

## 6.3100: Dynamic System Modeling and Control Design

### CT System Functions

merge with CT\_System\_Functions\_2

*October 09, 2024*

## Retrospective

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To date, we have used a number of different representations

- differential and difference equations
- block diagrams
- system (transfer) functions
- poles and zeros

to get a variety of perspectives on the control of systems.

These are most useful when you can readily translate between them.

## System Functions, Poles, and Natural Frequencies

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If a system contains only adders, gains, differentiators, and integrators, then its system function can be written as a **rational polynomial** in  $s$ :

$$H(s) = \frac{b_0 + b_1s + b_2s^2 + b_3s^3 + \dots}{a_0 + a_1s + a_2s^2 + a_3s^3 + \dots}$$

The numerator and denominator polynomials can be factored into products of terms: one for each **zero** in the numerator or **pole** in the denominator:

$$H(s) = K \frac{(s - z_0)(s - z_1)(s - z_2) \dots}{(s - p_0)(s - p_1)(s - p_2) \dots}$$

Contributions of each pole can be **isolated** by expanding the system function in partial fractions:

$$H(s) = \frac{c_0}{s - p_0} + \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \dots$$

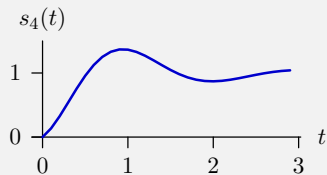
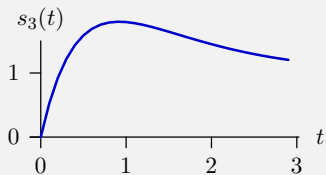
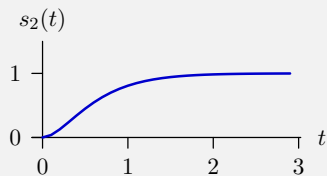
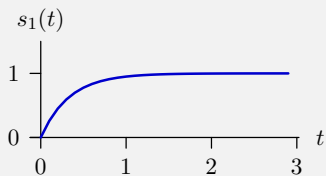
Each partial fraction term corresponds to a **first-order subsystem**:

$$H_i(s) = \frac{Y_i(s)}{X_i(s)} = \frac{c_i}{s - p_i}$$

with a **natural frequency** equal to the corresponding pole  $p_i$ .

## Check Yourself

Step responses of four systems that contain only adders, gains, differentiators, and integrators are shown below.

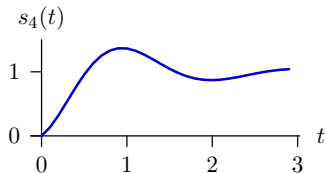
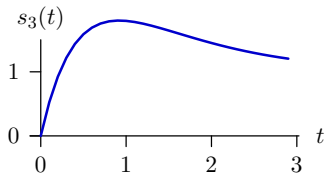
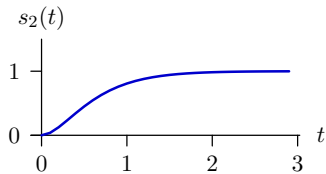
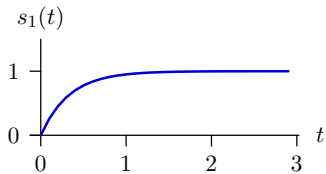


Which if any of the systems could have just one pole?  
Which must have  $> 1$  but no more than 2 poles (both real)?

## Check Yourself

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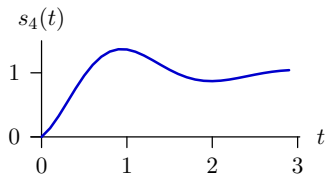
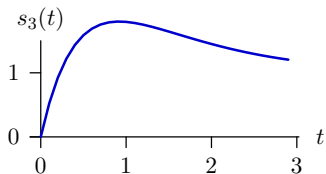
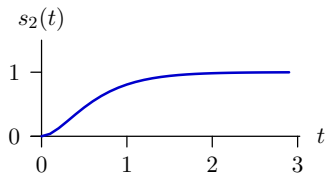
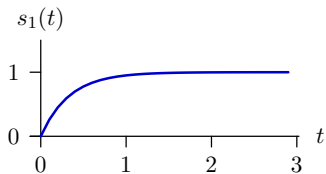
Only the first step response ( $s_1(t)$ ) has an exponential shape that corresponds to a single pole.



