2

THE UNILATERAL LAPLACE TRANSFORM

2.0 Introduction

An engineered dynamic system is typically characterized by the operation it is intended to perform—amplifier, motor controller, radio direction finder, etc. For a physical system observed in nature (for example, the solar system), it is enough to ask "How does it work?" For an engineered system, it is necessary also to ask "What is it supposed to do?" A similar interest in overall objectives arises in studies of biological and social systems, which are generally thought to have evolved under constraints that make them appear purposeful.

The study of engineered systems thus involves a continual interplay of *functional* (input-output, "black box") and *structural* (circuit, state, block diagram) system descriptions. To proceed from a structural to a functional description, we must *analyze* the given structure to determine its input-output behavior. Alternately, we may wish to *synthesize* a structure having a desired functional response.

Mathematically, a functional description of a system has the form of an *operator** such as

$$y(t) = f[x(t), \lambda_1(0), \lambda_2(0), \ldots, \lambda_n(0)] \qquad (2.0\text{–}1)$$

which explicitly assigns a unique output waveform $y(t)$, $t > 0$, to each input waveform $x(t)$, $t > 0$, and each initial state described by $\lambda_1(0), \lambda_2(0), \ldots, \lambda_n(0)$. A particular system may have several equivalent† functional descriptions, just as it may have several equivalent structural descriptions (for example, a circuit diagram, a set of node equations, a set of state equations). Our principal goal in

*In the sequel, when we wish to talk generally about the behavior of systems without implying any particular dimensions (voltage, current, displacement, temperature, etc.) for the input and output time functions, we shall usually designate the input by $x(t)$, the output by $y(t)$, and state variables by $\lambda_i(t)$. Systems can, of course, have more than one input or output; if so, appropriate formal changes in (2.0–1) would be required. Note that a system characterized by a (point) *function*, such as $y(t) = x^2(t)$, is more restricted than a system characterized by an *operator*, such as $y(t) = \int_0^t x(\tau)\,d\tau + y(0)$; in the first case the present value of the output depends only on the present value of the input, whereas in the second it depends on input values at many times—for example, the entire interval from $t = 0$ to the present, and even earlier times as reflected in $y(0)$.

†Two operators are *equivalent* if they give the same response to the same input and state for every input and state.

this chapter is to use Laplace transform techniques to derive what is called the frequency-domain form of functional description of an LTI system. In a later chapter, we shall study an equivalent time-domain form.

2.1 The Unilateral Laplace Transform

The *unilateral Laplace transform* (abbreviated $\mathcal{L}$-transform) is itself an operator that maps a function of time into a function of a complex variable $s = \sigma + j\omega$ according to the formula

$$\boxed{X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}\,dt\ .} \tag{2.1-1}$$

It must be emphasized that it is the fact that this integral maps $x(t)$ into a function of s that is important. We are interested in the values of $X(s)$ throughout some region of the complex s-plane, not just at a single point, because (as we shall discuss) given $X(s)$ in a region of the s-plane we can in general recover $x(t)$ uniquely for $t > 0$, that is, the Laplace transform as an operator is *biunique.*

This property of biuniqueness is critical to the usefulness of the Laplace transform.* Thus suppose we are interested in describing the operator $y(t) = f[x(t), \{\lambda_i(0)\}]$ characterizing some system. Instead of doing this directly, it often turns out to be simpler and more illuminating to describe the operator $Y(s) = F[X(s), \{\lambda_i(0)\}]$ relating the Laplace transforms of the input and output; because of biuniqueness this is equivalent to the description sought. For reasons to be described, s is called the *complex frequency*. The operator $Y(s) = F[X(s), \{\lambda_i(0)\}]$ is thus said to characterize the system in the *frequency domain,* whereas $y(t) = f[x(t), \{\lambda_i(0)\}]$ characterizes it in the *time domain*:

$$\begin{array}{lccc}
 & & \text{input} & \text{output} \\
 & & \downarrow & \downarrow \\
\text{time domain} & \rightarrow & x(t) \Rightarrow & y(t) = f[x(t), \{\lambda_i(0)\}] \\
 & & \Updownarrow & \Updownarrow \\
\text{frequency domain} & \rightarrow & X(s) \Rightarrow & Y(s) = F[X(s), \{\lambda_i(0)\}]
\end{array}$$

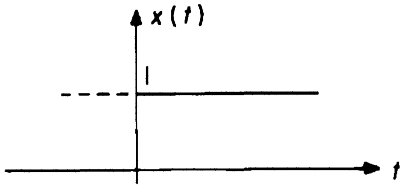
As an analytical tool for computing the response to specific simple inputs, the Laplace transform is primarily useful for systems that are equivalent functionally

*As one might expect from the name, the $\mathcal{L}$-transform was one of the many contributions to mathematics and physics of the Marquis Pierre Simon de Laplace (1749–1827), who pointed out the biunique relationship between the two functions and applied the results to the solution of differential equations in a paper published in 1779 with the rather cryptic title "On what follows." The real value of the $\mathcal{L}$-transform in applications seems not to have been appreciated, however, for over a century, until it was essentially rediscovered and popularized by the eccentric British engineer Oliver Heaviside (1850–1925), whose studies had a major impact on many aspects of modern electrical engineering.

to a lumped LTI circuit of intermediate complexity— say 3^{rd}-order. For nonlinear or time-varying systems $F[\cdot]$ is usually no easier to describe or evaluate than $f[\cdot]$. Very small LTI systems (1^{st}-order) with simple inputs can generally be solved most efficiently by direct methods—as in Example 1.7–1. Very large systems require mechanical aids in any event; Laplace transform techniques require the manipulation of matrices with algebraic elements (functions of s) and the extraction of the roots of high-order polynomials—both of which are clumsy operations for a computer. But within its limited and important domain, the Laplace transform is an astonishingly effective tool for solving circuit problems, and the insights gained into general system properties and behavior are even more important.

2.2 Examples of $\mathcal{L}$-Transforms and Theorems

Example 2.2–1



$$x(t) = 1, \quad t > 0$$

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt = \int_0^\infty 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

provided that $\Re e[s] > 0$ so that $e^{-st} \to 0$ as $t \to \infty$.

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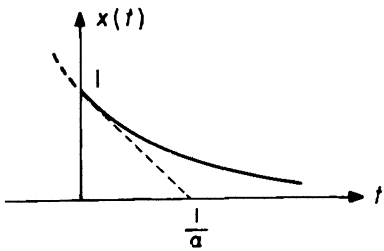
Example 2.2–2



$$x(t) = e^{-\alpha t}, \quad t > 0$$

$$X(s) = \int_0^\infty e^{-\alpha t}e^{-st}\,dt = \frac{-1}{s+\alpha}e^{-(s+\alpha)t}\Big|_0^\infty = \frac{1}{s+\alpha}$$

provided that $\Re e[s] > -\alpha$ so that $e^{-(s+\alpha)t} \to 0$ as $t \to \infty$.

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Note that the values of $x(t)$ for $t < 0$ do not influence $X(s)$ (and thus obviously cannot be recovered from $X(s)$). Consequently

a) $\mathcal{L}[1] = \mathcal{L}[u(t)] = \dfrac{1}{s}, \quad \Re e[s] > 0;$

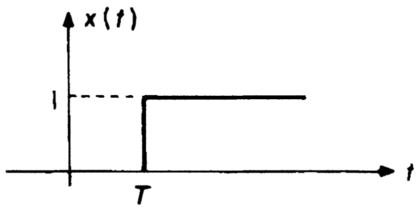
b) $\mathcal{L}[e^{-\alpha t}] = \mathcal{L}[e^{-\alpha t}u(t)] = \mathcal{L}[u(-t) + e^{-\alpha t}u(t)] = \dfrac{1}{s+\alpha}, \quad \Re e[s] > -\alpha.$

(The *bilateral* Laplace transform, defined as $\int_{-\infty}^{\infty} x(t)e^{-st}\,dt$, is affected by $x(t)$, $t < 0$, and has a number of other properties that will be briefly explored in an appendix to Chapter 14.)

It is also important to note that the integral defining $X(s)$ often exists only in a limited region of the s-plane called the *domain of (absolute) convergence.* It should be obvious that, provided $x(t)$ grows no faster than $e^{\sigma_0 t}$ for some finite value of σ_0 and is otherwise well-behaved, the product $e^{-st}x(t)$ will be absolutely integrable in the right half-plane, $\Re e[s] > \sigma_0$. The smallest (real) number σ_0 such that $e^{-\sigma t}x(t)$ is absolutely integrable for all $\sigma > \sigma_0$ is called the *abscissa of (absolute) convergence.* The domain of absolute convergence may be the entire s-plane, so that $\sigma_0 = -\infty$; this will be true, for example, if $x(t)$ is a *pulse*, that is, if $x(t)$ is nonzero only for a finite time interval. On the other hand, if $x(t)$ grows faster than any exponential, for example, $x(t) = e^{t^2}$, then there will be no domain of convergence and Laplace transform methods cannot be applied.

