8

THE UNILATERAL Z-TRANSFORM AND ITS APPLICATIONS

8.0 Introduction

$\mathcal{L}$-transforms and system functions provided the most convenient method for finding the zero state response of continuous-time systems. A similar technique is applicable to discrete-time systems. The discrete-time exponential function, z^n, plays the role of the kernel, e^{st}, and the Z-transform replaces the $\mathcal{L}$-transform. Our development of these topics will be both brief and restricted in scope, but it will provide an orderly method for solving linear time-invariant difference equations, as well as introducing such useful notions as the discrete-time *system function* and the characterization of discrete-time systems in terms of pole-zero locations in the complex z-plane.

8.1 The Z-Transform

The (unilateral) *Z-transform* of a sequence $x[n]$ is defined by the formula

$$\tilde{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n} . \tag{8.1-1}$$

If $|x[n]|$ grows no faster than exponentially, this series will converge for all z outside some circle in the complex z-plane whose radius r_0 is called the *radius of convergence* (see Figure 8.1-1). As in the case of the $\mathcal{L}$-transform, the usefulness of the Z-transform depends on the fact that the relationship between $\tilde{X}(z)$ and the sequence $x[n]$ is *biunique*—to each $x[n]$ defined for $n \geq 0$ there corresponds one and only one $\tilde{X}(z)$ defined for $|z| > r_0$, and vice versa. For Z-transforms,

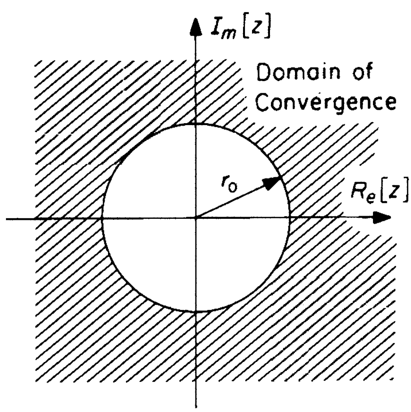


Figure 8.1-1. Typical domain of convergence.

this uniqueness theorem is basically a reinterpretation of the central theorem concerning the uniqueness and convergence of power-series expansions of analytic functions of a complex variable.*

Example 8.1–1

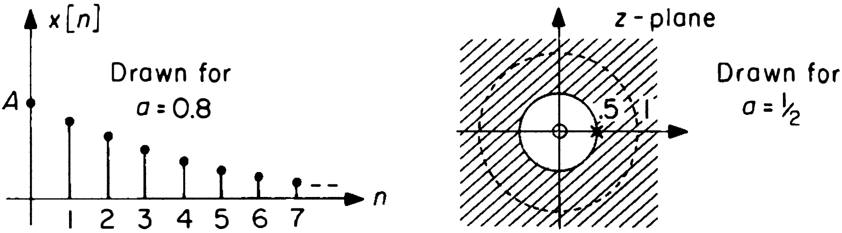


Figure 8.1–2. A DT exponential function and its domain of convergence.

Suppose that $x[n]$ is a DT exponential function

$$x[n] = Aa^n, \quad n \geq 0 \tag{8.1–2}$$

as shown in Figure 8.1–2. Then

$$\tilde{X}(z) = A \sum_{n=0}^{\infty} a^n z^{-n}$$

which converges to

$$\tilde{X}(z) = \frac{A}{1 - az^{-1}} \tag{8.1–3}$$

if $|az^{-1}| < 1$ or $|z| > |a|$. These formulas remain valid if a is complex. We note that $\tilde{X}(z)$ has a zero at $z = 0$ and a pole at $z = a$ on the circle bounding the region of convergence.

One important special case results if $a = 1$ so that $x[n] = A =$ constant, $n \geq 0$. The pole of $\tilde{X}(z)$ is now located at $z = 1$. This situation is illustrated in Figure 8.1–3.

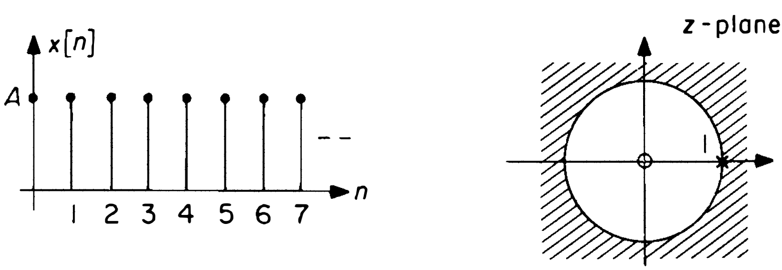


Figure 8.1–3. Z-transform of $x[n] = A$, $n \geq 0$.

*See, e.g., E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering* (New York, NY: Prentice-Hall, 1976).

Another interesting situation arises if a is negative, since then the signs of successive values of $x[n]$ alternate. In this case the pole is on the negative real axis. This situation is illustrated in Figure 8.1–4.

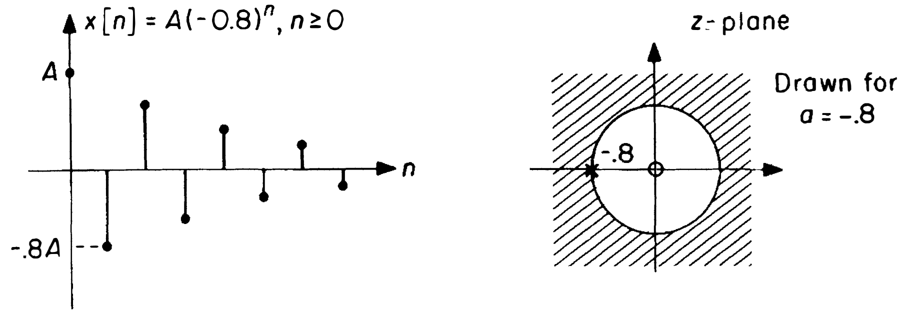


Figure 8.1–4. Z-transform of $x[n] = A(-0.8)^n$, $n \geq 0$.

▶▶▶

Example 8.1–2

Consider, for $|z| > 1/2$,

$$\tilde{X}(z) = \frac{30z^2}{6z^2 - z - 1} = \frac{5}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}.$$

By the uniqueness theorem this should correspond to a unique $x[n]$, $n \geq 0$. How can we carry out the inverse Z-transformation? One way to find $x[n]$, $n \geq 0$, is to expand $\tilde{X}(z)$ in a power series in z^{-1}. This can be accomplished, for example, by long division:

$$\begin{array}{r|llll}
 & 5 + \frac{5}{6}z^{-1} + \frac{35}{36}z^{-2} + \cdots & & & \\
\hline
1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2} & 5 & & & \\
 & 5 & -\frac{5}{6}z^{-1} & -\frac{5}{6}z^{-2} & \\
\cline{2-4}
 & & \frac{5}{6}z^{-1} & +\frac{5}{6}z^{-2} & \\
 & & \frac{5}{6}z^{-1} & -\frac{5}{36}z^{-2} & -\frac{5}{36}z^{-3} \\
\cline{3-5}
 & & & \frac{35}{36}z^{-2} & +\frac{5}{36}z^{-3} \\
 & & & \frac{35}{36}z^{-2} & -\frac{35}{216}z^{-3} - \frac{35}{216}z^{-4} \\
\cline{4-5}
 & & & & \cdots
\end{array}$$

(Note that both the numerator and the denominator of $\tilde{X}(z)$ are written as series of descending powers of z.) Thus for sufficiently large $|z|$ (in fact, for $|z| > 0.5$) we can write

$$\tilde{X}(z) = 5 + \frac{5}{6}z^{-1} + \frac{35}{36}z^{-2} + \cdots .$$

Since in general $\tilde{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$, we conclude that

$$x[0] = 5,$$
$$x[1] = \frac{5}{6},$$
$$x[2] = \frac{35}{36},$$
$$\text{etc.}$$

Although some variant of this procedure will always work to recover to $x[n]$ from $\tilde{X}(z)$, it is obviously clumsy. A more powerful technique parallels the partial-fraction method for $\mathcal{L}$-transforms. Thus we may write

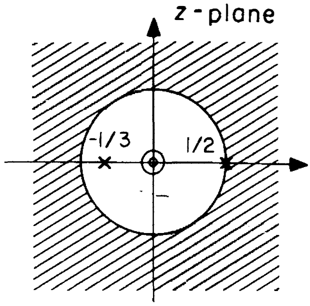


Figure 8.1–5. Pole-zero plot.

$$\tilde{X}(z) = \frac{30z^2}{6z^2 - z - 1} = \frac{5}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$$
$$= \frac{5}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}$$
$$= \frac{3}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 + \frac{1}{3}z^{-1}}$$

where the coefficients of each fraction are obtained as before, that is,

$$\left.\frac{5\cancel{\left(1 - \frac{1}{2}z^{-1}\right)}}{\cancel{\left(1 - \frac{1}{2}z^{-1}\right)}\left(1 + \frac{1}{3}z^{-1}\right)}\right|_{z^{-1}=2} = 3; \qquad \left.\frac{5\cancel{\left(1 + \frac{1}{3}z^{-1}\right)}}{\left(1 - \frac{1}{2}z^{-1}\right)\cancel{\left(1 + \frac{1}{3}z^{-1}\right)}}\right|_{z^{-1}=-3} = 2.$$

