MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

# 6.3100: Intro to Modeling and Control—Fall 2023

## Code of Arms Postlab - Due 9:00am, 10/18/23

*We suggest writing your solutions in a separate document and then turning it into a pdf (scan it or take pictures if you like to use pencil and paper). And PLEASE INCLUDE YOUR DERIVATIONS!! We have no way of verifying your understanding with just a numerical answer, particularly if there were a minor calculation error.*

## Problem One

A clinician is trying to determine if feedback can be used for dosage control of a particular medication. Using daily measurements for hundreds of patients, the clinician has generated a simple model for the relationship between daily dosage, denoted $c[n]$ where $n$ is the day index, and daily-measured bloodstream concentration, denoted $x[n]$ (we assume the data has been appropriately normalized for patient age, weight, etc).

The clinician's simple model is

$$x[n] = \zeta x[n-1] + \frac{1}{2}c[n-1]$$

where $\zeta$ is a degradation factor that varies between 0.5 to 0.9, depending on the patient. In words, the clinician's model says that in one day, somewhere between ten to fifty percent of the medication already in the bloodstream decays away, and about half of the previous day's dosage is in the bloodstream.

### Part A)

- 1) Suppose $c[n] = c_0$, a fixed dosage. If the desired steady-state bloodstream concentration is $x_d$, as a function of $x_d$, what should $c_0$ be if $\zeta = 0.5$? What if $\zeta = 0.9$?

- 2) If you use $c_0$ concentration for $\zeta = 0.5$ on a patient with $\zeta = 0.9$, what will the steady-state bloodstream concentration be for that patient? If you do not know the degradation factor for the patient, should you use the $c_0$ assuming $\zeta = 0.9$ or $c_0$ assuming $\zeta = 0.5$?

### Part B)

Suppose we try using proportional control, so that

$$c[n] = K_p\left(x_d - x[n]\right).$$

- 1) If we never want the bloodstream concentration to oscillate regardless of $\zeta$ (so the natural frequency should NOT be negative), what is the maximum $K_p$ we can use?

- 2) Using the maximum $K_p$ you just determined, what will the steady-state bloodstream concentration be, as a fraction $x_d$, over the range of $\zeta$ from 0.5 to 0.9. Did the feedback control approach improve the situation?

**Part C)**

In order to improve the approach to steady-state, consider including integral control as in

$$c[n] = K_p (x_d - x[n]) + K_i s[n]$$

where

$$s[n] = s[n-1] + (x_d - x[n-1]).$$

Notice that the formula above is NOT the same as the formula in lab. We will examine the differences below.

We can write a $2 \times 2$ system of equations to describe the model and control, using the matrix $\mathbf{A}$ and the vector $\mathbf{B}$ as in

$$\begin{bmatrix} x[n] \\ s[n] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x[n-1] \\ s[n-1] \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x_d$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

- 1) Determine the values of the $\mathbf{A}$ matrix and the $\mathbf{B}$ vector in terms of $K_p$, $K_i$, $\zeta$ and $x_d$.
- 2) Recall that when the eigenvalues of $\mathbf{A}$ (aka natural frequencies) are less than one in magnitude, then the formula for the steady-state is $(\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$. Show that if $K_i > 0$ (and the eigencondition holds) that the steady-state bloodstream concentration matches $x_d$ for any $\zeta$.
- 3) How would you decide if a pair of values for $K_p$ and $K_i$ are good? What criteria would you use? How would you search for them?

**Part D)** You may have noticed that in lab, we stated a formula for $s[n]$ that was a little different than the one above, it was

$$s[n] = s[n-1] + (x_d - x[n]).$$

The reason the formula directly above might be preferable is that one might expect the sum of all past errors on the $n^{th}$ step should include the $n^{th}$ error, and our earlier formula did not (and in fact, in lab we steered you to an implementation that was inconsistent with the lab formula).

If we want to implement the above formula, we will need $x[n]$, but that variable is on the left side of the equation. So lets invent a $2 \times 2$ $E$ matrix,

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} x[n] \\ s[n] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x[n-1] \\ s[n-1] \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x_d$$

where $e_{11} = e_{22} = 1$, and we can pick $e_{21}$ and $e_{12}$ to implement our equation for $s[n]$ that includes $x[n]$. Then when we are done, we can invert that $E$ matrix and form $\tilde{\mathbf{A}} = \mathbf{E}^{-1}\mathbf{A}$ and $\tilde{\mathbf{B}} = \mathbf{E}^{-1}\mathbf{B}$.

- 1) Determine the values $e_{12}$ and $e_{21}$ to implement the improved equation for $s[n]$.
- 2) Show that the steady-state match when $K_i > 0$ still holds.

**Problem Two**

For this problem, based on your implementation of the $\mathbf{A}$ matrix from lab, determine two $\mathbf{B}$ vectors, a $\mathbf{B}_{in}$ that is scaled by $\theta_d[n]$ in the state space description, and a $\mathbf{B}_{disturb}$ that is scaled by $u_{disturb}$, the disturbance associated with dropping the weight on the arm. That is, find the values in the $5 \times 1$ vectors in

$$\vec{x}[n] = \mathbf{A}\vec{x}[n-1] + \mathbf{B}_{in}\theta_d[n-1] + \mathbf{B}_{disturb}u_{disturb}[n-1].$$

Think carefully about what state equation is directly affected by the disturbance, and then show that when $K_i > 0$, the steady-state value of propeller arm angle is unaffected by the disturbance.

**Hint:** *In determining* $\mathbf{B}_{disturb}$ *vector, which entry is non-zero? You can answer that by determining which equation is affected by dropping the weight onto the arm. Then since* $\mathbf{B}_{disturb}$ *is scaled by* $u_{disturb}$*, you can normalize the nonzero entry in* $\mathbf{B}_{disturb}$ *to one.*

This problem does not require much work (let the computer invert the matrices), but it does require some careful thought.