Most of the $X(s)$ of interest to us will be *rational functions* (ratios of polynomials in s). The roots of the denominator polynomial are values of s for which the function $X(s) \to \infty$, and these are called *poles.* The roots of the numerator are called *zeros.* By definition, there can be no poles in the region of convergence; for a rational function $X(s) = \mathcal{L}[x(t)]$, the domain of convergence is the region to the right of the rightmost pole. Non-rational transforms will often appear in our studies, however, as the following examples illustrate.

Example 2.2–3



$x(t) = u(t-T), \quad t > 0,\ T > 0$

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt = \int_T^\infty 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_T^\infty = \frac{1}{s}e^{-sT}$$

provided that $\Re e[s] > 0$ so that $e^{-st} \to 0$ as $t \to \infty$.

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Together, Examples 2.2–3 and 2.2–1 illustrate a general theorem:

DELAY THEOREM:

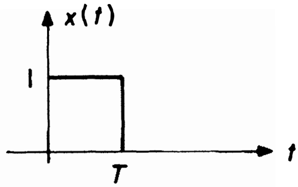
Let $\mathcal{L}[x(t)] = X(s)$. Then for $T > 0$,

$$\mathcal{L}[x(t-T)u(t-T)] = X(s)e^{-sT}.$$

(The proof of the Delay Theorem follows immediately from a simple change of variable in (2.1–1). Note that the Delay Theorem is not in general true for $T < 0$, as can be seen by considering the $\mathcal{L}$-transforms of $x(t) = u(t)$ and of $x(t+1)u(t+1) = u(t+1)$. Can you give a statement of the conditions under which it will be true for $T < 0$?)

Example 2.2–4

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & t > T. \end{cases}$$



$$X(s) = \int_0^\infty x(t)e^{-st}\,dt = \int_0^T 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^T = \frac{1}{s}\left(1 - e^{-sT}\right), \quad \text{all } s.$$

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Note that the pulse $x(t)$ in Example 2.2–4 can be written as a difference of two time functions:

$$x(t) = 1 - u(t-T), \quad t > 0, T > 0.$$

The resulting $X(s)$ is the difference of the transforms of the individual time functions. This, too, is an example of a general theorem:

LINEARITY THEOREM:

Let $\mathcal{L}[x_1(t)] = X_1(s)$ and $\mathcal{L}[x_2(t)] = X_2(s)$. Then

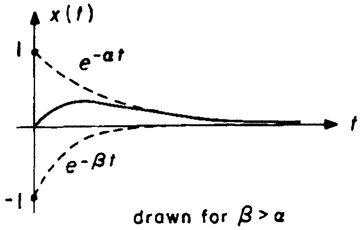
$$\mathcal{L}[ax_1(t) + bx_2(t)] = aX_1(s) + bX_2(s).$$

(Again, the proof of the Linearity Theorem follows immediately from (2.1–1).)

Of course, neither of the transforms obtained in Examples 2.2–3 and 2.2–4 is a rational function because of the presence of the e^{-sT} terms. Nevertheless, $(1/s)e^{-sT}$ behaves very much like $1/s$ near $s = 0$; that is, it has a pole at $s = 0$. Since e^{-sT} is well-behaved for all finite s, the domain of convergence in Example 2.2–3 is $\Re e[s] > 0$. Despite appearances, the transform of the pulse in Example 2.2–4, $(1/s)(1 - e^{-sT})$, is well-behaved for all s, including $s = 0$; indeed, no constraints are necessary on s for the $\mathcal{L}$-transform of the pulse waveform in Example 2.2–4 to exist, so the domain of convergence is the entire plane.

The Delay and Linearity Theorems are useful for finding, without integrating, the $\mathcal{L}$-transforms of many interesting functions. Some examples are:

Example 2.2–5



$$x(t) = e^{-\alpha t} - e^{-\beta t}, \quad t > 0$$

$$X(s) = \frac{1}{s+\alpha} - \frac{1}{s+\beta} = \frac{\beta-\alpha}{(s+\alpha)(s+\beta)}.$$

The domain of convergence is $\Re e[s] > -\alpha$ or $\Re e[s] > -\beta$, whichever is further to the right.

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Example 2.2–6

$$x(t) = \sin\omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right], \quad t > 0$$

$$X(s) = \frac{1}{2j}\frac{1}{s-j\omega_0} - \frac{1}{2j}\frac{1}{s+j\omega_0} = \frac{\omega_0}{s^2+\omega_0^2}, \quad \Re e[s] > 0.$$

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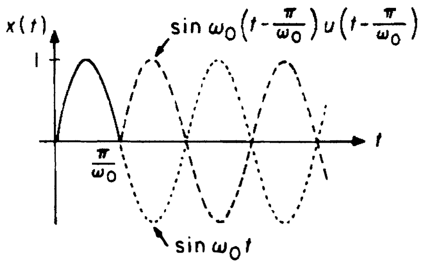
Example 2.2–7



$$x(t) = \begin{cases} \sin\omega_0 t, & 0 < t < \pi/\omega_0 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \sin\omega_0 t + \sin\omega_0\left[t - \frac{\pi}{\omega_0}\right]u\left[t - \frac{\pi}{\omega_0}\right]$$

$$X(s) = \frac{\omega_0}{s^2+\omega_0^2} + \frac{\omega_0}{s^2+\omega_0^2}e^{-s\pi/\omega_0} = \frac{\omega_0}{s^2+\omega_0^2}\left(1 + e^{-s\pi/\omega_0}\right), \quad \text{all } s.$$

This $X(s)$ has no poles anywhere in the finite s-plane. (The apparent poles at $s = \pm j\omega_0$ are cancelled by zeros of $\left(1 + e^{-s\pi/\omega_0}\right)$ at those points.)

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Henceforth we shall usually drop any explicit indication of domains of convergence, relying implicitly on the fact that there is *some* value of σ_0 such that for $\Re e[s] > \sigma_0$, *all* transforms arising in a particular problem are well-defined.

2.3 The Inverse Laplace Transform

The scheme proposed in Section 2.1 for exploiting Laplace transforms in system characterization requires that we be able to reverse the process described above and recover $x(t)$ given $X(s)$. It can be shown (it is not at all obvious) that this can be accomplished through the *inverse Laplace transform*

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi j}\int_C X(s)e^{st}\,ds \tag{2.3–1}$$

where the integral is a line integral along an appropriate contour C in the complex plane. This integral can be remarkably effective in many cases, but its significance and efficient handling depend on an appreciation of the theory of functions of a complex variable, which puts it beyond our scope. (We shall, however, have a bit more to say about this topic after introducing the Fourier transform in Chapter 14.)

For our purposes it will be adequate to carry out the inverse transformation by manipulating $X(s)$ into a sum of terms, each of which we can recognize as the direct transform of a simple time function. Exploiting the Linearity Theorem and the Uniqueness Theorem (one form of which we now state) gives the desired inversion.

UNIQUENESS THEOREM:

If $X_1(s) = \mathcal{L}[x_1(t)]$ and $X_2(s) = \mathcal{L}[x_2(t)]$ exist and are equal in any small region of the s-plane, then $X_1(s) = X_2(s)$ throughout their common region of convergence and $x_1(t) = x_2(t)$ for almost all $t > 0$.

This theorem implies that for practical purposes the inverse transformation is unique.*

The inverse transform technique that we are going to describe will work in any case in which $X(s)$ is a rational function, and in some other cases as well. The key is to appreciate that every rational function can be expanded in *partial fractions*. Specifically, if the rational function is *proper* (degree of numerator less than the degree of the denominator—we shall remove this restriction in Chapter 11) and if the roots of the denominator— the poles of $X(s)$—are *simple*

*Of course, nothing can be learned from $X(s)$ about $x(t)$ for $t < 0$. "Almost all" means that $x_1(t)$ and $x_2(t)$ might differ at isolated points, for example, $x_1(t) = 1$, $x_2(t) = \left\{\begin{matrix} 1, & t \neq 1 \\ 0, & t = 1 \end{matrix}\right\}$; but such differences generally have no practical consequences for reasons we shall explore in Chapter 11. For a proof of the uniqueness theorem, see, e.g., D. V. Widder, *The Laplace Transform* (Princeton, NJ: Princeton Univ. Press, 1946) p. 63. Uniqueness is closely related to the fact that $X(s)$ is an *analytic function* throughout its domain of convergence—that is, $X(s)$ can be expanded in a convergent Taylor's series about any point in its domain.

or *distinct* (we shall remove this restriction shortly), then we may always write

$$\begin{aligned} X(s) &= \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_0} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{(s - s_{p_1})(s - s_{p_2})\ldots(s - s_{p_m})} \\ &= \frac{k_1}{s - s_{p_1}} + \frac{k_2}{s - s_{p_2}} + \cdots + \frac{k_m}{s - s_{p_m}} \end{aligned} \tag{2.3–2}$$

where $k_1, k_2, \ldots, k_m$ are an appropriate set of constants called *residues.** Once the k_i are found, we may write the corresponding $x(t)$ immediately as

$$x(t) = k_1 e^{s_{p_1} t} + k_2 e^{s_{p_2} t} + \cdots + k_m e^{s_{p_m} t}, \quad t > 0. \tag{2.3–3}$$

This, or its generalization to include multiple-order poles, is often called the *Heaviside Expansion Theorem.*

It should be evident from (2.3–2) that the residues for $X(s)$ with simple poles can be easily found from the formula

$$k_i = [X(s)(s - s_{p_i})]_{s=s_{p_i}}. \tag{2.3–4}$$

The following examples illustrate the partial-fraction procedure for finding inverse Laplace transforms for $X(s)$ with simple poles.

