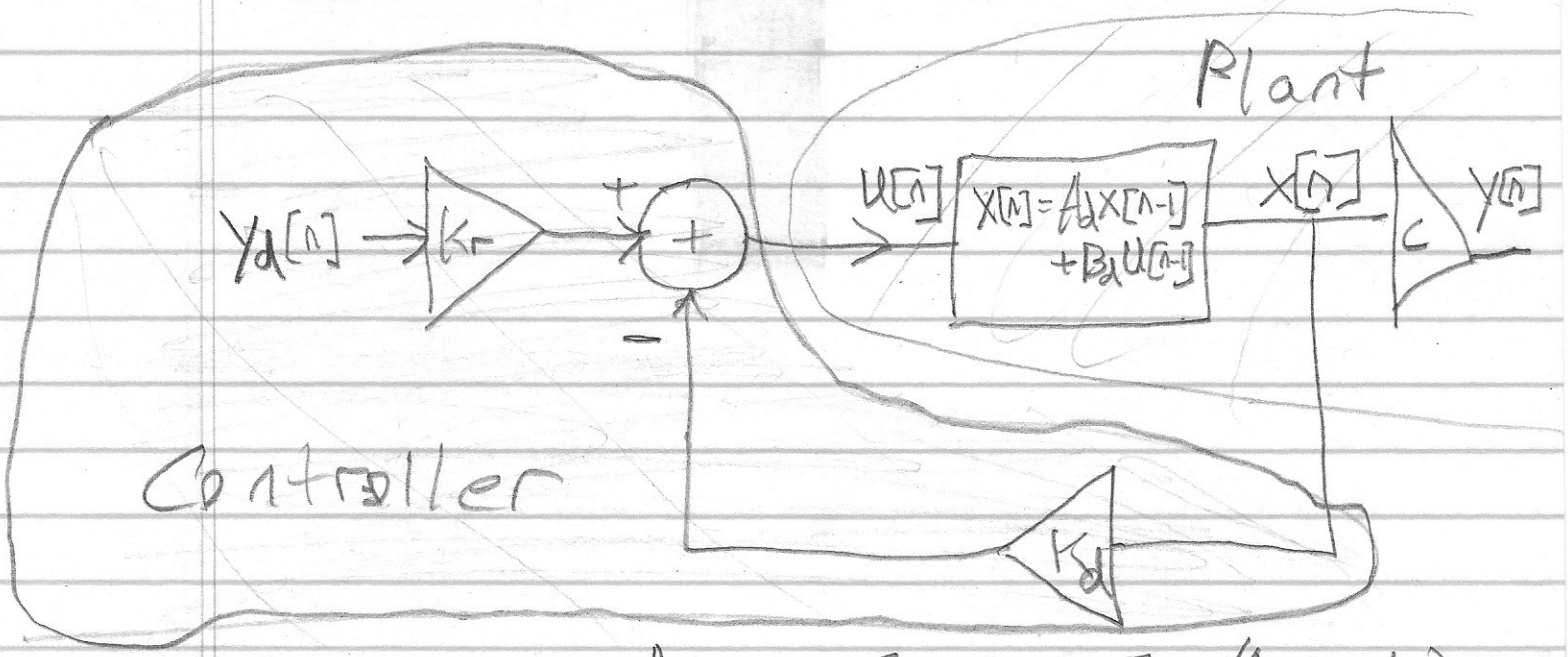
3) Update estimates of state & observable