## Check Yourself

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Only the first step response ( $s_1(t)$ ) has an exponential shape that corresponds to a single pole.

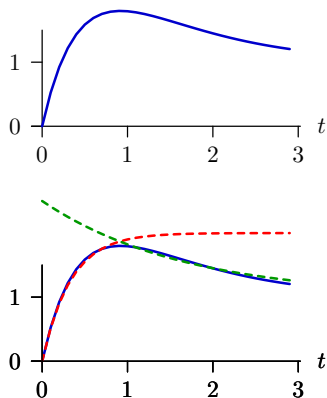


Which must have  $> 1$  but no more than 2 poles (both real)?

## Sums of Exponentials

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Although the third step response ( $s_3(t)$ ) “overshoots” its final value, it is actually the sum of two exponentials.

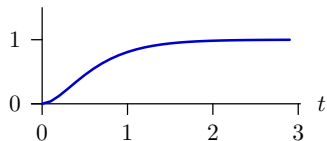


The faster pole contributes significantly to the fast initial rise of the step response and the slower pole contributes to the subsequent decline.

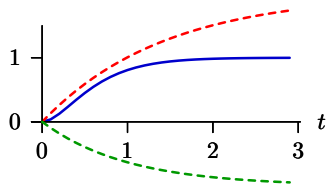
## Sums of Exponentials

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The second step response ( $s_2(t)$ ) is also the sum of two exponentials.



The two components have equal and opposite initial slopes, so the initial slope of the step response is zero – unlike the exponential response of  $s_1(t)$ .



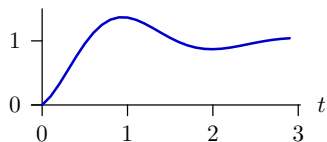
The red component dominates the final value.



## Complex Exponentials

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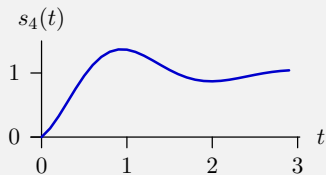
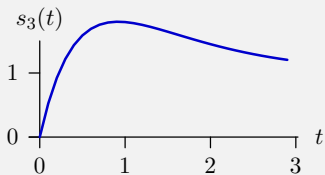
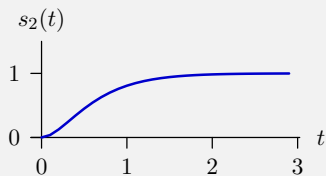
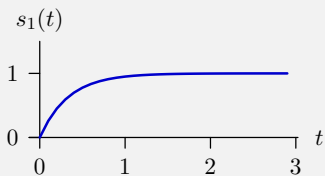
Only the fourth step response ( $s_4(t)$ ) is oscillatory in the sense that the oscillations would persist forever if there were no damping in the system.



If the system has just two poles, they must have non-zero imaginary parts.

## Check Yourself

Step responses of four systems that contain only adders, gains, differentiators, and integrators are shown below.

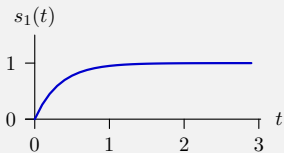


Which if any of the systems could have just one pole? **1**

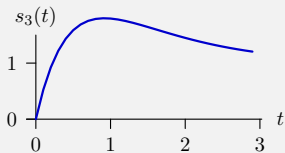
Which must have  $> 1$  but no more than 2 poles (both real)? **2,3**

## Check Yourself

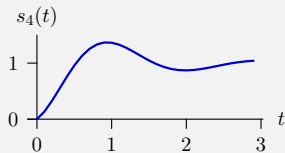
Match the step responses to their system functions below.



$$H_a(s) = \frac{6s + 2}{s^2 + 3s + 2}$$



$$H_b(s) = \frac{s + 10}{s^2 + 2s + 10}$$



$$H_c(s) = \frac{3}{s + 3}$$

Which map shows the correct correspondence?