Example 2.3–1

$$X(s) = \frac{s+3}{s^2+s} = \frac{s+3}{s(s+1)} = \frac{k_0}{s} + \frac{k_1}{s+1}$$

where

$$k_0 = [X(s)s]_{s=0} = \left.\frac{s+3}{s+1}\right|_{s=0} = 3$$

$$k_1 = [X(s)(s+1)]_{s=-1} = \left.\frac{s+3}{s}\right|_{s=-1} = -2.$$

Thus

$$X(s) = \frac{3}{s} - \frac{2}{s+1} \implies x(t) = 3 - 2e^{-t}, \quad t > 0.$$

Since the partial fraction expansion is an algebraic identity, it can *and should* always be checked by multiplying the terms back together again:

$$X(s) = \frac{3(s+1) - 2s}{s(s+1)} = \frac{s+3}{s^2+s}.$$

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*Note carefully the way in which $X(s)$ in (2.3–2) is normalized so that the highest power of s in the denominator has coefficient unity. Failure to carry out this normalization is a common source of error in evaluating residues.

Example 2.3–2

$$X(s) = \frac{s+1}{s^2+1} = \frac{s+1}{(s+j)(s-j)} = \frac{k_+}{s+j} + \frac{k_-}{s-j}$$

where

$$k_+ = [X(s)(s+j)]_{s=-j} = \left.\frac{s+1}{s-j}\right|_{s=-j} = \frac{1-j}{-2j} \quad \frac{1}{\sqrt{2}}$$

$$k_- = [X(s)(s-j)]_{s=j} = \left.\frac{s+1}{s+j}\right|_{s=j} = \frac{1+j}{2j} = \frac{1}{\sqrt{2}}e^{-j\pi/4}$$

Thus

$$x(t) = \frac{1}{\sqrt{2}}e^{j\pi/4}e^{-jt} + \frac{1}{\sqrt{2}}e^{-j\pi/4}e^{jt} = \sqrt{2}\cos(t-\pi/4), \quad t>0.$$

Note that because the coefficients of the numerator and denominator polynomials are real, the poles occur in conjugate complex pairs and have conjugate complex residues.
▶▶▶

Example 2.3–3

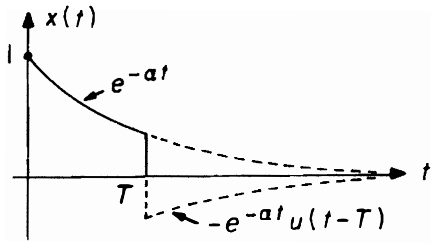
$$X(s) = \frac{1-e^{-(s+\alpha)T}}{s+\alpha}.$$

This is not a rational function, but it can be written as a sum of products of rational functions and e^{-sT} factors:

$$X(s) = \frac{1}{s+\alpha} - \frac{e^{-\alpha T}}{s+\alpha}e^{-sT}.$$

Hence, from the Delay Theorem,

$$x(t) = e^{-\alpha t} - e^{-\alpha T}\left[e^{-\alpha(t-T)}u(t-T)\right] = e^{-\alpha t} - e^{-\alpha t}u(t-T), \quad t>0.$$



This is a pulse-type waveform, and we note that $X(s)$ does not in fact have a pole at $s=-\alpha$ or anywhere else in the finite s-plane; the domain of convergence is the entire plane.
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2.4 Multiple-Order Poles

The partial-fraction expansion procedure becomes slightly more complex if any roots of the denominator polynomial are repeated or multiple-order. Extra terms must then be added to the expansion corresponding to all powers of the repeated factor up to the order of the pole. The following example should make the ideas clear.

Example 2.4–1

$$X(s) = \frac{1}{(s+1)^3(s+2)} = \frac{k_1''}{(s+1)^3} + \frac{k_1'}{(s+1)^2} + \frac{k_1}{s+1} + \frac{k_2}{s+2}.$$

The residue k_2 and the coefficient k_1'' can easily be found by an obvious extension of the previous method:

$$k_2 = [X(s)(s+2)]_{s=-2} = \left.\frac{1}{(s+1)^3}\right|_{s=-2} = -1$$

$$k_1'' = \left[X(s)(s+1)^3\right]_{s=-1} = \left.\frac{1}{s+2}\right|_{s=-1} = 1.$$

These two terms by themselves are not enough to represent $X(s)$ (as can readily be seen by multiplying the expansion out with $k_1' = k_1 = 0$). To find k_1 and k_1' several methods are possible:

a) Multiply back together and match coefficients with the original function:

$$\frac{1}{(s+1)^3} + \frac{k_1'}{(s+1)^2} + \frac{k_1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)^3(s+2)}$$
$$= \frac{(s+2) + k_1'(s+1)(s+2) + k_1(s+1)^2(s+2) - (s+3)^3}{(s+1)^3(s+2)}.$$

Matching highest powers gives $k_1 s^3 - s^3 = 0$ or $k_1 = 1$. Matching next highest powers gives $k_1' s^2 + k_1(2s^2 + 2s^2) - 3s^2 = 0$ or $k_1' = -1$.

b) Expand $X(s)(s+1)^3$ in a power series in $(s+1)$:

$$X(s)(s+1)^3 = \frac{1}{s+2} = \frac{1}{(s+1)+1} = 1 - (s+1) + (s+1)^2 - \cdots$$

(where we have used the expansion $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ which is valid for $|x| < 1$, that is, for s near -1). Then

$$X(s) = \frac{1 - (s+1) + (s+1)^2 - \cdots}{(s+1)^3} = \frac{1}{(s+1)^3} - \frac{1}{(s+1)^2} + \frac{1}{s+1} - \cdots.$$

The coefficients of the negative powers are the desired terms, that is, $k_1'' = 1$, $k_1' = -1$, and $k_1 = 1$. (Since the coefficients of the Taylor's-series expansion of

$X(s)(s+1)^3$ are related to the derivatives of this expression, formulas are often given in mathematics books for k_1 and k_1' in terms of derivatives, but these are rarely the most effective way to compute these coefficients.)

c) Subtract away the $k_1''/(s+1)^3$ term, leaving a function with only a second-order pole. Repeating gives a function with only a first-order pole, which can be expanded as before. Thus

$$X(s) - \frac{k_1''}{(s+1)^3} = \frac{1}{(s+1)^3(s+2)} - \frac{1}{(s+1)^3} = \frac{1-(s+2)}{(s+1)^3(s+2)}$$
$$= \frac{-1}{(s+1)^2(s+2)} = \frac{k_1'}{(s+1)^2} + \frac{k_1}{s+1} - \frac{1}{s+2}.$$

Then

$$k_1' = \left[\frac{-1}{\cancel{(s+1)^2}(s+2)}\cancel{(s+1)^2}\right]_{s=-1} = -1$$

etc.

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To evaluate the time functions corresponding to $\mathcal{L}$-transforms with multiple-order poles, we need to know the time function whose transform is $1/(s+\alpha)^n$. This can be found from repeated application of the following theorem.

MULTIPLY-BY-t THEOREM:

Let $\mathcal{L}[x(t)] = X(s)$. Then

$$\mathcal{L}[t\,x(t)] = -\frac{dX(s)}{ds}. \qquad (2.4\text{–}1)$$

The proof follows at once on differentiating the basic defining formula

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt.$$

Thus we conclude that

$$\mathcal{L}\left[te^{-\alpha t}\right] = -\frac{d}{ds}\frac{1}{s+\alpha} = \frac{1}{(s+\alpha)^2} \qquad (2.4\text{–}2)$$

so that, by induction,

$$\mathcal{L}\left[t^n e^{-\alpha t}\right] = \frac{n!}{(s+\alpha)^{n+1}}. \qquad (2.4\text{–}3)$$

In particular, completing Example 2.4–1, if

$$X(s) = \frac{1}{(s+1)^3(s+2)} = \frac{1}{(s+1)^3} - \frac{1}{(s+1)^2} + \frac{1}{s+1} - \frac{1}{s+2}$$

then

$$x(t) = \frac{1}{2}t^2e^{-t} - te^{-t} + e^{-t} - e^{-2t}, \quad t > 0.$$

The appendix to this chapter contains a brief table of $\mathcal{L}$-transform pairs and a list of important properties. More extensive tables are widely available.