Then, since the Z-transform is a linear operation, we conclude from uniqueness and Example 8.1–1 that

$$x[n] = 3\left(\frac{1}{2}\right)^n + 2\left(-\frac{1}{3}\right)^n, \quad n \geq 0.$$

This formula checks our preceding results for $n = 0$, 1, and 2 but is clearly much more effective than the power-series method if we are interested in values of $x[n]$ for n much greater than 2. Notice the special way in which $\tilde{X}(z)$ is written in terms of negative powers of z with the constant term in the denominator equal to 1. Note also the particular form of the partial-fraction expansion, which is chosen so that terms of the form $1/(1 - az^{-1})$ can be recognized as corresponding to the sequence a^n, $n \geq 0$. ▶▶▶

Example 8.1–3

The partial-fraction expansion procedure of the preceding example apparently fails if the numerator of $\tilde{X}(z)$ is of equal or higher degree in z^{-1} than the denominator. Consider, for example,

$$\tilde{X}_1(z) = \frac{11 - z^{-1} - z^{-2}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{11 - z^{-1} - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}.$$

If we attempt as above to write (incorrectly, as we shall see)

$$\tilde{X}_1(z) = \frac{k_1}{1 - \frac{1}{2}z^{-1}} + \frac{k_2}{1 + \frac{1}{3}z^{-1}}$$

with

$$k_1 = \left.\frac{\left(11 - z^{-1} - z^{-2}\right)\cancel{\left(1 - \frac{1}{2}z^{-1}\right)}}{\cancel{\left(1 - \frac{1}{2}z^{-1}\right)}\left(1 + \frac{1}{3}z^{-1}\right)}\right|_{z^{-1}=+2} = 3$$

$$k_2 = \left.\frac{\left(11 - z^{-1} - z^{-2}\right)\cancel{\left(1 + \frac{1}{3}z^{-1}\right)}}{\left(1 - \frac{1}{2}z^{-1}\right)\cancel{\left(1 + \frac{1}{3}z^{-1}\right)}}\right|_{z^{-1}=-3} = 2$$

we obtain the same coefficients as in Example 8.1–2, which corresponds to the partial-fraction expansion of $\dfrac{5}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$ rather than the expression we sought to describe.*

One clue to the difficulty is that each of the terms in the attempted expansion vanishes as $z^{-1} \to \infty$, whereas the given $\tilde{X}_1(z) \to 6$ as $z^{-1} \to \infty$. Indeed,

$$\tilde{X}_1(z) = \frac{11 - z^{-1} - z^{-2}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{5}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} + 6$$

so we could write (correctly)

$$\tilde{X}_1(z) = \frac{3}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 + \frac{1}{3}z^{-1}} + 6.$$

In general, we can obtain an expansion of this kind by first dividing the denominator into the numerator, reducing the degree in z^{-1} of the remainder until it is less than the degree of the denominator. Thus if we seek an expansion of

$$\tilde{X}_2(z) = \frac{1 + z^{-1} + z^{-2}}{1 - z^{-1}}$$

*This same difficulty arises, of course, if we attempt to use partial fractions to take the inverse transform of an improper $\mathcal{L}$-transform. The correct partial-fraction expansion is readily obtained in that case as described here. The interpretation of the results, however, requires special techniques, as we shall explain in Chapter 11.

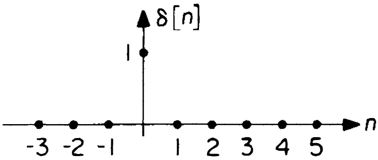
we divide

$$
\begin{array}{r|l}
 & -z^{-1}-2 \\
\cline{2-2}
-z^{-1}+1 & z^{-2}+\ z^{-1}+1 \\
 & \underline{z^{-2}-\ z^{-1}} \\
 & \phantom{z^{-2}+}\ 2z^{-1}+1 \\
 & \phantom{z^{-2}+}\ \underline{2z^{-1}-2} \\
 & \phantom{z^{-2}+2z^{-1}-}\ 3
\end{array}
$$

to obtain the expansion

$$\tilde{X}_2(z) = -z^{-1} - 2 + \frac{3}{1 - z^{-1}}\,.$$

The general result is thus a polynomial in z^{-1} plus a proper fraction in z^{-1} that can be expanded in partial fractions in the ordinary way.



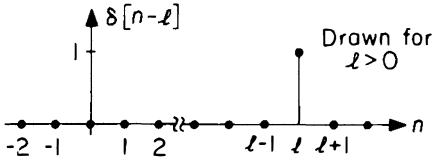


Figure 8.1–6. Unit sample function. **Figure 8.1–7.** Delayed sample function.

It should be evident from the basic definition of the Z-transform (8.1–1) that the inverse transform of $k_\ell z^{-\ell}$ is a DT function $f[n]$ that is zero for all n except $n = \ell$, at which point $f[\ell] = k_\ell$. It is more convenient, however, to introduce a special function, the *unit sample function* $\delta[n]$, defined* by

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0\,. \end{cases} \tag{8.1–4}$$

The Z-transform of $\delta[n]$ is obviously

$$\delta[n] \Longleftrightarrow 1\,. \tag{8.1–5}$$

The *delayed* unit sample function,† $\delta[n-\ell]$, is defined by

$$\delta[n-\ell] = \begin{cases} 1, & n = \ell \\ 0, & n \neq \ell \end{cases} \tag{8.1–6}$$

and has the Z-transform

$$\delta[n-\ell] \Longleftrightarrow z^{-\ell}. \tag{8.1–7}$$

We can write the inverse transforms of $\tilde{X}_1(z)$ and $\tilde{X}_2(z)$ in terms of unit sample functions since

$$\tilde{X}_1(z) = \frac{11 - z^{-1} - z^{-2}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{3}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 + \frac{1}{3}z^{-1}} + 6$$

*Note that $\delta[n]$ is defined for *all* n, $-\infty < n < \infty$, not just for $n \geq 0$.

†The delayed unit sample function is often called *Kronecker's delta* and indicated by the notation $\delta_{n\ell}$.

so that for $n \geq 0$,

$$x_1[n] = 3\left(\frac{1}{2}\right)^n + 2\left(-\frac{1}{3}\right)^n + 6\delta[n]$$
$$= \begin{cases} 11, & n = 0 \\ 3\left(\frac{1}{2}\right)^n + 2\left(-\frac{1}{3}\right)^n, & n > 0 \end{cases}$$

and since

$$\tilde{X}_2(z) = \frac{1 + z^{-1} + z^{-2}}{1 - z^{-1}} = -z^{-1} - 2 + \frac{3}{1 - z^{-1}}$$

so that for $n \geq 0$,

$$x_2[n] = -\delta[n-1] - 2\delta[n] + 3$$
$$= \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 3, & n > 1\,. \end{cases}$$

▸▸▸

As with $\mathcal{L}$-transforms, most applications of Z-transforms involve manipulation of a few basic transform pairs using a small number of properties and theorems. Some important theorems are:

SUPERPOSITION (LINEARITY):

$$ax[n] + by[n] \Longleftrightarrow a\tilde{X}(z) + b\tilde{Y}(z)\,. \tag{8.1–8}$$

MULTIPLICATION BY AN EXPONENTIAL:

$$a^n x[n] \Longleftrightarrow \tilde{X}(a^{-1}z)\,. \tag{8.1–9}$$

MULTIPLICATION BY n:

$$nx[n] \Longleftrightarrow -z\frac{d\tilde{X}(z)}{dz}\,. \tag{8.1–10}$$

DELAY BY $N \geq 0$:

$$x[n-N]u[n-N] \Longleftrightarrow z^{-N}\tilde{X}(z)\,. \tag{8.1–11}$$

In the Delay Theorem, $u[n]$ is the discrete-time *unit step function*

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0\,. \end{cases} \tag{8.1–12}$$

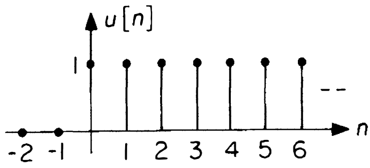


Figure 8.1–8. Unit step function.

For each theorem the proof follows almost immediately from the basic definition (8.1–1). A table of simple Z-transforms and basic Z-transform theorems is given in the appendix to this chapter.

Example 8.1–4

From Example 8.1–1 and the Multiplication-by-n Theorem, the Z-transform of

$$x[n] = na^n, \quad n \geq 0$$

is

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2}.$$

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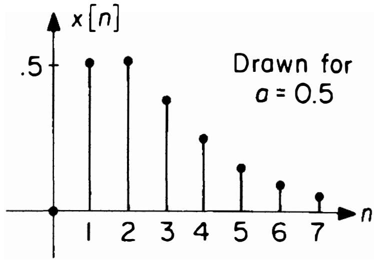


Figure 8.1–9. $x[n] = na^n$, $n \geq 0$.

