6.3100: Dynamic System Modeling and Control Design

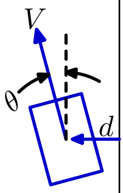
DT Poles and Stability

September 30, 2024

Last Time

We introduced an **operator** representation for discrete time systems.

Example: **robotic steering**



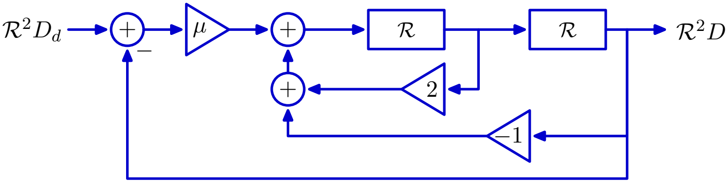
$$d[n] = d[n-1] + V\Delta T\theta[n-1] \qquad D = \mathcal{R}D + V\Delta T\mathcal{R}\Theta$$

$$\theta[n] = \theta[n-1] + \Delta T\omega[n-1] \qquad \Theta = \mathcal{R}\Theta + \Delta T\mathcal{R}\Omega$$

$$\omega[n] = \gamma u[n] \qquad \Omega = \gamma U$$

$$u[n] = K_p(d_d[n] - d[n]) \qquad U = K_p(D_d - D)$$

The operator representation (with $\mathcal{R}$ representing **right-shift**) retains the structure of the underlying difference equations, and allows us to manipulate and simplify difference equations using **polynomials** of $\mathcal{R}$.

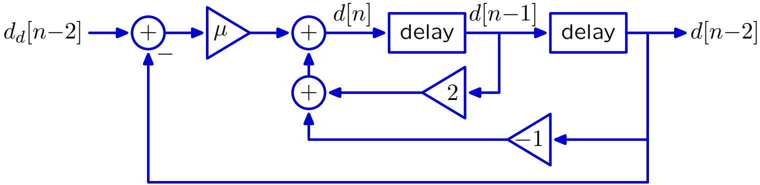


Last Time

By interpreting $\mathcal{R}$ as **delay** we get a description in the time domain.

Time domain: step-by-step calculation of **samples**:

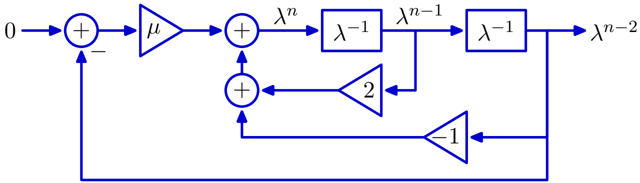
$$d[n] = 2d[n-1] - d[n-2] + (\Delta T)^2 V K_p \gamma \Big(d_d[n-2] - d[n-2] \Big)$$



Interpreting $\mathcal{R}$ as gain (λ^{-1}) provides a description in the frequency domain.

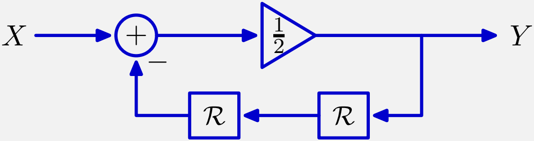
Frequency domain: constraints on the structure of the output **signal**:

$$\lambda^n = 2\lambda^{n-1} - \lambda^{n-2} - (\Delta T)^2 V K_p \gamma \lambda^{n-2}$$



Check Yourself

Consider the following system.



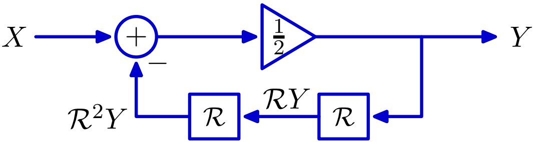
Find an expression of the form $\mathcal{G}_1(\mathcal{R})Y = \mathcal{G}_2(\mathcal{R})X$ for this system.

Use the operator expression to find the natural frequencies.

Is the system stable?

Check Yourself

Consider the following system.



Find an expression of the form $\mathcal{G}_1(\mathcal{R})Y = \mathcal{G}_2(\mathcal{R})X$ for this system.

$$Y = \frac{1}{2}\left(X - \mathcal{R}^2 Y\right)$$

$$\left(1 + \frac{1}{2}\mathcal{R}^2\right)Y = \frac{1}{2}X$$

Use the operator expression to find the natural frequencies.

Find values of λ for which Y is nonzero while X is zero.

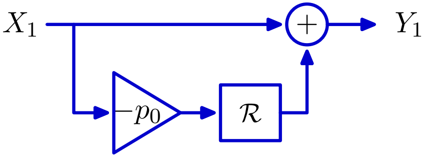
$$\left(1 + \frac{1}{2\lambda^2}\right)Y = \frac{1}{2}X = 0 \quad \rightarrow \quad \lambda = \pm\frac{j}{\sqrt{2}}$$

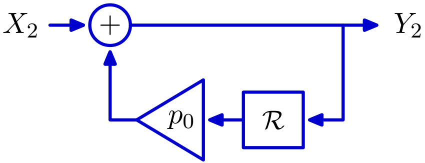
Is the system stable?

Yes. The magnitudes of the natural frequencies are both less than 1.