2.5 Circuit Analysis with the Laplace Transform

In the preceding chapter, we explained how the dynamic equations characterizing the behavior of a circuit are derived from two kinds of information—the constitutive relations between branch voltages and currents that describe the elements, and the constraints among the same variables that arise from their interconnections and Kirchhoff's Laws. Since the constitutive relations in general involve derivatives and/or integrals, the dynamic equations are in general differential equations. Alternatively, as suggested in Section 2.1, we could describe both the elements and the interconnection constraints in terms of relations between the unilateral Laplace transforms of the branch voltages and currents. The result for LTI circuits is a set of algebraic equations that are much easier than differential equations to manipulate and interpret.

Let's begin by replacing the constitutive relations for the simple 2-terminal lumped electrical elements of Figure 1.1–1 by equivalent relations in the frequency domain. Thus, a linear resistor is equally adequately described by the instantaneous version of Ohm's Law, $v(t) = Ri(t)$, or by the relation

$$V(s) = RI(s) \tag{2.5–1}$$

between the $\mathcal{L}$-transforms of the voltage and the current that follows from the Linearity Theorem. Mathematically, (2.5–1) has the same form as Ohm's Law for the "voltage" $V(s)$ across a "resistor" R carrying a "current" $I(s)$. Of course, $V(s)$ and $I(s)$ are not a voltage and a current, but rather the transforms of a voltage and a current. Nevertheless, it is often very convenient to draw a circuit diagram "in the frequency domain" in which the branch variables are the transforms of the actual branch time functions and the elements are described by constitutive relations between these transforms.

To derive similar frequency-domain constitutive relations for linear capacitors and inductors, we need the following theorem.

DIFFERENTIATION THEOREM:

Let $\mathcal{L}[x(t)] = X(s)$. Then

$$\boxed{\mathcal{L}\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0)\,.} \tag{2.5–2}$$

The proof of this theorem proceeds as follows. By definition*

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt.$$

Integrate by parts, that is, use the formula $\int u\, dv = uv - \int v\, du$ with $u = e^{-st}$, $du = -se^{-st}\, dt$, $dv = \dfrac{dx(t)}{dt} dt$, $v = x(t)$, to give

*We assume $dx(t)/dt$ is well-behaved. It is interesting to note, however, that the right-hand side of (2.5–2) is apparently meaningful even when $x(t)$ has discontinuities, suggesting that some sort of significance can be attached to $dx(t)/dt$ even in such cases—as we shall explore further in Chapter 11.

$$\int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt = x(t)e^{-st}\Big|_0^\infty + s\int_0^\infty x(t)e^{-st}\, dt$$
$$= -x(0) + sX(s)$$

where we have assumed $x(t)e^{-st}|^\infty = 0$, as it generally must if s is in the domain of convergence of $X(s) = \mathcal{L}[x(t)]$. This completes the proof.

Using this theorem to transform the equation $i(t) = C\, dv(t)/dt$ characterizing a capacitor C yields

$$I(s) = CsV(s) - Cv(0) \tag{2.5–3}$$

or

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}. \tag{2.5–4}$$

Mathematically, (2.5–4) has the same form as Ohm's Law for the "voltage" $V(s)$ across a "resistor" $1/Cs$ carrying a "current" $I(s)$ in series with a "voltage source" $v(0)/s$. In the "frequency-domain" circuit, then, a capacitor will be represented by an *impedance*, $Z(s) = 1/Cs$, in series with a "voltage" source $v(0)/s$ reflecting the initial state. Similarly, transforming the equation $v(t) = L\,di(t)/dt$ characterizing an inductor L yields

$$V(s) = LsI(s) - Li(0) \tag{2.5–5}$$

or

$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}. \tag{2.5–6}$$

Hence in the "frequency-domain" circuit, an inductor is represented by an impedance $Z(s) = Ls$ in parallel with a "current source" $i(0)/s$ describing the initial state.

Resistor: $V(s) = RI(s)$

Capacitor: $V(s) = \dfrac{1}{Cs}I(s) + \dfrac{v(0)}{s}$

Inductor: $I(s) = \dfrac{1}{Ls}V(s) + \dfrac{i(0)}{s}$

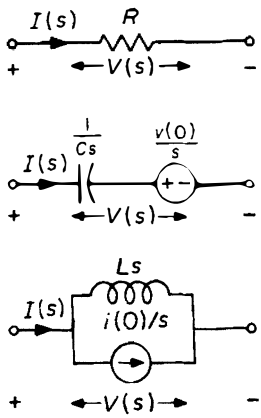


Figure 2.5–1. Simple LTI elements in the frequency domain.

The frequency-domain element descriptions derived above are summarized in Figure 2.5–1. Note that the "branch variables" in these elements are the transforms of the voltages and currents that are actually present. The "element values" are the impedances associated with these elements, and the "sources" appear only in the frequency-domain descriptions. Note also that impedance is analogous to resistance, that is, it is the ratio of a voltage (transform) to a current (transform); the reciprocal of impedance—analogous to conductance—is called *admittance*. In general, the symbol $Z(s)$ is used to represent the impedance of an arbitrary element; $Y(s)$ is used to represent the admittance of an arbitrary element.

As a result of the Linearity Theorem, the constraints imposed on the time-domain branch voltages and currents by Kirchhoff's Laws carry over without alteration into the frequency domain:

$$\begin{array}{ccc} \textit{time domain} & & \textit{frequency domain} \\ \sum\limits_{cut\,set} i_j(t) = 0 & \Longleftrightarrow & \sum\limits_{cut\,set} I_j(s) = 0 \\ \sum\limits_{loop} v_j(t) = 0 & \Longleftrightarrow & \sum\limits_{loop} V_j(s) = 0\,. \end{array} \tag{2.5–7}$$

These constraints can be represented schematically in the frequency-domain circuit by connecting the elements in exactly the same way as they are connected in the time-domain circuit. How impedance methods and the $\mathcal{L}$-transform are combined to yield efficient solution procedures for LTI circuit problems is best explained through several examples.

Example 2.5–1

In Example 1.7–1 we computed by "classical" methods the response of the simple first-order circuit shown in Figure 2.5–2 to a constant input. The result, including an arbitrary initial voltage $v(0)$ on the capacitor, was

$$v(t) = RI + (v(0) - RI)e^{-t/RC}, \quad t > 0\,. \tag{2.5–8}$$

This is plotted in Figure 2.5–3.

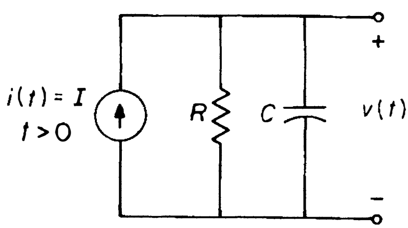


Figure 2.5–2. Example 2.5–1 circuit.

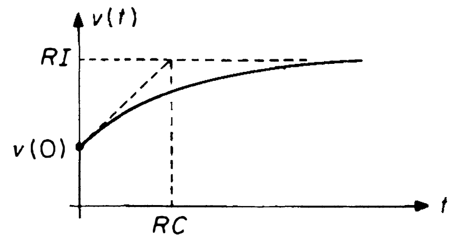


Figure 2.5–3. Constant input response.

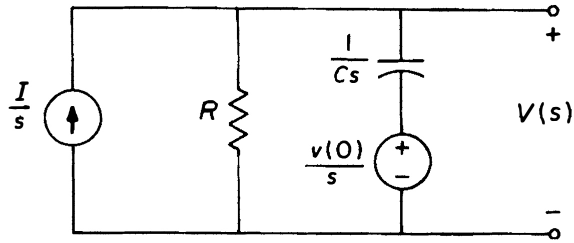


Figure 2.5–4. Frequency-domain form of circuit of Figure 2.5–2.

The frequency-domain form of this circuit is shown in Figure 2.5–4. The resistor and the capacitor have been replaced by their impedances, R and $1/Cs$, respectively. A source $v(0)/s$ has been inserted in series with the capacitor to describe the effects of the initial state. The constant current source I in the time-domain circuit has been replaced in the frequency-domain circuit by its transform I/s. We seek to determine $V(s)$, the transform of the voltage $v(t)$. Treating $V(s)$ as a node voltage, we can use elementary resistive circuit theory to derive the node equation

$$\frac{V(s)}{R} + \left(V(s) - \frac{v(0)}{s}\right)Cs - \frac{I}{s} = 0\,. \tag{2.5–9}$$

Solving (2.5–9) for $V(s)$ and expanding in partial fractions gives

$$V(s) = \frac{(I/C) + v(0)s}{s(s + (1/RC))} = \frac{RI}{s} + \frac{v(0) - RI}{s + (1/RC)} \tag{2.5–10}$$

which we recognize immediately from Examples 2.2–1 and 2.2–2 as the transform of

$$v(t) = RI + (v(0) - RI)e^{-t/RC}, \quad t > 0 \tag{2.5–11}$$

which is identical with the result obtained in Example 1.7–1. Of course, this is such a simple problem that the answer can be written down directly—which is obviously preferable to using transforms.