$H_a H_b H_c$

Map 1



$H_a H_b H_c$

Map 2



$H_a H_b H_c$

Map 3



$H_a H_b H_c$

Map 4



$H_a H_b H_c$

Map 5



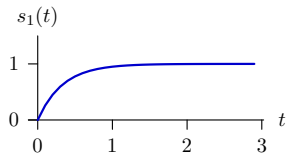
$H_a H_b H_c$

Map 6

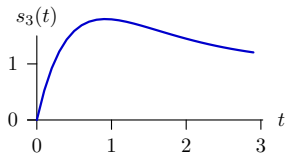
## Check Yourself

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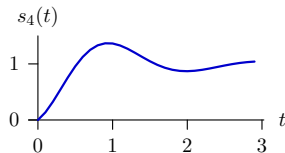
Match the step responses to their system functions below.



$$H_a(s) = \frac{6s + 2}{s^2 + 3s + 2}$$



$$H_b(s) = \frac{s + 10}{s^2 + 2s + 10}$$



$$H_c(s) = \frac{3}{s + 3}$$

Only  $s_1(t)$  can be written as a single exponential, and only  $H_c(s)$  has a single pole. Therefore  $H_c(s)$  corresponds to  $s_1(t)$ .

The denominator of  $H_a(s)$  can be factored as  $(s+1)(s+2)$ .

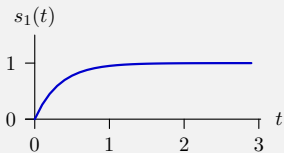
The denominator of  $H_b(s)$  can be factored as  $(s+1+j3)(s+1-j3)$ .

The complex poles of  $H_b(s)$  correspond to the oscillations in  $s_4(t)$ .

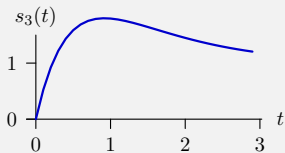
The real poles of  $H_a(s)$  correspond to the sum of two exponentials in  $s_3(t)$ .

## Check Yourself

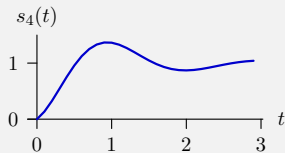
Match the step responses to their system functions below.



$$H_a(s) = \frac{6s + 2}{s^2 + 3s + 2}$$

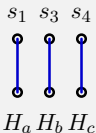


$$H_b(s) = \frac{s + 10}{s^2 + 2s + 10}$$

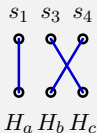


$$H_c(s) = \frac{3}{s + 3}$$

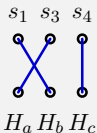
Which map shows the correct correspondence? **Map 5**



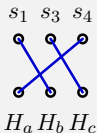
Map 1



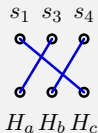
Map 2



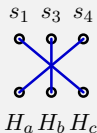
Map 3



Map 4



Map 5



Map 6

## Complex-Valued Roots

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Poles can have non-zero imaginary components (e.g.,  $H_b(s)$ , previous slide).

$$H_b(s) = \frac{s + 10}{s^2 + 2s + 10} = \frac{s + 10}{(s + 1 + j3)(s + 1 - j3)}$$

$$p = -1 \pm j3$$

Each pole is associated with a complex-valued eigenfunction:  $e^{(-1 \pm j3)t}$  but the step response of this system must have no imaginary part.

How is the step response real-valued when the poles have imaginary parts?

## Complex-Valued Roots

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The numerator and denominator polynomials of the system functions of systems that contain just adders, gains, differentiators, and integrators have real-valued coefficients.

If  $p$  is a root of a polynomial with constant real-valued coefficients, then its complex conjugate  $p^*$  is also a root.

---

Let  $D(s)$  represent a polynomial in  $s$  with constant, real-valued coefficients. If  $p$  is a root of  $D(s)$  then  $D(p) = 0$ .<sup>1</sup>

Since all of the coefficients of  $D(p)$  are real-valued,

$$D(p^*) = (D(p))^* = 0^* = 0.$$

Thus  $p^*$  is also a root.

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<sup>1</sup> That's the definition of a root.

## Complex Roots

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If we pair the factors corresponding to complex-conjugate roots, the resulting polynomial has real-valued coefficients.

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Let  $p = \sigma + j\omega_0$ . Then  $p^* = \sigma - j\omega_0$ .