Example 8.1–5

From Example 8.1–1 and the Delay Theorem, we have the Z-transform pair

$$a^{n-N}u[n-N] \Longleftrightarrow \frac{z^{-N}}{1-az^{-1}}.$$

In particular, the transform of the delayed unit step function is

$$u[n-N] \Longleftrightarrow \frac{z^{-N}}{1-z^{-1}}.$$

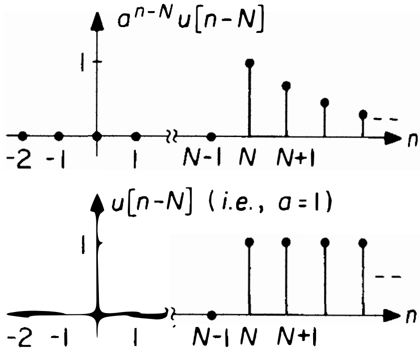


Figure 8.1–10. $a^{n-N}u[n-N]$.

As an application of this result, note that we may write the discrete-time pulse function

$$p_N[n] = \begin{cases} 1, & 0 \leq n < N \\ 0, & n \geq N \end{cases}$$

in the form

$$p_N[n] = u[n] - u[n-N].$$

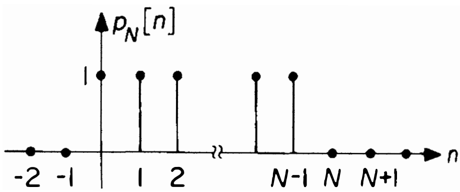


Figure 8.1–11. The function $p_N[n]$.

We may then apply the Linearity Theorem and the Delay Theorem to obtain

$$p_N[n] \Longleftrightarrow \frac{1-z^{-N}}{1-z^{-1}} = \tilde{P}_N(z)$$

which we recognize as precisely formula (7.3–4) for the partial sum of the geometric series,

$$\tilde{P}_N(z) \stackrel{\Delta}{=} \sum_{n=0}^{\infty} p_N[n]z^{-n} = \sum_{n=0}^{N-1} z^{-n}$$

which, of course, is also the result obtained if $\tilde{P}_N(z)$ were evaluated directly by (8.1–1).

▶▶▶

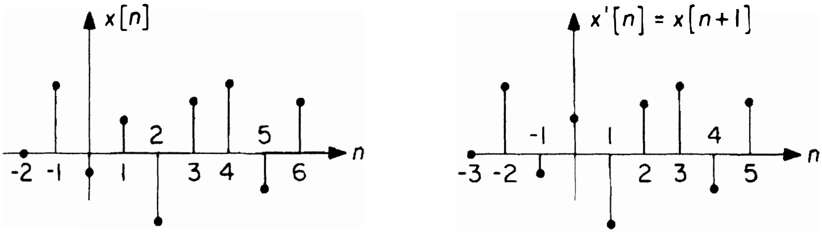
8.2 The Z-Transform Applied to LTI Discrete-Time Systems

To apply the Z-transform to the analysis of discrete-time systems, we need another theorem which plays much the same role for Z-transforms that the Differentiation Theorem does for $\mathcal{L}$-transforms:

FORWARD-SHIFT THEOREM:

If $\tilde{X}(z)$ is the (unilateral) Z-transform of $x[n]$,
then $z(\tilde{X}(z) - x[0])$ is the Z-transform of $x[n+1]$.

The proof is immediate from the basic definition of $\tilde{X}(z)$ and the pictures in Figure 8.2–1. The following examples illustrate the broad usefulness of this theorem.



$$\tilde{X}(z) = \sum_{0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$\tilde{X}'(z) = \sum_{0}^{\infty} x'[n]z^{-n} = \sum_{0}^{\infty} x[n+1]z^{-n} = x[1] + x[2]z^{-1} + x[3]z^{-2} + \cdots = z\big(\tilde{X}(z) - x[0]\big)$$

Figure 8.2–1. Illustration of the Forward-Shift Theorem.

Example 8.2–1

The mortgage problem of Example 7.1–1 led to the difference equation

$$P[n+1] = (1+r)P[n] - p, \quad n \geq 0 .$$

Taking the Z-transforms of both sides, using the Forward-Shift Theorem, yields

$$z(\tilde{P}(z) - P[0]) = (1+r)\tilde{P}(z) - \frac{p}{1-z^{-1}} .$$

Solving for $\tilde{P}(z)$ and expanding in partial fractions yields

$$\tilde{P}(z) = \frac{p/r}{1-z^{-1}} + \frac{P[0] - p/r}{1-(1+r)z^{-1}} .$$

Inverse transforming yields

$$P[n] = p/r + (P[0] - p/r)(1+r)^n, \quad n \geq 0$$

which is the result derived by induction in Example 7.2–1.
▶▶▶

Example 8.2–2

If $\tilde{X}(z)$ is the Z-transform of $x[n]$, then $z^2\tilde{X}(z) - z^2x[0] - zx[1]$ is the Z-transform of $x[n+2]$. This is easy to show, either directly or by considering $x[n+2]$ as the forward shift of $x[n+1]$ and applying the Forward-Shift Theorem twice to obtain $z\big(z[\tilde{X}(z) - x[0]] - x[1]\big)$. Extensions to still larger forward shifts proceed similarly. Such extensions permit Z-transforms to be applied directly to the solution of input-output LTI difference equations of arbitrary order. Thus, suppose we have a system described by

$$y[n+2] - \frac{1}{6}y[n+1] - \frac{1}{6}y[n] = 2x[n].$$

We seek the response to $x[n] = 1$, $n \geq 0$, with initial conditions $y[0] = 0$, $y[1] = 1$. Taking the Z-transforms of both sides, using the Forward-Shift Theorem and its extension, we find

$$\big(z^2\tilde{Y}(z) - z^2y[0] - zy[1]\big) - \frac{1}{6}\big(z\tilde{Y}(z) - zy[0]\big) - \frac{1}{6}\tilde{Y}(z) = 2\tilde{X}(z).$$

Substituting

$$\tilde{X}(z) = \frac{1}{1 - z^{-1}}$$

and inserting the given values for $y[0]$ and $y[1]$ yields

$$\begin{aligned}\tilde{Y}(z) &= \frac{z^{-1} + z^{-2}}{\left(1 - z^{-1}\right)\left(1 - \dfrac{z^{-1}}{6} - \dfrac{z^{-2}}{6}\right)} \\ &= \frac{3}{1 - z^{-1}} - \frac{3.6}{1 - \dfrac{1}{2}z^{-1}} + \frac{0.6}{1 + \dfrac{1}{3}z^{-1}}.\end{aligned}$$

Inverse transforming gives

$$y[n] = 3 - 3.6\left(\frac{1}{2}\right)^n + 0.6\left(-\frac{1}{3}\right)^n, \quad n \geq 0.$$

It is easy to check directly that this satisfies the difference equation and has the required values at $n = 0$ and $n = 1$.

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Transforming the input-output difference equation directly (as above) may lead to some difficulties of interpretation if the equation contains terms proportional to $x[n+1]$, $x[n+2]$, …, corresponding to shifted input sequences. In this case, knowledge of $y[0]$, $y[1]$, …, $y[N-1]$, together with $x[n]$, $n \geq 0$, determines a unique response, but the values of $y[0]$, $y[1]$, …, $y[N-1]$ do not define the state of the system at $n = N-1$. (It is necessary to know $x[0]$, $x[1]$, …, $x[N-1]$ as well.) The situation is precisely analogous to that discussed in Problem 3.3 for continuous-time systems.

8.3 Frequency-Domain Representations of Discrete-Time Systems

The description or analysis of CT LTI circuits or block diagrams in preceding chapters was much simplified by transforming from the time domain, in which elements are described by differential equations, to the frequency domain, in which elements are described by impedances or system functions. Structural constraints among subsystems (Kirchhoff's Laws, cascade connections, feedback, etc.) then lead to algebraic equations that can readily be solved to give system-function descriptions of the overall input-output ZSR behavior. And various attributes of system functions—particularly pole-zero locations—permit one to say a great deal about the general characteristics of system behavior even without explicitly solving for the response.

System functions and frequency-domain methods have similar advantages for discrete-time systems, as we can illustrate by developing a frequency-domain form of the delay-adder-gain diagrams of Section 7.2. To begin, recall that the unit delay element was defined in Section 7.2 by the difference equation

$$y[n+1] = x[n]\,. \tag{8.3–1}$$

Z-transforming, using the Forward-Shift Theorem, leads to an equivalent description:

$$z\Big(\tilde{Y}(z) - y[0]\Big) = \tilde{X}(z)$$

or

$$\tilde{Y}(z) = \frac{\tilde{X}(z)}{z} + y[0]\,. \tag{8.3–2}$$

We may thus replace the delay block in block diagrams by the transform representation shown in Figure 8.3–1. Adder and gain elements transform into the frequency domain without alteration.