▶▶▶

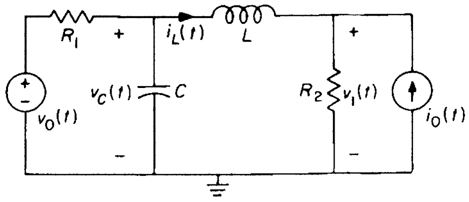
Example 2.5–2

The kind of situation in which the Laplace transform is most useful is illustrated by the problem described in Figure 2.5–5. To solve it, write node equations for the frequency-domain circuit as if it were a resistive circuit with branch resistances equal to the impedances. Selecting $V_C(s)$ and $V_1(s)$ as node voltage variables, we obtain by summing currents away from the nodes

$$\frac{V_C(s) - V_0(s)}{R_1} + \left(V_C(s) - \frac{v_C(0)}{s}\right)Cs + \frac{V_C(s) - V_1(s)}{Ls} + \frac{i_L(0)}{s} = 0 \tag{2.5–12}$$

$$\frac{V_1(s) - V_C(s)}{Ls} + \frac{V_1(s)}{R_2} - \frac{i_L(0)}{s} - I_0(s) = 0\,. \tag{2.5–13}$$

Time Domain



$v_0(t) = 3,\ t > 0$

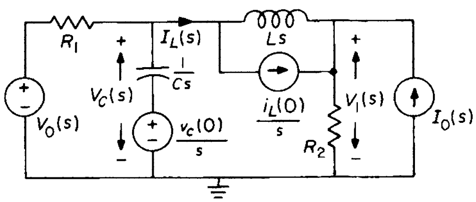
$i_0(t) = 4e^{-t},\ t > 0$

$v_C(0) = -1$

$i_L(0) = 2$

$R_1 = 0.5\,\Omega,\quad R_2 = 1\,\Omega,\quad L = 1\,\mathrm{H},\quad C = 0.5\,\mathrm{F}$

Frequency Domain



$V_0(s) = 3/s$

$I_0(s) = 4/(s+1)$

$v_C(0) = -1$

$i_L(0) = 2$

Figure 2.5–5. Time- and frequency-domain circuits for Example 2.5–2.

Solving (2.5—12, 13) for $V_1(s)$ gives (after some algebra)

$$\begin{aligned} V_1(s) \;=\; & \frac{R_2\left(s^2 + s\frac{1}{R_1C} + \frac{1}{LC}\right)}{s^2 + s\left(\frac{1}{R_1C} + \frac{R_2}{L}\right) + \frac{1}{LC}\left(1 + \frac{R_2}{R_1}\right)} I_0(s) \\ & + \frac{\frac{R_2}{LCR_1}}{s^2 + s\left(\frac{1}{R_1C} + \frac{R_2}{L}\right) + \frac{1}{LC}\left(1 + \frac{R_2}{R_1}\right)} V_0(s) \\ & + \frac{s\frac{R_2}{L}}{s^2 + s\left(\frac{1}{R_1C} + \frac{R_2}{L}\right) + \frac{1}{LC}\left(1 + \frac{R_2}{R_1}\right)} \frac{v_C(0)}{s} \\ & + \frac{s^2R_2 + s\frac{R_2}{R_1C}}{s^2 + s\left(\frac{1}{R_1C} + \frac{R_2}{L}\right) + \frac{1}{LC}\left(1 + \frac{R_2}{R_1}\right)} \frac{i_L(0)}{s}. \end{aligned} \tag{2.5—14}$$

To catch algebraic mistakes, it is good practice to carry circuit analyses about this far in literal form so that the dimensions of terms being added can be checked for consistency (using the facts that s has dimensions t^{-1}, RC and L/R have dimensions t, and LC has dimensions t^2). But to proceed further, it is helpful to substitute the element values given in Figure 2.5–5 to obtain

$$\begin{aligned} V_1(s) \;=\; & \frac{s^2+4s+2}{s^2+5s+6} I_0(s) \;+\; \frac{4}{s^2+5s+6} V_0(s) \\ & + \frac{s}{s^2+5s+6} \frac{v_C(0)}{s} \;+\; \frac{s(s+4)}{s^2+5s+6} \frac{i_L(0)}{s}. \end{aligned} \tag{2.5—15}$$

Substituting the appropriate functions for the transforms of the sources and the initial conditions as given in Figure 2.5–5, and factoring $s^2 + 5s + 6 = (s+2)(s+3)$, yields

$$\begin{aligned} V_1(s) &= \frac{4(s^2}{(s+1)(s+2)(s+3)} + \frac{12}{s(s+2)(s+3)} - \frac{1}{(s+2)(s+3)} + \frac{2(s+4)}{(s+2)(s+3)} \\ &= \frac{6s^3 + 25s^2 + 27s + 12}{s(s+1)(s+2)(s+3)} = \frac{2}{s} - \frac{2}{s+1} + \frac{5}{s+2} + \frac{1}{s+3}. \end{aligned} \qquad (2.5\text{–}16)$$

Inverse transforming yields

$$v_1(t) = 2 - 2e^{-t} + 5e^{-2t} + e^{-3t}, \quad t > 0 \qquad (2.5\text{–}17)$$

which is the final answer sought.

▶ ▶ ▶

Clearly the $\mathcal{L}$-transform is a powerful and efficient solution procedure for problems of the type illustrated in Example 2.5–2. But the algebra (which we did not display above) can be much reduced by exploiting one or more of the simplification techniques studied in earlier courses for linear resistive circuits—equivalent series and parallel resistances, voltage- and current-divider formulas, superposition, Thévenin and Norton theorems, ladder method, etc. These apply just as well to impedances in frequency-domain circuits, as shown in the following example.

Example 2.5–3

Suppose we replace the left-hand part of the circuit of Figure 2.5–5 by its Thévenin equivalent as shown in Figure 2.5–6.

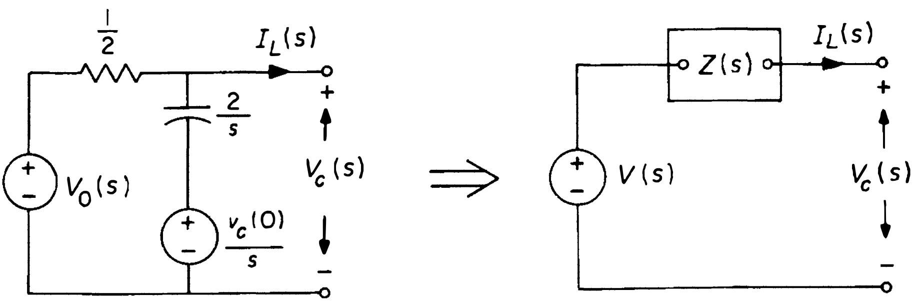


Figure 2.5–6. Thévenin equivalent for part of the circuit of Figure 2.5–5.

If we assume an open circuit (that is, $I_L(s) = 0$), we have by superposition and use of the voltage-divider formula,

$$\begin{aligned} V_C(s) &= \frac{2/s}{(1/2)+(2/s)} V_0(s) + \frac{1/2}{(1/2)+(2/s)} \frac{v_C(0)}{s} \\ &= \frac{4V_0(s)}{s+4} + \frac{v_C(0)}{s+4} = V(s) \end{aligned}$$

where $V(s)$ is the Thévenin voltage source. To find the Thévenin impedance $Z(s)$, set the sources $V_0(s)$ and $v_C(0)/s$ to zero and compute the parallel combination of the two impedances $1/2$ and $2/s$:

$$Z(s) = \frac{(1/2)\cdot(2/s)}{(1/2)+(2/s)} = \frac{2}{s+4}.$$

The equivalent circuit now appears as in Figure 2.5–7.

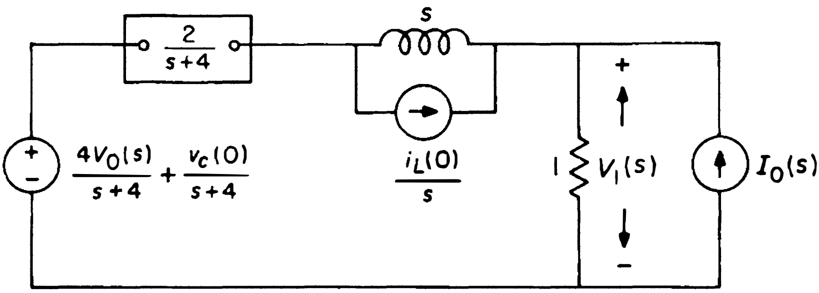


Figure 2.5–7. Equivalent circuit after Thévenin replacement of left part.