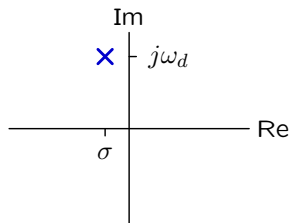
$$(s - p)(s - p^*) = s^2 - 2 \operatorname{Re}(p) s + |p|^2$$

Both the real part of  $p$  and the squared magnitude of  $p$  are real-valued. Therefore the product of these factors has real-valued coefficients.



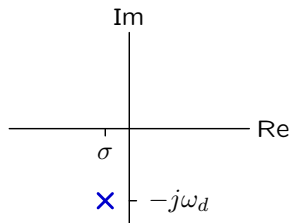
## Natural Frequencies

Eigenfunctions of a system with a single complex pole are complex valued.



$$H_0(s) = \frac{1}{s - \sigma - j\omega_d}$$

$$h_0(t) = e^{(\sigma + j\omega_d)t} = e^{\sigma t}(\cos(\omega_d t) + j \sin(\omega_d t))$$



$$H_1(s) = \frac{1}{s - \sigma + j\omega_d}$$

$$h_1(t) = e^{(\sigma - j\omega_d)t} = e^{\sigma t}(\cos(\omega_d t) - j \sin(\omega_d t))$$

$$H_0(s) + H_1(s) = \frac{1}{s^2 - 2\sigma s + \sigma^2 + \omega_d^2}$$

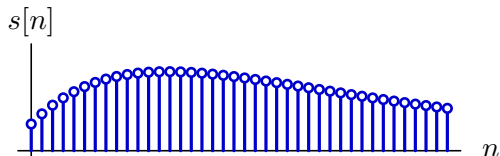
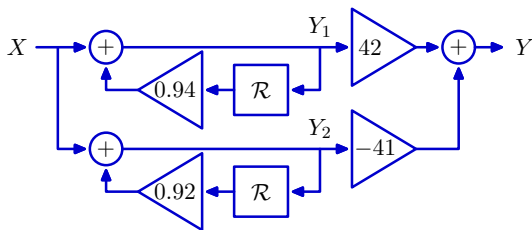
$$h_0(t) + h_1(t) = 2e^{\sigma t} \cos(\omega_d t)$$

Responses of a system with complex poles in conjugate pairs are real-valued.

## Comparison to DT

While the step response of a second order DT system with real-valued poles is a sum of geometrics, it may “overshoot” the final value.

$$\frac{Y}{X} = \frac{42}{1 - 0.94\mathcal{R}} - \frac{41}{1 - 0.92\mathcal{R}}$$



As in CT, this overshoot should not be confused with oscillations that occur with complex valued poles.

## Complex Poles

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What if a pole has a non-zero imaginary part?

Example:

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} + \mathcal{R}^2} = \frac{1}{1 - \frac{1}{z} + \frac{1}{z^2}} = \frac{z^2}{z^2 - z + 1}$$

The poles are at  $z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ , corresponding eigenfunctions are  $(\frac{1}{2} \pm j\frac{\sqrt{3}}{2})^n$ .

Powers of complex numbers are easy to compute using polar forms.

Express the pole at  $z = a + jb$  as  $re^{j\theta}$  where  $r^2 = a^2 + b^2$  and  $\tan \theta = \frac{b}{a}$ .

Then the eigenfunction is  $(re^{j\theta})^n = r^n e^{j\theta n}$ :

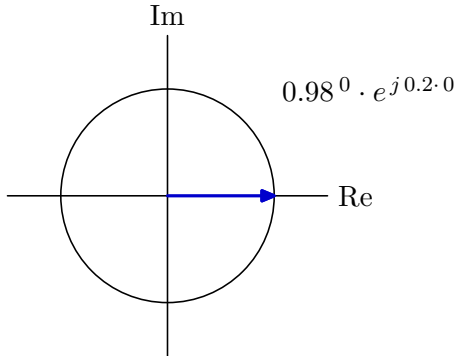
- geometric growth of magnitude
- linear growth of angle

## Complex Pole Example

---

Consider a complex pole at  $re^{j\theta}$  where  $r = 0.98$  and  $\theta = 0.2$ .