$$\hat{x} \Leftarrow A_d \hat{x} + B_d u + \text{correction}$$

$$\hat{y} \Leftarrow C \hat{x}$$

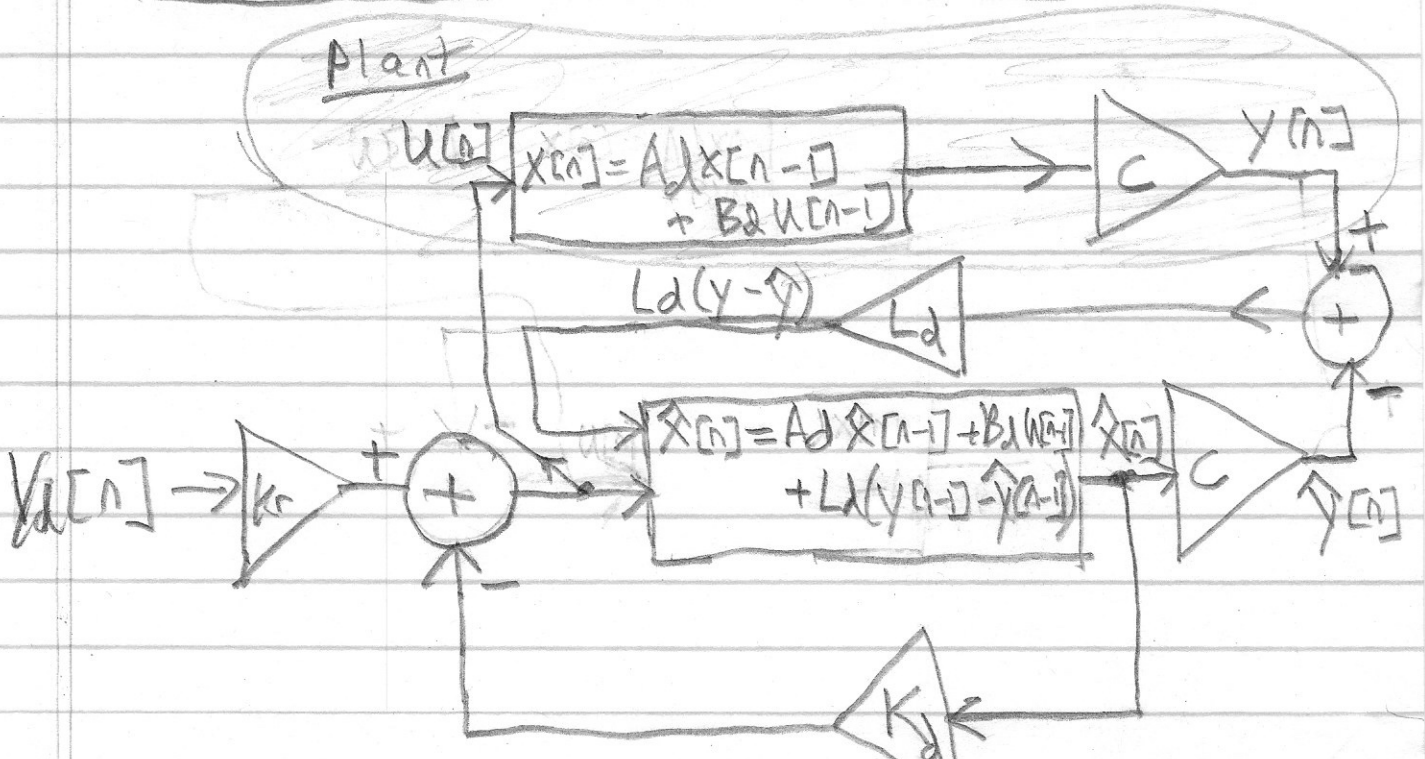
5A

Measured-State Feedback



$$x[n] = A_d x[n-1] + B_d u[n-1] = (A_d - B_d K_d) x[n-1] + B_d K_r y_d[n]$$

Observer-State Feedback



$$x[n] = A_d x[n-1] - B_d K_d \hat{x}[n-1] + B_d K_r r[n-1]$$
$$\hat{x}[n] = (A_d - B_d K_d - L_d C)\hat{x}[n-1] + L_d C x[n-1] + B_d K_r y_d[n-1]$$

Analyze Coupled System

Actual: $x[n] = A_d x[n-1] + B_d u[n-1]$

Estimate: $\hat{x}[n] = A_d \hat{x}[n-1] + B_d u[n-1] + L_d(y[n-1] - \hat{y}[n-1])$

$e_{rr}[n] \equiv x[n] - \hat{x}[n]$

$\Rightarrow \hat{x}[n] = x[n] - e[n]$

Subtracting

$$\underbrace{x[n] - \hat{x}[n]}_{e_{rr}[n]} = A_d\Big(\overbrace{x[n-1] - \hat{x}[n-1]}^{e_{rr}[n-1]}\Big) - L_d\Big(\underbrace{y[n-1] - \hat{y}[n-1]}_{C\,e_{rr}[n-1]}\Big)$$

$$e_{rr}[n] = A_d e_{rr}[n-1] - L_d C_d e_{rr}[n-1]$$

$$= (A_d - L_d C_d)\, e_{rr}[n-1]$$

e_{rr} does not depend on k_d!

Coupled System Continued

Closed-Loop Control using Observer

$$x[n] = A_d x[n-1] + B_d\left(K_{r_d} Y_d[n-1] - K_d \hat{x}[n-1]\right)$$

Since $\hat{x}[n-1] = x[n-1] - e[n-1]$

$$\Rightarrow x[n] = (A_d - B_d K_d)\, x[n-1] + B_d K_d\, e[n-1] + B_d K_{r_d} Y_d[n-1]$$

$e[n] \equiv \hat{x}[n] - x[n]$

Compare to Closed loop measured-state

Assuming $u = K_{r_d} Y_d - F_d x$ (measured state)

$$x_r[n] = (A_d - B_d K_d)\, x_r[n-1] + B_d K_{r_d} Y_d[n-1]$$

x_r is state if measured state feedback is used

$$(x[n] - x_r[n]) = (A_d - B_d K_d)(x[n-1] - x_r[n-1]) + B_d K_d\, e[n-1]$$

$x[n]$: state using observed state feedback

$x_r[n]$: state using measured state feedback

Observer Feedback like "Real" State Feedback if $e \to 0$ fast

Relation between K_d & L_d

Picking L's by pole placement has similar issues as picking K's by pole placement.

If eig $(A_d - L_d C_d)$ are smaller than the eig $(A_d - B_d K_d)$

Then State-Feedback with estimated state will perform like state-feedback with measured state.

Why not pick L_d such that

$$\max_i |\lambda_i (A_d - L_d C_d)| \cong 0?$$

Because large values of L_d are bad!

Noise Example with (measurement noise)

With noise

$$e_{rr}[n] = A_d\, e_{rr}[n-1] - L_d\left(y[n-1] - \hat{y}[n-1] + W[n-1]\right)$$

(Noise)

Suppose $W[n] = W_0$ (constant noise)

$$e_{rr}[n] = \left(A_d - L_d C_d\right) e_{rr}[n-1] + L_d W_0$$

In steady-state $e_{rr}[n] = e_{rr}[n-1] = e_{rr}[\infty]$

$$\left(I - \left(A_d - L_d C_d\right)\right) e_{rr}[\infty] = L_d W_0$$

$$e_{rr}[\infty] = \left(I - \left(A_d - L_d C_d\right)\right)^{-1} L_d W_0$$

Even in Best case Large L_d's mean high sensitivity to measurement Noise!

Best Case! If L_d's chosen for very fast decay

$$A_d - L_d C_d \simeq 0 \Rightarrow \left(I - \left(A_d - L_d C_d\right)\right)^{-1} \simeq I$$

and $e_{rr}[\infty] \simeq L_d W_0$

Big L_d / Big Errors