Again using superposition, divider formulas, etc., we can write $V_1(s)$ by inspection:

$$V_1(s) \;=\; \underbrace{\frac{1}{1+s+\dfrac{2}{s+4}}}_{\substack{\text{voltage}\\ \text{divider}}}\left[\frac{v_C(0)}{s+4}+\frac{4V_0(s)}{s+4}\right] + \underbrace{\frac{s\cdot 1}{1+s+\dfrac{2}{s+4}}}_{\substack{\text{current}\\ \text{divider}}}\frac{i_L(0)}{s} + \underbrace{\frac{\left(s+\dfrac{2}{s+4}\right)\cdot 1}{1+s+\dfrac{2}{s+4}}}_{\substack{\text{parallel}\\ \text{impedances}}} I_0(s)$$

or

$$\begin{aligned} V_1(s) \;=\; & \frac{s^2+4s+2}{s^2+5s+6} I_0(s) \;+\; \frac{4}{s^2+5s+6} V_0(s) \\ & + \frac{s}{s^2+5s+6}\frac{v_C(0)}{s} \;+\; \frac{s(s+4)}{s^2+5s+6}\frac{i_L(0)}{s} \end{aligned}$$

which is exactly the same as before.

▶ ▶ ▶

2.6 Summary

Because of the biunique relationship between a time function and its Laplace transform, one can characterize the input-output behavior of a system either in the time domain—by describing how to find the output time function given the input time function and appropriate information about the initial state—or in the frequency domain—by describing how to find the transform of the output time function given the transform of the input time function and appropriate state information. For LTI circuits, the time-domain description is in terms of differential equations, but the frequency-domain description leads to linear algebraic equations that are much easier to manipulate. Moreover, applying Laplace transforms to the constitutive relations for LTI elements and to Kirchhoff's Laws leads to the idea of replacing the time-domain circuit with a frequency-domain circuit in which the elements are replaced by their impedances. Solving for desired branch variables then reduces to an elementary problem in resistive circuit theory.

The Laplace transform is a very powerful "crank" for "turning out" in this way solutions to transient circuit problems, as shown in Examples 2.5–1, 2, 3. But the value of impedance methods and the frequency domain goes well beyond their usefulness in solving transient circuit problems, as we shall begin to see in the next chapter.

APPENDIX TO CHAPTER 2

Table II.1—Short Table of Unilateral $\mathcal{L}$-Transforms

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt$$

$x(t) = \mathcal{L}^{-1}[X(s)]$		$X(s) = \mathcal{L}[x(t)]$
$\delta(t)$*	$\Longleftrightarrow$	1
$u(t) = 1$	$\Longleftrightarrow$	$\frac{1}{s}$
$e^{-\alpha t}$	$\Longleftrightarrow$	$\frac{1}{s+\alpha}$
t^n	$\Longleftrightarrow$	$\frac{n!}{s^{n+1}}$
$t^n e^{-\alpha t}$	$\Longleftrightarrow$	$\frac{n!}{(s+\alpha)^{n+1}}$
$\sin \omega_0 t$	$\Longleftrightarrow$	$\frac{\omega_0}{s^2+\omega_0^2}$
$\cos \omega_0 t$	$\Longleftrightarrow$	$\frac{s}{s^2+\omega_0^2}$
$e^{-\alpha t}\cos \omega_0 t$	$\Longleftrightarrow$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$

Note: $x(t)$ is defined by $\mathcal{L}^{-1}[X(s)]$ for $t \geq 0$ only.

Table II.2—Important Unilateral $\mathcal{L}$-Transform Theorems

Linearity	$ax_1(t) + bx_2(t)$	$\Longleftrightarrow$	$aX_1(s) + bX_2(s)$
Delay	$x(t-T)u(t-T)$	$\Longleftrightarrow$	$X(s)e^{-sT}, \quad T > 0$
Time Multiplication	$tx(t)$	$\Longleftrightarrow$	$-\frac{dX(s)}{ds}$
$e^{-\alpha t}$ Multiplication	$e^{-\alpha t}x(t)$	$\Longleftrightarrow$	$X(s+\alpha)$
Scaling	$x(at)$	$\Longleftrightarrow$	$\frac{1}{a}X\left[\frac{s}{a}\right], \quad a > 0$
Differentiation	$\frac{dx(t)}{dt}$	$\Longleftrightarrow$	$sX(s) - x(0)$
Integration	$\int_0^t x(\tau)\,d\tau$	$\Longleftrightarrow$	$\frac{X(s)}{s}$
Initial-Value	$x(0)$	$=$	$\lim_{s\to\infty} sX(s)$
Final-Value†	$x(\infty)$	$=$	$\lim_{s\to 0} sX(s)$
Convolution‡	$\int_0^t x_1(\tau)x_2(t-\tau)\,d\tau$	$\Longleftrightarrow$	$X_1(s)X_2(s)$

*See Chapter 11 †Provided $sX(s)$ has no poles in $\Re e[s] \geq 0$ ‡See Chapter 10

EXERCISES FOR CHAPTER 2

Exercise 2.1

Show that the $\mathcal{L}$-transforms of each of the following time functions are as given. Sketch each time function and show on a diagram the locations of all finite poles and zeros of the transform. Indicate by shading on this diagram the region of absolute convergence. Throughout, assume that α and ω_0 are positive quantities.

	$\underline{x(t),\ t>0}$		$\underline{X(s)}$
a)	$1-e^{-\alpha t}$	$\Longleftrightarrow$	$\dfrac{\alpha}{s(s+\alpha)},\quad \Re e[s]>0$
b)	$e^{+\alpha t}\sin\omega_0 t$	$\Longleftrightarrow$	$\dfrac{\omega_0}{(s-\alpha)^2+\omega_0^2},\quad \Re e[s]>\alpha$
c)	$e^{-\alpha t}\cos(\omega_0 t+\pi/4)$	$\Longleftrightarrow$	$\dfrac{s+\alpha-\omega_0}{\sqrt{2}[(s+\alpha)^2+\omega_0^2]},\quad \Re e[s]>-\alpha$
d)	$te^{-\alpha t}\cos\omega_0 t$	$\Longleftrightarrow$	$\dfrac{(s+\alpha)^2-\omega_0^2}{[(s+\alpha)^2+\omega_0^2]^2},\quad \Re e[s]>-\alpha$
e)	$\begin{cases}1, & 1<t<2\\ 0, & \text{elsewhere}\end{cases}$	$\Longleftrightarrow$	$\dfrac{1}{s}\left[e^{-s}-e^{-2s}\right],\quad \text{all } s$
f)	$\begin{cases}t, & 0<t<1\\ 2-t, & 1<t<2\\ 0, & \text{elsewhere}\end{cases}$	$\Longleftrightarrow$	$\left(\dfrac{1-e^{-s}}{s}\right)^2,\quad \text{all } s$
g)	$\begin{cases}1, & -1<t<0\\ 0, & \text{elsewhere}\end{cases}$	$\Longleftrightarrow$	0
h)	$\cosh\alpha t$	$\Longleftrightarrow$	$\dfrac{s}{s^2-\alpha^2},\quad \Re e[s]>\alpha$
i)	$e^{-\alpha t}u(t-1)$	$\Longleftrightarrow$	$\dfrac{e^{-(s+\alpha)}}{s+\alpha},\quad \Re e[s]>-\alpha$
j)	$1-(1+\alpha t)e^{-\alpha t}$	$\Longleftrightarrow$	$\dfrac{\alpha^2}{s(s+\alpha)^2},\quad \Re e[s]>0$

Exercise 2.2

Show that the inverse unilateral $\mathcal{L}$-transforms of the following functions of s are as given. Sketch each time function and show on a diagram the locations of all finite poles and zeros of the transform. Indicate by shading on this diagram the region of absolute convergence.

	$\underline{X(s)}$		$\underline{x(t),\ t>0}$
a)	$\dfrac{1}{(s+1)(s+2)}$	$\Longleftrightarrow$	$e^{-t}-e^{-2t}$
b)	$\dfrac{s+3}{(s+1)(s+2)}$	$\Longleftrightarrow$	$2e^{-t}-e^{-2t}$

(Exercise 2.2 continued on the next page)

Exercise 2.2 (cont.)