The  $n^{\text{th}}$  sample of the corresponding eigenfunction is  $(re^{j\theta})^n = r^n e^{j\theta n}$ .

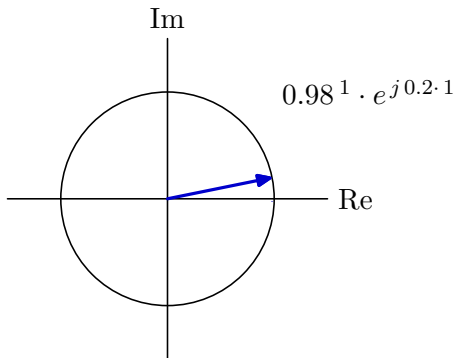


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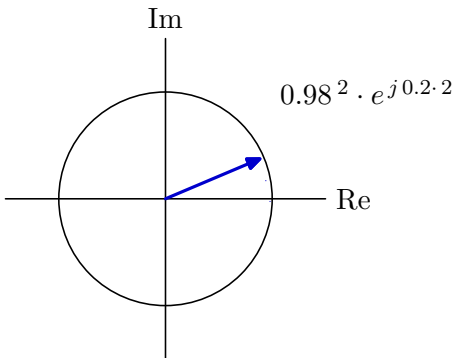


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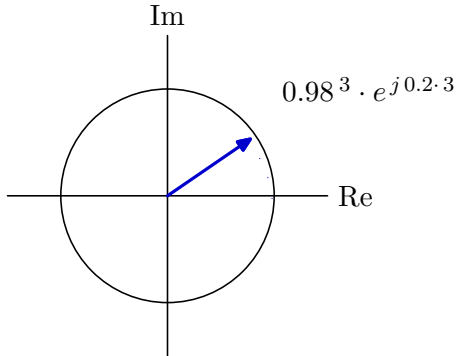


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## Complex Roots

---

If  $p$  is a root of a polynomial with constant real-valued coefficients, then its complex-conjugate  $p^*$  is also a root.

---

Let  $D(z)$  represent a polynomial in  $z$  with constant real-valued coefficients.

If  $p$  is a root of  $D(z)$  then  $D(p) = 0$ .

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$$D(p^*) = (D(p))^* = 0^* = 0.$$

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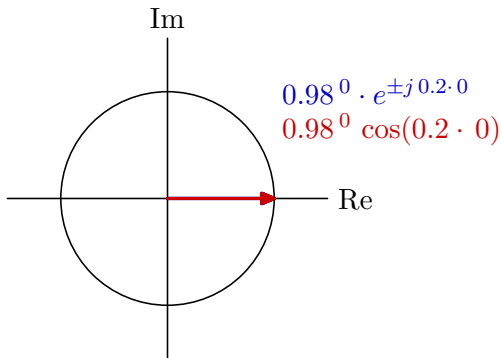
## Complex Conjugates

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The sum of the eigenfunctions associated with complex conjugates is real.