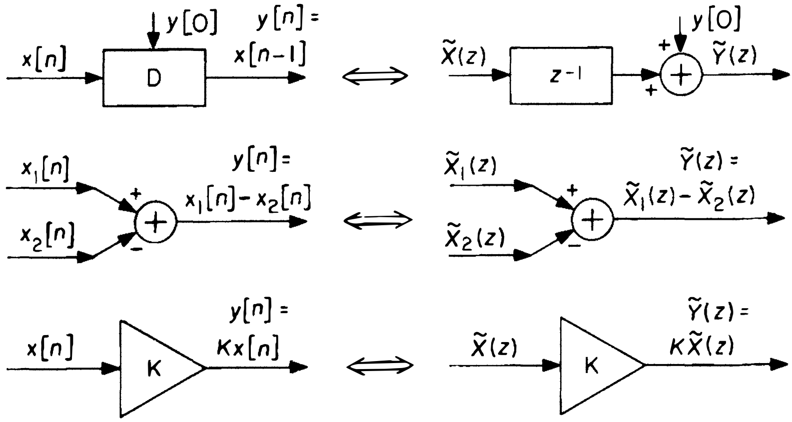


Figure 8.3–1. Frequency-domain representations of delay, adder, and gain elements.

Replacing each block as above, any delay-adder-gain block diagram becomes a frequency-domain representation of the DT system having precisely the same properties as frequency-domain representations of CT systems. Some of the principal consequences are illustrated in the following example.

Example 8.3–1

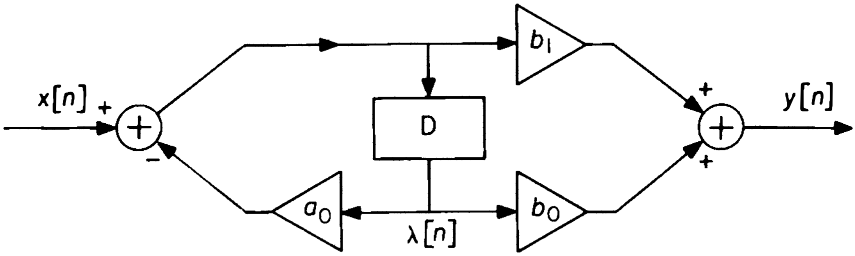


Figure 8.3–2. 1^{st}-order system for Example 8.3–1.

The block diagram of Figure 8.3–2 is the form taken by the general canonical block diagram of Figure 7.2–7 in the 1^{st}-order case. It is equivalent to the difference equation

$$y[n+1] + a_0 y[n] = b_1 x[n+1] + b_0 x[n] .$$

Replacing the blocks and variables by their frequency-domain equivalents yields the block diagram of Figure 8.3–3.

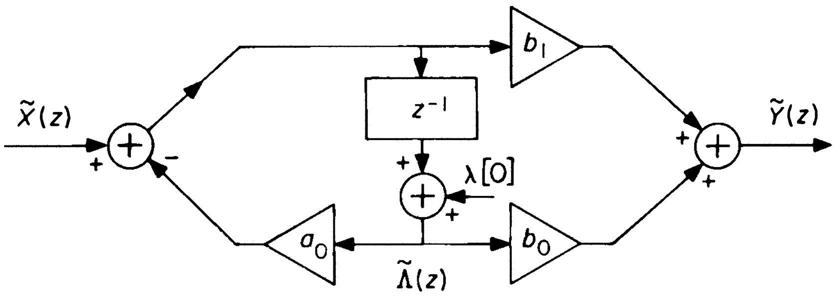


Figure 8.3–3. Frequency-domain equivalent of Figure 8.3–2.

Using superposition and Black's formula for the feedback loop, we obtain, almost by inspection,

$$\tilde{\Lambda}(z) = \frac{z^{-1}\tilde{X}(z)}{1+a_0 z^{-1}} + \frac{\lambda[0]}{1+a_0 z^{-1}}$$

and

$$\tilde{Y}(z) = b_0\tilde{\Lambda}(z) + b_1(\tilde{X}(z) - a_0\tilde{\Lambda}(z)) .$$

Eliminating $\tilde{\Lambda}(z)$ yields the input-output relation

$$\tilde{Y}(z) = \frac{b_0 + b_1 z}{a_0 + z}\tilde{X}(z) + \frac{(b_0 - b_1 a_0)z}{a_0 + z}\lambda[0]$$

which can, of course, be checked by applying the Forward-Shift Theorem directly to the difference equation, using the fact (derivable from Figure 8.3–2) that

$$y[0] - b_1 x[0] = (b_0 - b_1 a_0)\lambda[0]\,.$$

To obtain a complete solution for $y[n]$, we need only substitute an appropriate expression for $\tilde{X}(z)$ and inverse transform. This is most conveniently done for a numerical example. Thus suppose

$$a_0 = 0.5, \quad b_0 = 2, \quad b_1 = 2, \quad \lambda[0] = 1$$

and

$$x[n] = \left(\frac{1}{3}\right)^n, \quad n \geq 0\,.$$

Then

$$\tilde{X}(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}$$

and

$$\tilde{Y}(z) = \frac{2(1 + z^{-1})}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} + \frac{1}{1 + \frac{1}{2}z^{-1}}$$
$$= \frac{-0.2}{1 + \frac{1}{2}z^{-1}} + \frac{3.2}{1 - \frac{1}{3}z^{-1}}$$

so that

$$y[n] = -0.2\left(-\frac{1}{2}\right)^n + 3.2\left(\frac{1}{3}\right)^n, \quad n \geq 0\,.$$

▸▸▸

It should be reasonably obvious from this example that the following comments apply to DT linear time-invariant systems in general. In each case the close parallel between DT LTI system behavior and CT LTI system behavior should be carefully noted.

1. The total response time can be considered as the sum of the *zero state response* (ZSR)—the first term in the equation for $\tilde{Y}(z)$ in the example above—and the *zero input response* (ZIR)—the remaining term. The ZSR depends only on the input $x[n]$, $n \geq 0$; the ZIR depends only on the initial state, for example, the outputs of the delay elements at $n = 0$.

2. If there is more than one input, the ZSR is a superposition of terms describing the separate effects of each input. Each such term is a product of the Z-transform of that input and a *system function* $\tilde{H}_i(z)$ relating the i^{th} input to the output. Thus, in Example 8.3–1 the system function relating the output to the external input $x[n]$ is

$$\tilde{H}(z) = \frac{b_0 + b_1 z}{a_0 + z}.$$

The inverse transform of the product $\tilde{H}(z)\tilde{X}(z)$ is the ZSR response to the input $x[n]$.

3. The system function and the input-output difference equation imply one another through the replacement

$$z \Longleftrightarrow \text{forward shift.}$$

In general the finite-order DT LTI system described by the difference equation

$$\sum_{k=0}^{N} a_k y[n+k] = \sum_{\ell=0}^{N} b_\ell x[n+\ell] \tag{8.3–3}$$

is also described by the system function

$$\tilde{H}(z) = \frac{\sum_{\ell=0}^{N} b_\ell z^\ell}{\sum_{k=0}^{N} a_k z^k}. \tag{8.3–4}$$

Since $\tilde{H}(z)$ is a rational function of z, it is characterized (except for a multiplicative constant) by the locations of its *poles* and *zeros*.

4. The poles of $\tilde{H}(z)$ are also the roots of the characteristic equation describing the solutions of the homogeneous difference equation

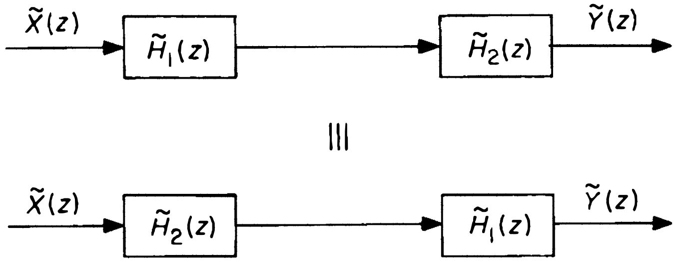
$$\sum_{k=0}^{N} a_k z^k = 0. \tag{8.3–5}$$

If the N roots of this equation, that is, the N poles of $\tilde{H}(z)$, are labelled z_1, z_2, ..., z_N, then the ZIR has the form (assuming no multiple-order roots)

$$y[n] = \sum_{k=1}^{N} A_k z_k^n \qquad \text{(ZIR)} \tag{8.3–6}$$

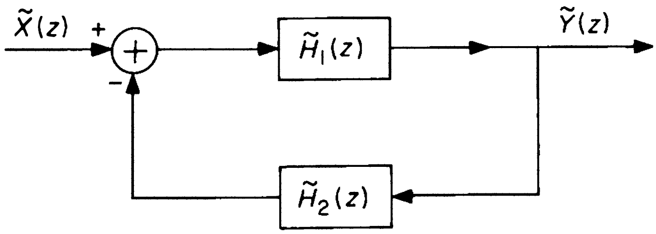
where the values of the N constants A_k depend on the initial state. If $|z_k| < 1$ for all poles, then the ZIR dies away and the system described by $\tilde{H}(z)$ is *stable* (in the input-output sense).