	$X(s)$		$x(t),\quad t>0$
c)	$\dfrac{1}{s^2-\alpha^2}$	$\Longleftrightarrow$	$\dfrac{1}{\alpha}\sinh \alpha t$
d)	$\dfrac{1}{(s+\alpha)^2+\omega_0^2}$	$\Longleftrightarrow$	$\dfrac{1}{\omega_0}e^{-\alpha t}\sin \omega_0 t$
e)	$\dfrac{1}{s}(1-e^{-s})^2$	$\Longleftrightarrow$	$\begin{cases} 1, & 0<t<1 \\ -1, & 1<t<2 \\ 0, & \text{elsewhere} \end{cases}$
f)	$\dfrac{1}{s^2+1}\left(1-e^{-2\pi s}\right)$	$\Longleftrightarrow$	$\begin{cases} \sin t, & 0<t<2\pi \\ 0, & \text{elsewhere} \end{cases}$
g)	$\dfrac{s+2}{(s+1)^2}$	$\Longleftrightarrow$	$(1+t)e^{-t}$
h)	$\dfrac{1}{s^2(s-1)}$	$\Longleftrightarrow$	$e^t-(1+t)$
i)	$\dfrac{s}{s^2+2s+2}$	$\Longleftrightarrow$	$\sqrt{2}e^{-t}\cos(t+\pi/4)$
j)	$\dfrac{1}{s^3+2s^2+2s+1}$	$\Longleftrightarrow$	$e^{-t}+\dfrac{2}{\sqrt{3}}e^{-t/2}\cos\left(\dfrac{\sqrt{3}t}{2}-\dfrac{5\pi}{6}\right)$

Exercise 2.3

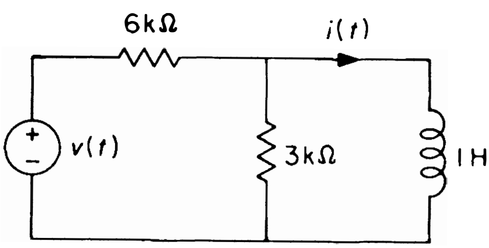
The state equation describing a system is

$$\frac{d\lambda(t)}{dt} = -2\lambda(t) + x(t).$$

Show that the response of this system to an input $x(t) = u(t-1)$ is described by

$$\lambda(t) = \frac{1}{2}\left(1-e^{-2(t-1)}\right)u(t-1).$$

Exercise 2.4



Show that the current in the circuit above is

$$i(t) = 2e^{-10^3 t} - e^{-2\times 10^3 t}\,\text{milliamp},\quad t \ge 0$$

if it is known that $v(t) = 6e^{-10^3 t}$ volt, $t>0$, and $i(0) = 1$ mamp. Sketch $i(t)$.

PROBLEMS FOR CHAPTER 2

Problem 2.1

a) Use the basic definition of the unilateral $\mathcal{L}$-transform and integration by parts to derive the Integration Theorem of Table II.2:

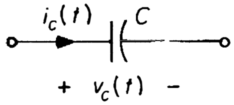
$$\mathcal{L}\left[\int_0^t x(\tau)\,d\tau\right] = \frac{1}{s}\mathcal{L}[x(t)].$$

b) Derive the same result from the Differentiation Theorem of Section 2.5 by setting

$$y(t) = \frac{dx(t)}{dt}, \quad x(t) = \int_0^t y(\tau)\,d\tau + x(0).$$

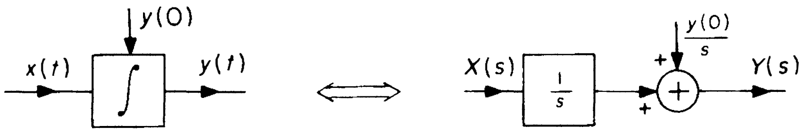
c) Use the Integration Theorem to devise an impedance representation for a capacitor described by the constitutive relation

$$v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)\,d\tau + v_C(0)$$



and show that it is the same as that given in Figure 2.5–1.

d) Use the Differentiation or Integration Theorem to show that an integrator has the frequency-domain representation shown below.



$$x(t) = \frac{dy(t)}{dt} \qquad Y(s) = \frac{X(s)}{s} + \frac{y(0)}{s}$$

or

$$y(t) = y(0) + \int_0^t x(\tau)\,d\tau$$

What are the corresponding frequency-domain representations for adders and gain elements as used in block diagrams such as Problem 1.5? Draw a frequency-domain transformation of the block diagram in Problem 1.5.

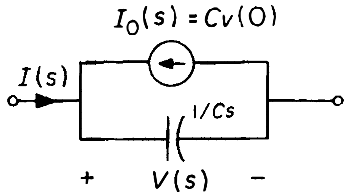
Problem 2.2

Inverse transforming the frequency-domain representation for the capacitor given in Figure 2.5–1 suggests that a capacitor initially charged to a voltage $v(0)$ at $t = 0$ should be indistinguishable in any circuit for $t > 0$ from an initially uncharged capacitor in series with a battery $v(0)$. Do you believe this? Analyze several simple circuit situations to convince yourself that this equivalence is reasonable.

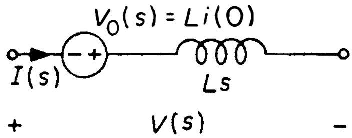
Problem 2.3

a) Argue that the frequency-domain descriptions of capacitors and inductors correspond to the equivalent circuits below.

$$I(s) = CsV(s) - \underbrace{Cv(0)}_{I_0(s)} \implies$$



$$V(s) = LsI(s) - \underbrace{Li(0)}_{V_0(s)} \implies$$



In these circuits note that $I_0(s) = Cv(0)$ (= constant) must not be thought of as a constant current source—it corresponds to a current source whose transform is a constant, and that is a very different thing. (A time function whose transform is a constant is called an *impulse*; we shall return to this topic in much detail in Chapter 11.)

b) Show that the circuits given in (a) are in impedance terms the Thévenin or Norton equivalents of the circuits given in Figure 2.5–1.

Problem 2.4

Two interesting theorems about $\mathcal{L}$-transforms are the Initial and Final Value Theorems:

INITIAL VALUE THEOREM:

If both $x(t)$ and $\dfrac{dx(t)}{dt}$ are $\mathcal{L}$-transformable and if $\lim_{s\to\infty} s\mathcal{L}[x(t)]$ exists, then

$$\lim_{s\to\infty} s\mathcal{L}[x(t)] = x(0).$$

FINAL VALUE THEOREM:

If both $x(t)$ and $\dfrac{dx(t)}{dt}$ are $\mathcal{L}$-transformable and if $s\mathcal{L}[x(t)]$ has no poles on the $j\omega$-axis or in the right half-plane, then

$$\lim_{s\to 0} s\mathcal{L}[x(t)] = \lim_{t\to\infty} x(t).$$

These theorems are easily proved from the Differentiation Theorem,

$$s\mathcal{L}[x(t)] = \mathcal{L}\left[\frac{dx(t)}{dt}\right] + x(0).$$

Thus as $s \to \infty$ with $\Re e[s] > 0$ (in the region of absolute convergence)

$$\lim_{s\to\infty} \mathcal{L}\left[\frac{dx(t)}{dt}\right] = \lim_{s\to\infty} \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt = \int_0^\infty \frac{dx(t)}{dt} \lim_{s\to\infty} e^{-st}\, dt = 0$$

which gives the first theorem. On the other hand as $s \to 0$ with $\Re e[s] \geq 0$ (in the region of absolute convergence)

$$\lim_{s\to 0} \mathcal{L}\left[\frac{dx(t)}{dt}\right] = \lim_{s\to 0} \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt = \int_0^\infty \frac{dx(t)}{dt} \lim_{s\to 0} e^{-st}\, dt$$
$$= \int_0^\infty \frac{dx(t)}{dt}\, dt = x(\infty) - x(0)$$

which gives the second. (For a further discussion of the Initial Value Theorem see Problem 11.15.)

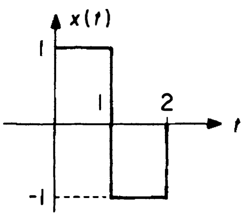
a) Apply these theorems to each of the following functions $X(s)$ to find (where possible) $x(0)$ and $\lim_{t\to\infty} x(t)$:

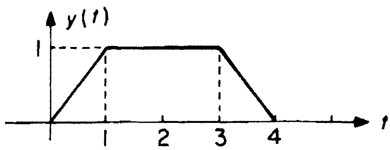
1. $\dfrac{1 - e^{-sT}}{s}$
2. $\dfrac{1}{s} e^{-sT}$
3. $\dfrac{1}{s(s+1)^2}$
4. $\dfrac{1}{s^2(s+1)}$
5. $\dfrac{s}{s^2+1}$
6. $\dfrac{(s+1)^2 - 1}{[(s+1)^2+1]^2}$

b) In each case find $x(t)$ and show that the results of the theorems agree.

Problem 2.5

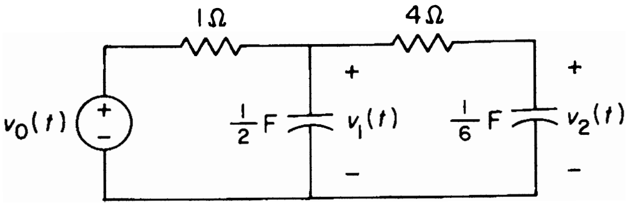
a) Find the Laplace transform of each of the time functions shown below. (HINT: It may sometimes be easier to find the transform of the derivative of the given function and use the Integration Theorem to find the desired transform.)





b) Show that your results are consistent with the Initial and Final Value Theorems of Problem 2.4.

Problem 2.6



a) Use impedance methods to find $V_2(s) = \mathcal{L}[v_2(t)]$ in terms of $V_0(s) = \mathcal{L}[v_0(t)]$, $v_1(0)$, and $v_2(0)$.

b) Use the Differentiation and Initial Value Theorems to evaluate $\dot{v}_2(0) = \left.\frac{dv_2(t)}{dt}\right|_{t=0}$ from the answer to (a). Express your answer in terms of $v_1(0)$ and $v_2(0)$. Show that the result does not depend on $v_0(t)$ as long as $v_0(0)$ is finite.