$$(re^{j\theta})^n + (re^{-j\theta})^n = r^n 2 \cos \theta n$$

Example:  $r = 0.98$ ,  $\theta = 0.2$



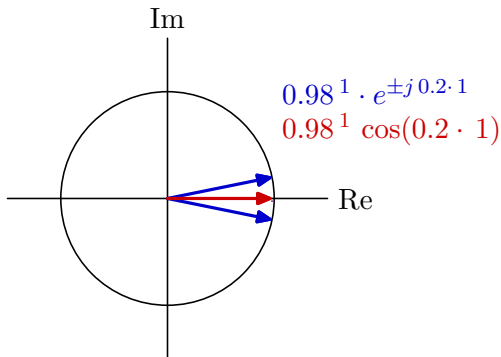
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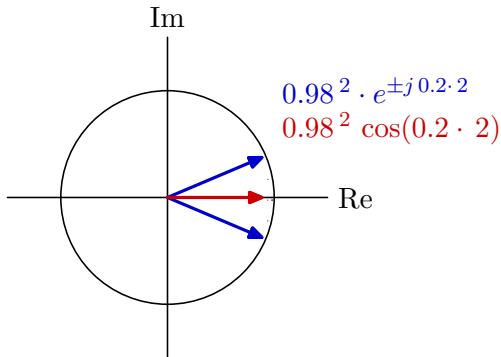
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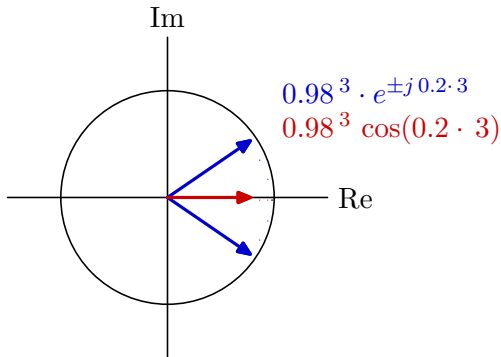
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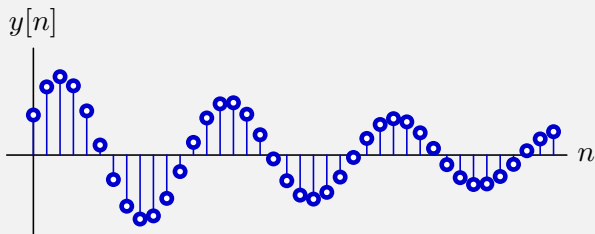
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Example:  $r = 0.98$ ,  $\theta = 0.2$



## Check Yourself

Output of a system with poles at  $z = re^{\pm j\Omega}$ .



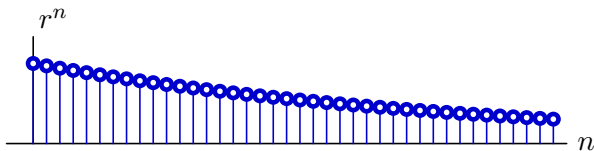
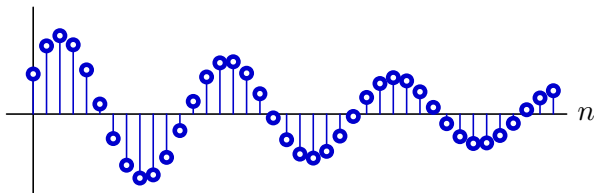
Which statement is true?

1.  $r < 0.5$  and  $\Omega \approx 0.5$
2.  $0.5 < r < 1$  and  $\Omega \approx 0.5$
3.  $r < 0.5$  and  $\Omega \approx 0.08$
4.  $0.5 < r < 1$  and  $\Omega \approx 0.08$
5. none of the above

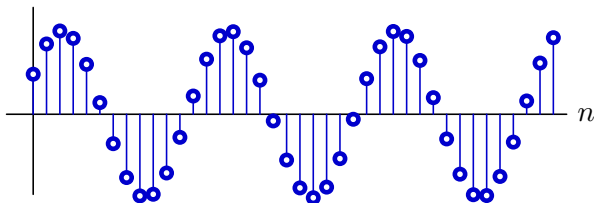
## Check Yourself

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$$y[n] = Az_0^n + Bz_1^n = Ar^n e^{jn\Omega} + Br^n e^{-jn\Omega}$$

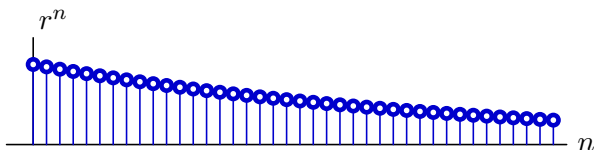
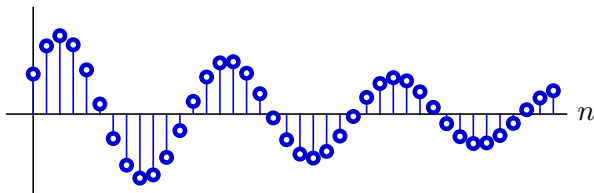