5. The domain of convergence of $\tilde{H}(z)$ is the region *outside* the smallest circle enclosing all the poles of $\tilde{H}(z)$. The system is stable if the domain of convergence includes the unit circle, $|z| = 1$.
6. DT LTI systems can be combined in cascade, in parallel, in feedback arrangements, etc., just as CT LTI systems can be. The rules for determining the combined system functions are the same for systems described by Z-transforms as for systems described by $\mathcal{L}$-transforms. For example, the system function of the cascade connection of two DT systems is the product of their individual system functions and is independent of the order of the cascade as shown in Figure 8.3–4. And Black's feedback formula applies to the arrangement shown in Figure 8.3–5.



$$\tilde{H}(z) = \frac{\tilde{Y}(z)}{\tilde{X}(z)} \text{ (ZSR)} = \tilde{H}_1(z)\tilde{H}_2(z) = \tilde{H}_2(z)\tilde{H}_1(z)$$

Figure 8.3–4. Cascade connection of DT systems.



$$\tilde{H}(z) = \frac{\tilde{Y}(z)}{\tilde{X}(z)} \text{ (ZSR)} = \frac{\tilde{H}_1(z)}{1 + \tilde{H}_1(z)\tilde{H}_2(z)}$$

Figure 8.3–5. Feedback connection of DT systems.

Example 8.3–2

To illustrate further some of these features of the frequency-domain description of

DT system behavior, consider once again the numerical integration of the equations describing the circuit of Examples 1.3–3 and 7.1–2, redrawn in Figure 8.3–6.

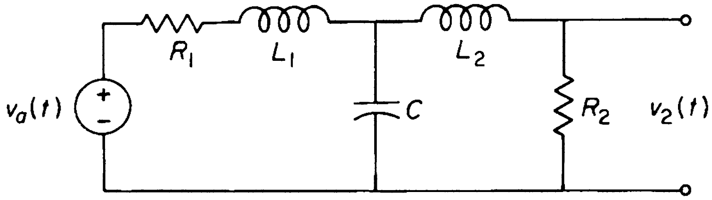


Figure 8.3–6. Circuit of Example 1.3–3.

In Example 7.2–1 we derived block diagrams both for the CT state equations of this circuit and for a set of DT state equations that were approximately equivalent. The diagrams differed only in that each integrator in the CT diagram was replaced by the cascade of a gain element, Δt, and an accumulator in the DT version. In the frequency domain, the corresponding representations are shown in Figure 8.3–7; the representation of the accumulator follows directly from the Forward-Shift Theorem applied to the defining difference equation. Hence the frequency-domain diagrams for the DT and CT state equations are identical under ZSR conditions, except for the replacement of $1/s$ by $\Delta t/(z-1)$ in each integrator-accumulator block. Thus, any system function relating two transforms in the DT diagram will be identical with the corresponding system function in the CT diagram provided that s is replaced by $(z-1)/\Delta t$ wherever it appears. In particular we readily conclude from impedance methods

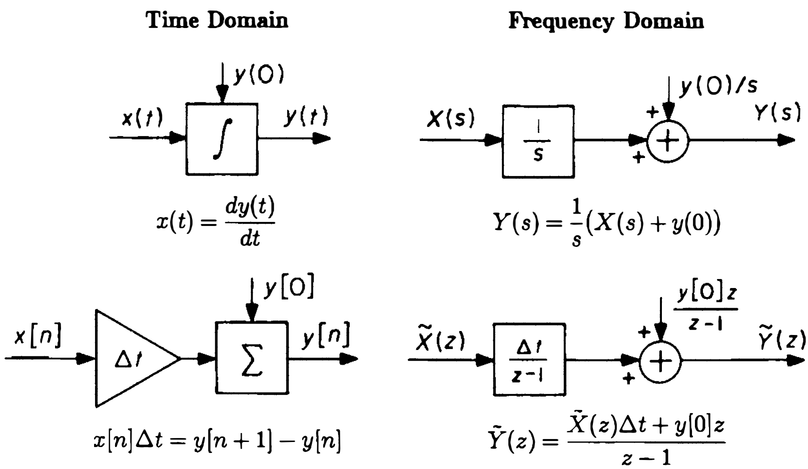


Figure 8.3–7. Integrators and accumulators in the time and frequency domains.

applied to the circuit diagram above that the input-output CT system function is

$$H(s) = \frac{V_2(s)}{V_a(s)} \quad \text{(ZSR)}$$

$$= \frac{\dfrac{R_2}{R_2 + L_2 s} \dfrac{(R_2 + L_2 s)\dfrac{1}{Cs}}{R_2 + L_2 s + \dfrac{1}{Cs}}}{\dfrac{(R_2 + L_2 s)\dfrac{1}{Cs}}{R_2 + L_2 s + \dfrac{1}{Cs}} + R_1 + L_1 s}$$

$$= \frac{0.5}{\left(\frac{s}{10^4} + 1\right)\left(\left(\frac{s}{10^4}\right)^2 + \left(\frac{s}{10^4}\right) + 1\right)} \tag{8.3–7}$$

where we have substituted the element values given in Example 7.1–2. The pole locations are thus as shown in Figure 8.3–8. Obviously, the CT circuit is stable; its slowest normal modes have a decay time constant of 2×10^{-4} sec. Following the scheme suggested above, we need only replace s wherever it appears in (8.3–7) by $(z-1)/\Delta t$ to obtain the input-output DT system function

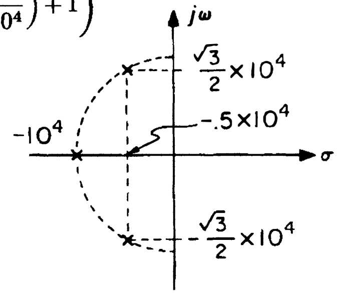


Figure 8.3–8. Pole locations for $H(s)$.

$$\tilde{H}(z) = \frac{\tilde{V}_2(z)}{\tilde{V}_a(z)} \quad \text{(ZSR)} = H\left(\frac{z-1}{\Delta t}\right)$$

$$= \frac{0.5}{\left(\frac{z-1}{10^4\Delta t} + 1\right)\left(\left(\frac{z-1}{10^4\Delta t}\right)^2 + \left(\frac{z-1}{10^4\Delta t}\right) + 1\right)} \tag{8.3–8}$$

The poles of $H(s)$ are located at

$$s = -10^4, \quad -0.5\times10^4 \pm j\frac{\sqrt{3}}{2}\times10^4$$

Consequently the poles of $\tilde{H}(z)$ are located at

$$\frac{z-1}{\Delta t} = -10^4, \quad -0.5\times10^4 \pm j\frac{\sqrt{3}}{2}\times10^4$$

or

$$z = 1 - 10^4\Delta t, \quad 1 - 0.5\times10^4\Delta t \pm j\frac{\sqrt{3}}{2}\times10^4\Delta t.$$

The pole locations thus depend on Δt, as shown in Figure 8.3–9.

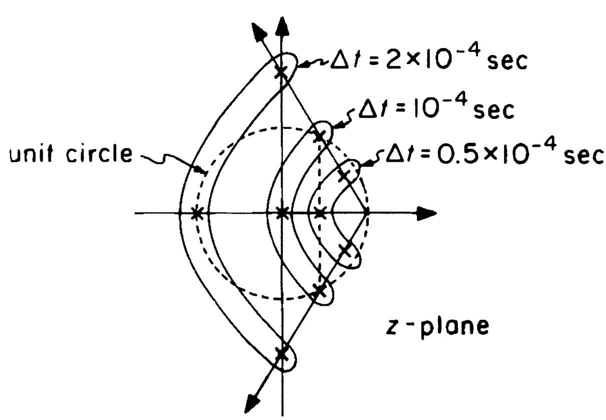


Figure 8.3–9. Pole locations for $\tilde{H}(z)$.

We saw in Example 7.3–2 that the DT system accurately describes the behavior of the CT system for $\Delta t \ll 10^{-4}$ sec. For very small Δt, the geometrical relationship of the poles of $\tilde{H}(z)$ to the point $z = 1$ and the circle $|z| = 1$ is similar to the relationship of the poles of $H(s)$ to the point $s = 0$ and the line $s = j\omega$. Such a similarity is in fact necessary and sufficient for the step responses of the DT and CT systems to be similar, as is discussed more fully in Problem 8.9. For larger Δt, the similarity deteriorates, and for $\Delta t > 10^{-4}$ sec the poles of $\tilde{H}(z)$ have magnitude greater than 1 so that the DT "approximation" actually becomes unstable. (The step response of the DT system with $\Delta t = 1.11 \times 10^{-4}$ sec is shown in Figure 8.3–10. Compare this with the step response for small Δt derived in Example 7.3–2.) As stated in Example 7.1–2, this numerical instability is a result of the choice of the simple forward Euler algorithm to characterize the discrete approximation to the integrator; the effect of other choices is discussed in Problem 8.7.