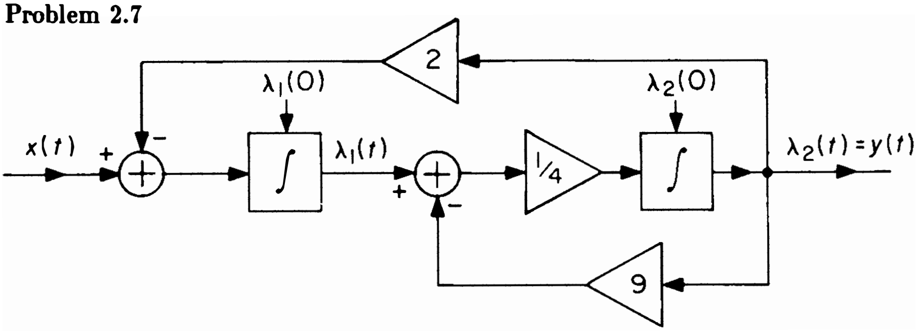
c) Check your answer to (b) directly from the circuit. (HINT: How is the current in the 4 Ω resistor related to $\dot{v}_2(t)$?)

d) Find by any means the input-output differential equation relating $v_2(t)$ and $v_0(t)$. Transform this equation to check the result obtained in (a). (HINT: To transform the second-order equation you may find it useful to derive first the general relation

$$\mathcal{L}\left[\frac{d^2x(t)}{dt^2}\right] = s^2X(s) - sx(0) - \dot{x}(0).\ )$$

e) Find $v_2(t)$, $t > 0$, if $v_0(t) = e^{-2t}$ volt, $v_2(0) = 1$ volt, $\dot{v}_2(0) = -2$ volts/sec.

Problem 2.7



The block diagram above describes the setup of an analog computer for solving a particular problem.

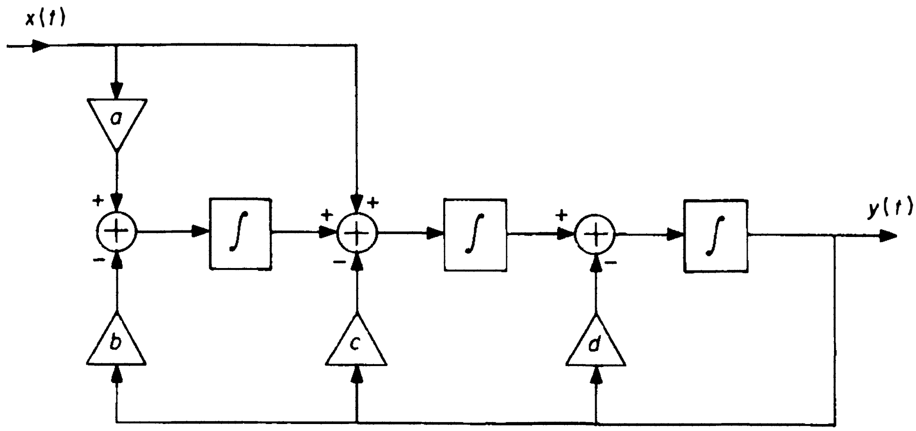
a) Write the dynamic equations for this system in state form.

b) Transform and solve these equations to obtain an expression for $Y(s)$ in terms of $X(s)$, $\lambda_1(0)$, and $\lambda_2(0)$.

c) Find $y(t)$, $t > 0$, for $x(t) = 2$, $t > 0$, $\lambda_1(0) = 4$, and $\lambda_2(0) = -0.5$.

d) Repeat (c) for the same initial state and the input $x(t) = 7e^{-2t}$, $t > 0$.

Problem 2.8



Determine how the transform $Y(s)$ of $y(t)$ is related to the transform $X(s)$ of $x(t)$ if the system above is in the zero state at $t = 0$.

Problem 2.9

Stated loosely, the *Maximum Principle of Pontryagin* says that to take a system from one state to another in the shortest possible time, subject to constraints on the magnitude of certain variables, one should operate continuously at the extremes, shifting from one extreme condition to another in a systematic way dependent on the initial and final states. For example, to take your car from rest at one place to rest at another in the shortest possible time you should (obviously) apply maximum acceleration up to the last possible moment such that maximum braking will just bring you to a screeching halt at the desired spot. Control systems of this sort are often rather picturesquely called "bang-bang" systems. As an example, consider a second-order system characterized by state variables $\lambda_1(t)$ and $\lambda_2(t)$ satisfying the state equations

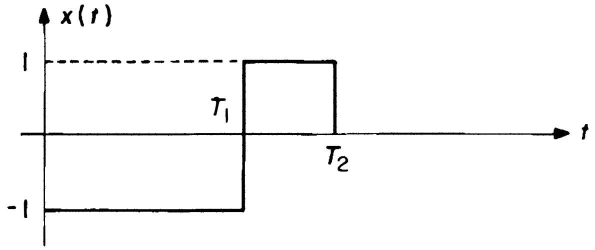
$$\frac{d\lambda_1(t)}{dt} = \lambda_2(t)$$
$$\frac{d\lambda_2(t)}{dt} = -3\lambda_2(t) - 2\lambda_1(t) + x(t)$$
$$y(t) = \lambda_1(t)$$

where $x(t)$ and $y(t)$ are the input and output, respectively.

a) Using $\mathcal{L}$-transforms, solve for $Y(s) = \Lambda_1(s) = \mathcal{L}[\lambda_1(t)]$ and $\Lambda_2(s) = \mathcal{L}[\lambda_2(t)]$ in terms of $X(s) = \mathcal{L}[x(t)]$, $\lambda_1(0)$, and $\lambda_2(0)$.

b) Consider the particular initial state $\lambda_1(0) = 1$, $\lambda_2(0) = 2$. Solve for $y(t)$ if $x(t) = 0$, $t > 0$. Under these conditions, how long would it take the system to come essentially to rest? For example, how long would it take for both the state variables to become less than 10% of their initial values?

c) Show that the system can be brought to rest at time T_2 (that is, $y(t) \equiv 0$, all $t > T_2$) by the input shown in the figure if the values of T_1 and T_2 are properly chosen. Find T_1 and T_2, and compare the resulting value of T_2 with the time to come essentially to rest when $x(t) = 0$, $t > 0$.



Problem 2.10

The useful application domain for $\mathcal{L}$-transforms is normally considered to be restricted to LTI systems. But the solution of some time-varying differential equations is also simpler in the frequency domain. An example is *Bessel's equation*

$$t\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + tx(t) = 0\,.$$

a) Use the Differentiation and Multiply-by-t Theorems to transform this equation into a first-order differential equation in $X(s) = \mathcal{L}[x(t)]$.

b) The equation in $X(s)$ can be integrated by separation of variables. Show (by substitution) that a solution is

$$X(s) = \frac{C}{\sqrt{s^2+1}}$$

where C is an arbitrary constant.

c) The zero-order Bessel function $x(t) = J_0(t)$ is a solution of the original equation. Indeed, the irrational function obtained in (b)—with an appropriate choice of C—is $\mathcal{L}[J_0(t)]$. $J_0(0)$ is normally defined to be 1; what is the appropriate choice of C?

Problem 2.11

This problem explores features of a model for freeway traffic. Experimentation indicates that the acceleration of a car in heavy traffic depends mostly on its velocity relative to the car immediately in front of it. Specifically, assume that

$$\frac{dv_1(t)}{dt} = k[v_0(t) - v_1(t)]$$

where $v_0(t)$ = velocity of leading car
$v_1(t)$ = velocity of following car
k = constant measuring the sensitivity of the driver of the following car. (Typically $0.2 < k < 0.8$ sec^{-1}.)

a) Determine the $\mathcal{L}$-transform ratio $\dfrac{V_1(s)}{V_0(s)}$ assuming $v_1(0) = 0$.

b) Assume that both cars are initally at rest. Let $v_0(t) = t$, $t > 0$. Find $v_1(t)$. Sketch $v_1(t)$ for $k = 1$.

c) Consider now a string of similar cars following one another in single file with velocities $v_0(t)$, $v_1(t)$, $v_2(t)$, etc. Assume also a more realistic relationship between the velocities of the n^{th} and $(n-1)^{\text{st}}$ cars:

$$\frac{dv_n(t)}{dt} = k[v_{n-1}(t-T) - v_n(t-T)]$$

where the reaction time, T, typically is 1-2 seconds in practice. Let the velocity of the lead car be

$$v_0(t) = \cos 2\pi f_0 t + V$$

where V is the common average velocity with which the whole file moves. Derive an expression for the sinusoidal steady-state velocity component of the n^{th} car.

d) For the conditions of (c), show that your result implies

$$|v_n(t) - V| \leq \frac{1}{\left|1 + \dfrac{j2\pi f_0}{k} e^{j2\pi f_0 T}\right|^n}.$$

Show from this that the amplitude of the velocity variations of the n^{th} car decreases with increasing n for all f_0 if and only if $kT < 0.5$ (a condition that requires the drivers to react reasonably fast but not to be unduly sensitive). What would be the consequences of $kT > 0.5$?