$$Ae^{jn\Omega} + Be^{-jn\Omega}$$

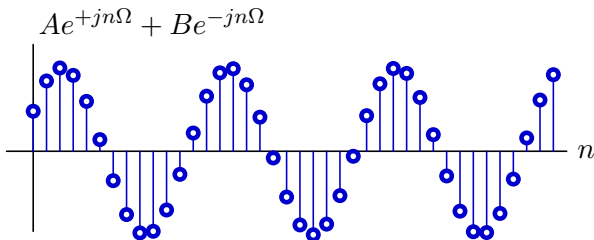


## Check Yourself

$$y[n] = Az_0^n + Bz_1^n = Ar^n e^{jn\Omega} + Br^n e^{-jn\Omega}$$



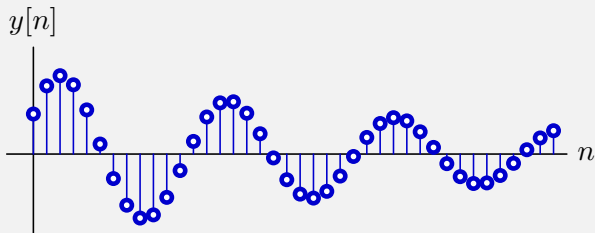
$$r > 0.95$$



$$\text{period} \approx 12 \rightarrow \Omega \approx 0.5$$

## Check Yourself

Output of a system with poles at  $z = re^{\pm j\Omega}$ .



Which statement is true? 2

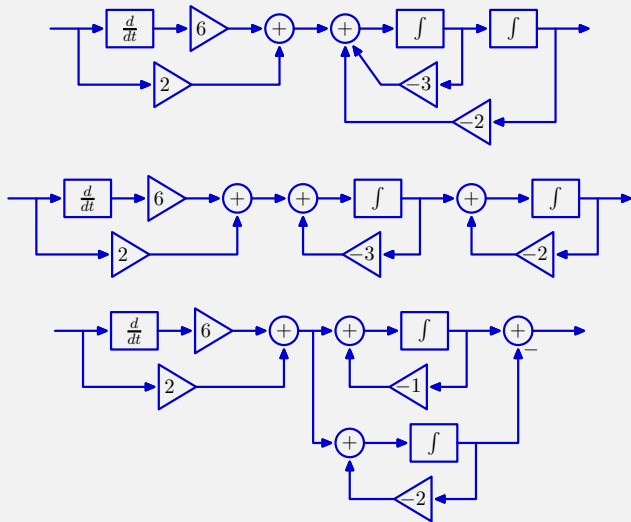
1.  $r < 0.5$  and  $\Omega \approx 0.5$
2.  $0.5 < r < 1$  and  $\Omega \approx 0.5$
3.  $r < 0.5$  and  $\Omega \approx 0.08$
4.  $0.5 < r < 1$  and  $\Omega \approx 0.08$
5. none of the above



## Block Diagrams

Except for 1 error, these 3 block diagrams represent the same  $H(s)$ :

$$H(s) = \frac{6s + 2}{s^2 + 3s + 2}$$

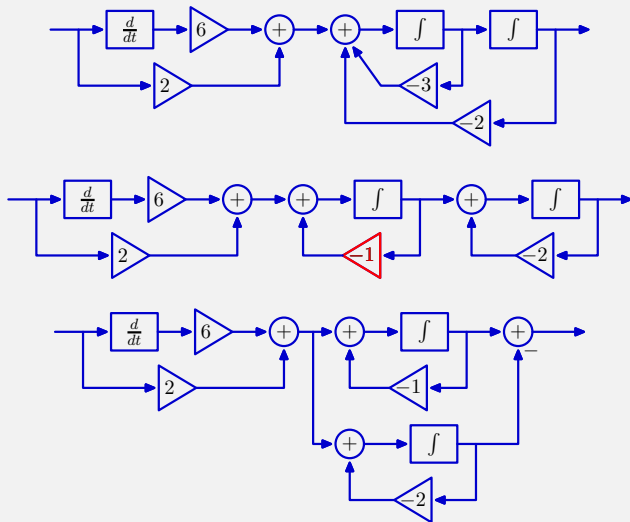


What is the error?

## Block Diagrams

Except for 1 error, these 3 block diagrams represent the same  $H(s)$ :

$$H(s) = \frac{6s + 2}{s^2 + 3s + 2}$$



What is the error?

The gain of  $-3$  near the center of middle figure should be  $-1$ .

## Summary

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Today we developed relations between a number of different representations of systems:

- differential and difference equations
- block diagrams
- system (transfer) functions
- poles and zeros

Next time we will add **frequency responses** to this list.