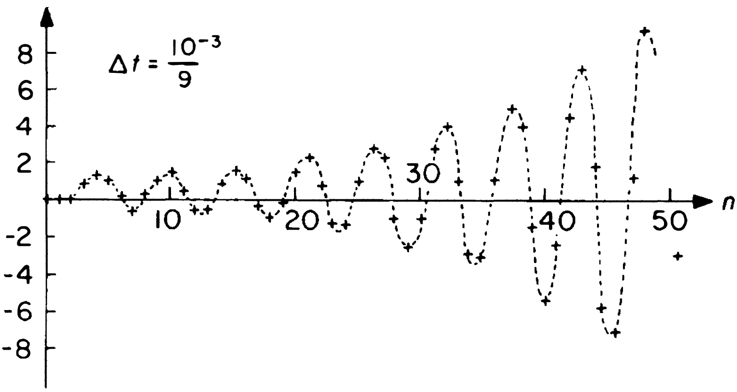


Figure 8.3–10. Step response of $\tilde{H}(z)$ for $\Delta t = 1.11 \times 10^{-4}$ sec.

▶▶▶

8.4 Summary

The unilateral Z-transform

$$\tilde{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

plays the same role for DT systems that the unilateral $\mathcal{L}$-transform plays for CT systems:

a) Because the relationship between $\tilde{X}(z)$ and $x[n]$, $n \geq 0$, is biunique, power-series or partial-fraction expansions may be used to rewrite $\tilde{X}(z)$ in a form from which $x[n]$ is evident.

b) A variety of theorems simplify the manipulation and derivation of transforms and their inverses.

c) The Forward-Shift Theorem reduces the analysis of DT systems characterized by difference equations or block diagrams to an algebraic process.

d) The total response of LTI DT systems can be considered as the sum of a ZSR and a ZIR. The Z-transform of the ZSR has the form

$$\tilde{Y}(z) = \tilde{H}(z)\tilde{X}(z)$$

where the system function, $\tilde{H}(z)$, for LTI systems of the type described in (c), is a rational function of z characterized by its poles and zeros. The poles also determine the form of the ZIR. The system is input-output stable if the poles lie inside the circle $|z| = 1$.

e) The input-output difference equation and the system function are closely related. An accumulator-adder-gain or delay-adder-gain block diagram is readily synthesized to correspond to any set of LTI difference equations or system functions.

The DT systems analyzed in the last two chapters are linear and time-invariant, but they are not the most general LTI discrete-time systems. The most general class requires extension to system functions that are not rational functions of z, and to a *convolution* rather than difference-equation characterization in the time domain. Such an extension is our goal in the next chapter.

APPENDIX TO CHAPTER 8

Table VIII.1—Short Table of Unilateral Z-Transforms

$$\tilde{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$x[n],\ n \geq 0$		$\tilde{X}(z)$
$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$	$\Longleftrightarrow$	1
$u[n-N] = \begin{cases} 1, & n \geq N \geq 0 \\ 0, & n < N \end{cases}$	$\Longleftrightarrow$	$\dfrac{z^{-N}}{1-z^{-1}}$
a^n	$\Longleftrightarrow$	$\dfrac{1}{1-az^{-1}}$
n	$\Longleftrightarrow$	$\dfrac{z^{-1}}{(1-z^{-1})^2}$
na^n	$\Longleftrightarrow$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$
$a^n \cos n\theta$	$\Longleftrightarrow$	$\dfrac{1-a\cos\theta z^{-1}}{1-2a\cos\theta z^{-1}+a^2z^{-2}}$
$a^n \sin n\theta$	$\Longleftrightarrow$	$\dfrac{a\sin\theta z^{-1}}{1-2a\cos\theta z^{-1}+a^2z^{-2}}$

Note: $x[n]$ is defined by $\tilde{X}(z)$ for $n \geq 0$ only.

Table VIII.2—Important Unilateral Z-Transform Theorems

Linearity	$ax[n]+by[n]$	$\Longleftrightarrow$	$a\tilde{X}(z)+b\tilde{Y}(z)$
Forward Shift	$x[n+1]$	$\Longleftrightarrow$	$z(\tilde{X}(z)-x[0])$
Delay	$x[n-N]u[n-N]$	$\Longleftrightarrow$	$z^{-N}\tilde{X}(z),\ N \geq 0$
Multiplication by a^n	$a^n x[n]$	$\Longleftrightarrow$	$\tilde{X}(a^{-1}z)$
Multiplication by n	$nx[n]$	$\Longleftrightarrow$	$-z\left(\dfrac{d\tilde{X}(z)}{dz}\right)$
Convolution*			
	$x[n]u[n] * h[n]u[n] = \sum_{m=0}^{n} x[m]h[n-m]$	$\Longleftrightarrow$	$\tilde{X}(z)\tilde{H}(z)$

*See Chapter 9.

EXERCISES FOR CHAPTER 8

Exercise 8.1

Derive the following Z-transform pairs, $\tilde{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$:

	$\underline{x[n],\ n \geq 0}$		$\underline{\tilde{X}(z)}$
a)	$2\delta[n-1] = \begin{cases} 2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$	$\Longleftrightarrow$	$2z^{-1}$
b)	$1 + \left(\frac{1}{2}\right)^n$	$\Longleftrightarrow$	$\dfrac{z(4z-3)}{2z^2 - 3z + 1}$
c)	$\left(\frac{1}{2}\right)^n \cos\frac{n\pi}{3}$	$\Longleftrightarrow$	$\dfrac{4 - z^{-1}}{4 - 2z^{-1} + z^{-2}}$
d)	n^2	$\Longleftrightarrow$	$\dfrac{z(z+1)}{(z-1)^3}$

Exercise 8.2

Complete the following table of unilateral Z-transforms by finding formulas for $x[n]$, $n \geq 0$, or $\tilde{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$ as required. For each of the pairs, sketch both $x[n]$, $n \geq 0$, and the pole-zero plot corresponding to $\tilde{X}(z)$.

	$\underline{x[n],\ n \geq 0}$		$\underline{\tilde{X}(z)}$
a)	2, all n	$\Longleftrightarrow$	?
b)	?	$\Longleftrightarrow$	$6z^{-1} - z^{-2}$
c)	$\cosh 2n$	$\Longleftrightarrow$	?
d)	$(n+1)3^{-n}$	$\Longleftrightarrow$	?
e)	?	$\Longleftrightarrow$	$\dfrac{1}{4z^2 - 1}$
f)	$x[n] = \begin{cases} 1, & n \leq 2 \\ (1/2)^{n-2}, & n > 2 \end{cases}$	$\Longleftrightarrow$	?

Answers: (a) $\dfrac{2z}{z-1}$ (b) $0, 6, -1, 0, 0, \ldots$ (c) $\dfrac{1 - (\cosh 2)z^{-1}}{1 - 2(\cosh 2)z^{-1} + z^{-2}}$

(d) $\dfrac{z^2}{(z - (1/3))^2}$ (e) $\begin{cases} 0, & n = 0,\ n \text{ odd} \\ (1/2)^n, & \text{otherwise} \end{cases}$ (f) $\dfrac{z^2 + (1/2)z + (1/2)}{z(z - (1/2))}$

PROBLEMS FOR CHAPTER 8

Problem 8.1

The delay-adder-gain synthesis schemes for discrete-time systems discussed in Section 7.2 are only a few of many ways to realize equivalent structures. Several other approaches are explored in this problem.

a) Determine a delay-adder-gain block diagram of the canonic type described in Section 7.2 for realizing the system function

$$\tilde{H}(z) = \frac{3 - 3z^{-1}}{1 + 0.5z^{-1} - 0.5z^{-2}}.$$

b) Develop an equivalent block diagram by first expanding $\tilde{H}(z)$ in partial fractions, then realizing each term separately as in Section 7.2, and finally connecting the separate parts in parallel (with adders) to realize $\tilde{H}(z)$.

c) Develop another equivalent block diagram by first writing $\tilde{H}(z)$ as a product of factors, that is,

$$\tilde{H}(z) = \left[\frac{K}{1 - \alpha z^{-1}}\right]\left[\frac{1 - \beta z^{-1}}{1 - \gamma z^{-1}}\right],$$

then realizing each factor separately as in Section 7.2, and finally connecting the separate parts in *cascade* to realize $\tilde{H}(z)$.

d) The example above had real poles and hence the gains required were all real. Suggest a modification of the procedures in (b) and (c) that will allow a realization of $\tilde{H}(z)$ having more than two, possibly *complex*, poles while still utilizing only amplifiers with *real* gains.

Problem 8.2

Each female in a certain rare species of insect lays eggs twice in her lifetime, one week apart, and then immediately dies. Careful experimental studies have uncovered the curious facts that all the females of this species lay their eggs on the same day of the week, Monday, and that each female lays precisely 80 eggs the first time and 500 eggs the second time. It has also been shown that 50% of the eggs hatch in a day or so (the remainder are eaten by turtles), and that half of these are females who reach maturity in time to lay eggs for the first time on Monday of the following week.

a) How many weeks does it take the population of mature female insects (ignore the males, who don't bite anyway) to increase by a factor of more than 10^6? (To be specific, make the count on Monday mornings.)

b) Suppose an insecticide (nonresidual, of course) is applied once a week (on Saturdays) and kills a fraction α of the insects who have just hatched. It has no effect on mature females who have already laid eggs once. How large must α be to hold the total population stationary?

Problem 8.3

A discrete-time system is described by the difference equation

$$y[n+2] = -y[n+1] + 2y[n] + x[n+2] + x[n+1]\,.$$

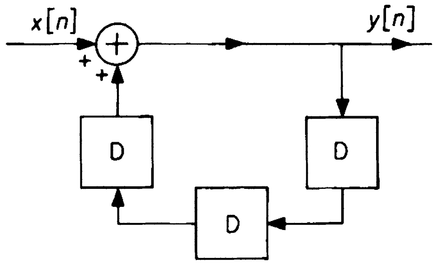
a) Find the system function $\tilde{H}(z)$ characterizing this system. Show its poles and zeros on a sketch of the z-plane.

b) Find the response of this system to the input

$$x[n] = 3^n u[n]\,.$$

c) Find the response of this system to the input $x[n] = 3^n$, $n \geq 0$, if $y[0] = 0$, $y[1] = 0$. Why is this not the same as the ZSR response to this same input found in (b)?

Problem 8.4

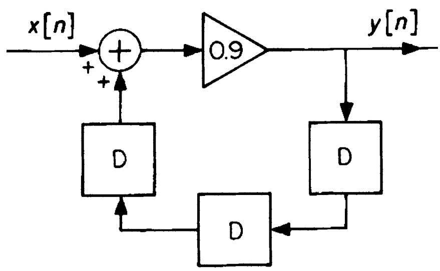


a) Find and sketch the response of the discrete-time system above to the input

$$x[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

b) For an input $x[n] = (1/2)^n u[n]$, find the output $y[n]$ for $n = 0, 1, 2, 3, 4, 5, 6$. Also find $y[n]$ for $n = 100, 101, 102, 103$. (Answers correct to 1% are acceptable.)

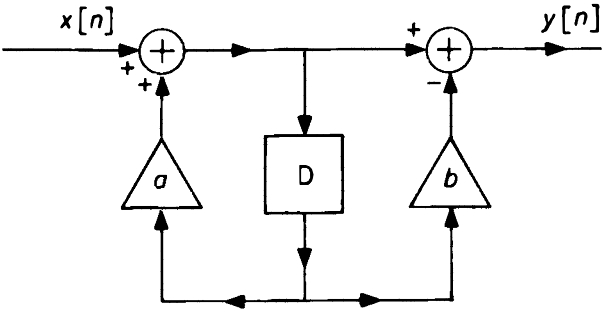
c) The system above is modified by inserting an amplifier with gain 0.9, as shown below. Find and sketch the response to the same input as in (a).



d) Write a difference equation relating $x[n]$ and $y[n]$ for the system of part (c).

e) Find the system function $\tilde{H}(z)$ and the transform of the output $\tilde{Y}(z)$ for the system of part (c) with the input of part (b).

Problem 8.5



a) What is the system function $\tilde{H}(z)$ of this discrete-time system?

b) Find by any means the response to the unit step, $x[n] = u[n]$.

Problem 8.6

a) Prove the following theorems for the (unilateral) Z-transform

INITIAL- AND FINAL-VALUE THEOREMS:

$$\text{i) } x[0] = \lim_{z\to\infty} \tilde{X}(z).$$

$$\text{ii) } x[\infty] = \lim_{z\to 1} (z-1)\tilde{X}(z) \quad \text{(if the limit exists).}$$

(HINT: (i) should give no trouble. (ii) follows by an argument that is essentially the discrete analog of the Final-Value Theorem for the $\mathcal{L}$-transformation. Argue first that the Z-transform of $x[n+1] - x[n]$ is

$$\lim_{N\to\infty} \sum_{n=0}^{N} [x[n+1] - x[n]]z^{-n} = z\tilde{X}(z) - zx[0] - \tilde{X}(z).$$

Taking the limit as $z \to 1$ on both sides and assuming orders of passing to the limit on the left may be interchanged, obtain

$$\lim_{z\to 1} (z-1)\tilde{X}(z) - x[0] = \lim_{N\to\infty} \sum_{n=0}^{N} [x[n+1] - x[n]].$$

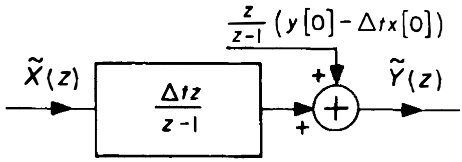
By writing out a few terms of the sum on the right, convince yourself that the limit (if it exists) is $x[\infty] - x[0]$, which gives the desired result.)

b) Test these theorems by applying them to all of the Z-transform pairs in the table in the appendix to this chapter.

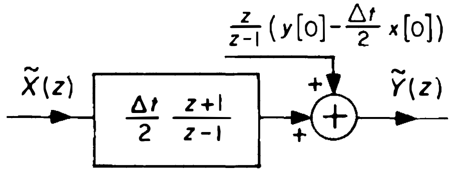
Problem 8.7

a) Section 7.1 and Problem 7.1 discuss several integration algorithms in addition to the forward Euler algorithm. Show that employing one of these other algorithms is equivalent to replacing the integrators in a block-diagram representation of the CT system with a variant of the DT accumulator whose frequency-domain description is one of the following:

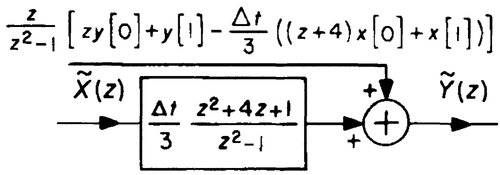
Backward Euler Algorithm



Trapezoid Rule



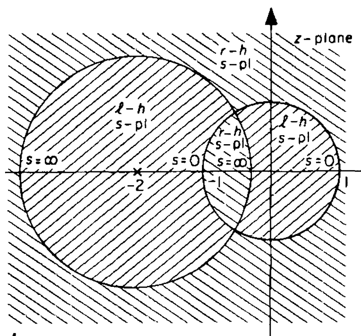
Simpson's Rule



b) Show that the difference equations derived in this way using the backward Euler or the trapezoid algorithms describe stable DT systems for any Δt and any stable CT system. (HINT: determine the region in the s plane corresponding to the region $|z| \leq 1$ if $s = 1/\tilde{H}_i(z)$ where $\tilde{H}_i(z)$ is the system function of the DT accumulator described by the algorithm. This is the region in which s-plane poles must lie if the DT approximation is to be stable. To determine these regions, exploit the fact that for mappings of this type circles (and straight lines, which are circles of infinite radius) map into circles or straight lines; three points determine a circle.)

c) Show that your results in (b) imply that the trapezoid rule has the important property that, for any value of Δt, the DT system is unstable if and only if the CT system from which it is derived is unstable.

d) If Simpson's rule is used in this way, each s-plane pole corresponds to two z-plane poles. Find formulas for the z poles in terms of the s pole. Show that one of these z poles always has $|z| \geq 1$ so that the DT system is always unstable, independent of the value of Δt or the nature of the CT system. Hence, Simpson's rule is not used for this purpose. (HINT: the left- and right-half s-planes map doubly into the z-plane regions shown to the right.)

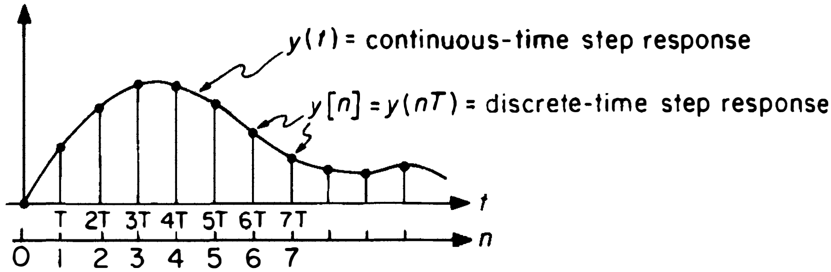


Problem 8.8

A useful technique for the design of DT systems is to choose $\tilde{H}(z)$ so that the response $y[n]$ to a DT step input, $x[n] = u[n]$, is the same as samples of the response $y(t)$ of some CT system $H(s)$ to a CT unit step, $x(t) = u(t)$. That is, we seek

$$y[n] = y(nT)$$

where T is some appropriate sampling interval. In this way, known desirable features of the CT system can be extended to the DT system. The relationship of $\tilde{H}(z)$ and $H(s)$ in this case is said to be a *step-invariant transformation.*



a) Derive the step-invariant $\tilde{H}(z)$ corresponding to $H(s) = \dfrac{1}{s+\alpha}$, $\alpha > 0$. Describe the locations of the poles and zeros of $\tilde{H}(z)$ as functions of α and T. Compare with a pole-zero plot of $H(s)$.

b) Draw a block-diagram realization of $\tilde{H}(z)$ using gain-adder-accumulator blocks as in Sections 7.2 and 8.3. Compare with a corresponding realization of $H(s)$ using gain-adder-integrator blocks.

c) Draw a block-diagram realization of $\tilde{H}(z)$ using gain-adder-delay blocks as in Sections 7.2 and 8.3.

d) Repeat (a) for $T = 0.02$ and a sharply resonant system, $H(s) = \dfrac{s}{s^2 + 2s + 101}$.

Problem 8.9

In the *sampled-data control system* shown in Figure 1 on the next page, the output $y(t)$ of a CT system $H(s)$ is sampled, some processing is done on the resulting sequence of samples, and the processed sequence $r[n]$ is converted back to a CT signal $r(t)$ that is subtracted from the control input $x(t)$ to generate the error signal $e(t)$, which becomes the input to $H(s)$. Usually the discrete-to-continuous converter (D/C) is a zero-order hold (see Figure 2 and Problem 7.4). Because of the effect of the clock on the converters, the system of Figures 1 and 2 is linear but time-varying. If, however, $x(t)$ has the same staircase character as $r(t)$ in Figure 2 (or if we are willing to approximate the actual smooth $x(t)$ by such a waveform), then it is easy to show that the system of Figure 1 is equivalent to the LTI DT system of Figure 3 (where $x[n]$ and $e[n]$ bear the same relationships to $x(t)$ and $e(t)$ that $r[n]$ bears to $r(t)$). Let $\tilde{H}(z)$ be the system function of the equivalent DT system in the dashed box of Figure 3.

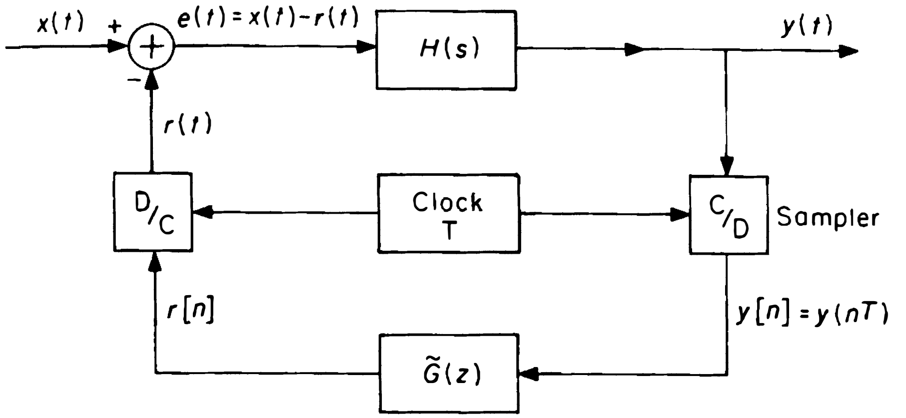


Figure 1.

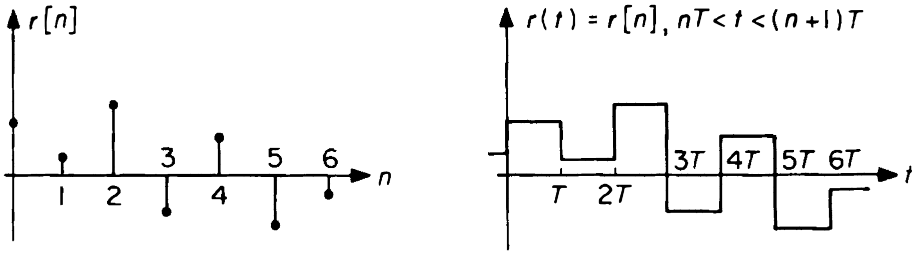


Figure 2.

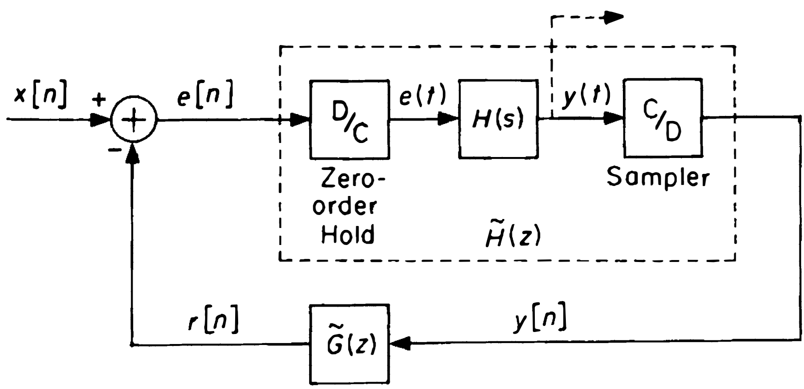


Figure 3.

a) Argue that $\tilde{H}(z)$ is related to $H(s)$ by a step-invariant transformation, that is, if $y_s(t)$ is the response of $H(s)$ to a CT unit step input, $e(t) = u(t)$, and if $y_s[n]$ is the response of $\tilde{H}(z)$ to a DT unit step, $e[n] = u[n]$, then $y_s[n] = y_s(nT)$, all n. See also Problem 8.8.

b) Suppose that $H(s)$ is unstable; specifically, suppose that

$$H(s) = \frac{1}{s-1}, \quad \Re e[s] > 1\,.$$

Show that

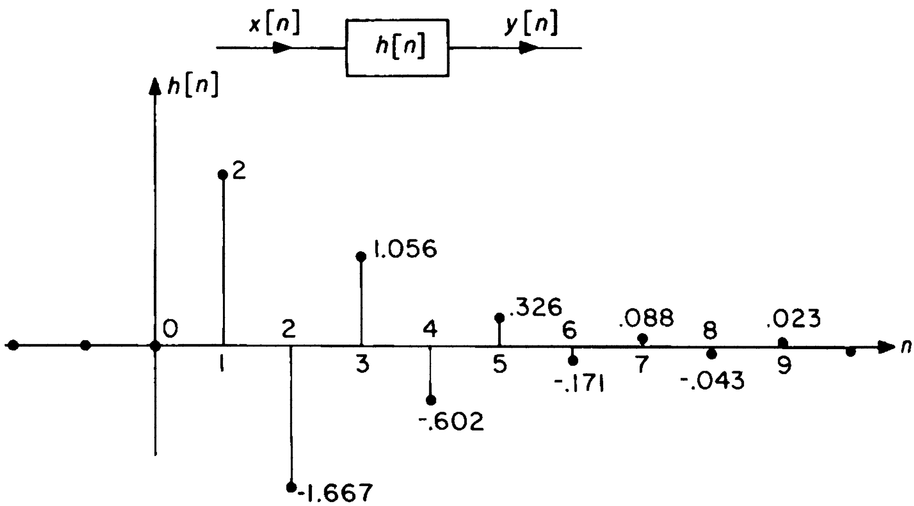
$$\tilde{H}(z) = \frac{(e^T - 1)z^{-1}}{1 - e^T z^{-1}}, \quad |z| > e^T$$

which is also unstable since $T > 0$.

c) Suppose that $H(s)$ is as in (b) and that $\tilde{G}(z) = K$. Find the range of values of K for which the closed-loop DT system of Figure 3 is stable.

Problem 8.10

a) The measured response $h[n]$ to a unit sample input $x[n] = \delta[n]$ of a certain DT system is shown in the figure below. Sketch the response $y[n]$ of this system to a unit step input $x[n] = u[n]$. Evaluate the response for $n \leq 8$.



b) A close fit to these observations is provided by the formula

$$h[n] = 12\left[\left(-\frac{1}{3}\right)^n - \left(-\frac{1}{2}\right)^n\right], \qquad n \geq 0.$$

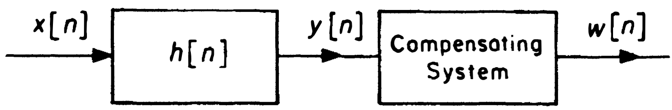
Find a closed-form expression for the system function $\tilde{H}(z)$ corresponding to this formula,

$$\tilde{H}(z) = \sum_{n=0}^{\infty} h[n]z^{-n}.$$

c) The overshoot in the step response of this system is troublesome in many applications. It is proposed to compensate the given system by cascading it (as shown below) with another system described by the difference equation

$$w[n] = ay[n] + by[n-1] + cy[n-2].$$

Find values of the constants a, b, and c such that the overall unit step response of the cascade is simply a delayed unit step, $u[n-1]$.



d) Devise a realization of the compensating system described by the difference equation in (c) in terms of delay lines, gain elements, and adders.

Problem 8.11

You are engaged in a game of "matching pennies" with your roommate. Each of you has a stack of pennies. You compare the pennies on the tops of the stacks. If they are both "heads" or both "tails" (i.e., if they "match"), you win and place both pennies on the bottom of your stack. Otherwise your roommate wins and places both pennies on the bottom of his or her stack. You then compare the next pennies in each stack. The game is over when either you or your roommate holds all the pennies.

Suppose at some point you have n pennies. At this point your roommate has $m = N - n$ pennies. Assume that N, the total number of pennies, is fixed. We seek to determine $p[n]$, the probability at this point that you will win the game. Evidently $p[0] = 0$ (you have lost) and $p[N] = 1$ (you have won). In general, $p[n]$ must satisfy the difference equation

$$p[n+1] = \frac{1}{2}p[n+2] + \frac{1}{2}p[n].$$

(Starting from the situation of having $n+1$ pennies, one either wins—with probability 0.5—and thus arrives at a point where one has $n+2$ pennies, or loses—with probability 0.5—and thus arrives at a point where one has n pennies.) Use Z-transform methods to solve this equation subject to the given boundary conditions, finding $p[n]$ for any n, $0 \leq n \leq N$.