Last Time: Analyzing Systems with Polynomials

Compare operator descriptions of these feedback and feedforward systems:



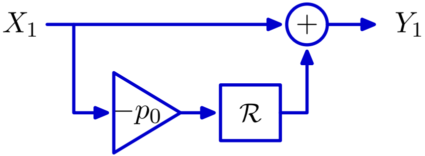


$$Y_1 = (1 - p_0\mathcal{R})X_1$$

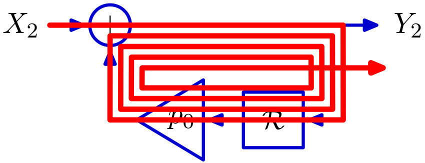
$$(1 - p_0\mathcal{R})Y_2 = X_2$$

Transient and Persistent Responses

Compare operator descriptions of these feedback and feedforward systems:



$$Y_1 = (1 - p_0\mathcal{R})X_1$$

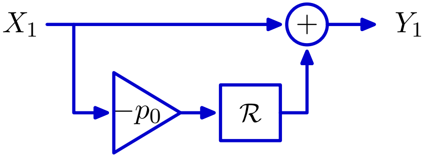


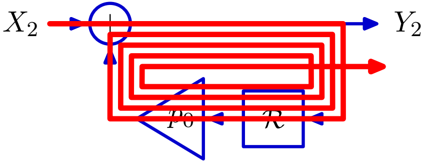
$$(1 - p_0\mathcal{R})Y_2 = X_2$$

$$Y_2 = (1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + \cdots)X_2$$

Reciprocal Relations

Compare operator descriptions of these feedback and feedforward systems:





$$Y_1 = (1 - p_0\mathcal{R})X_1$$

$$(1 - p_0\mathcal{R})Y_2 = X_2$$

$$Y_2 = (1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + \cdots)X_2$$

Substitute X_2 from the first equation into the second:

$$Y_2 = (1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + \cdots)(1 - p_0\mathcal{R})Y_2$$

and therefore

$$(1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + \cdots)(1 - p_0\mathcal{R}) = 1$$

The two factors $1+p_0\mathcal{R}+p_0^2\mathcal{R}^2+p_0^3\mathcal{R}^3+\cdots$ and $1-p_0\mathcal{R}$ must be **reciprocals**.

We can think of the operator representation of this feedback system as

$$\mathcal{H}(\mathcal{R}) = \frac{1}{1 - p_0\mathcal{R}} = 1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots$$

Polynomial Interpretation of Reciprocals

The reciprocal relation between the two representations

$$\mathcal{H}(\mathcal{R}) = \frac{1}{1-p_0\mathcal{R}} = 1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots$$

also follows from polynomial division.

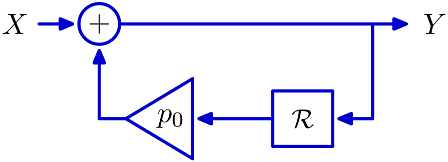
$$\begin{array}{r|llllll}
 & 1 & +p_0\mathcal{R} & +p_0^2\mathcal{R}^2 & +p_0^3\mathcal{R}^3 & +\cdots & \\
\hline
1-p_0\mathcal{R} & 1 & & & & & \\
 & 1 & -p_0\mathcal{R} & & & & \\
\cline{2-3}
 & & p_0\mathcal{R} & & & & \\
 & & p_0\mathcal{R} & -p_0^2\mathcal{R}^2 & & & \\
\cline{3-4}
 & & & p_0^2\mathcal{R}^2 & & & \\
 & & & p_0^2\mathcal{R}^2 & -p_0^3\mathcal{R}^3 & & \\
\cline{4-5}
 & & & & p_0^3\mathcal{R}^3 & & \\
 & & & & p_0^3\mathcal{R}^3 & -p_0^4\mathcal{R}^4 & \\
\cline{5-6}
 & & & & & \cdots &
\end{array}$$

The normal rules of polynomial algebra apply to system operators.

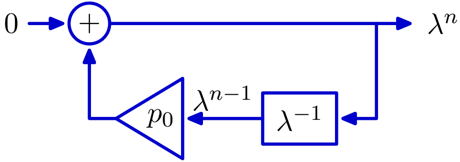
Using Operator Expressions in the Frequency Domain

Operators also simplify thinking in the frequency domain.

Find the natural frequency of this system.



The natural frequency has the form $Y = \lambda^n$ when the input $X = 0$.



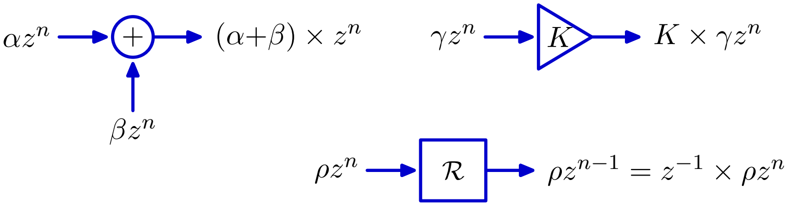
$$\lambda^n = 0 + p_0\lambda^{-1}\lambda^n$$

$$\lambda = p_0$$

The natural frequency λ is equal to p_0. This is a special case of a more general frequency result based on complex geometrics.

Geometric Signals

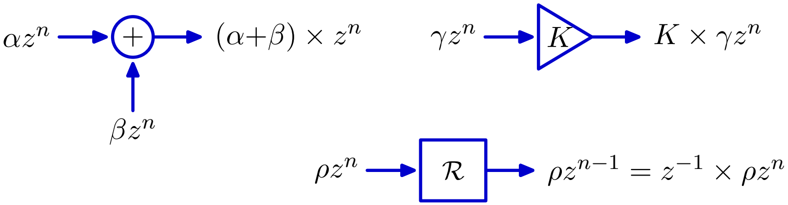
When the inputs to adders, gains, and delays are proportional to z^n, their outputs are also proportional to z^n.



If the output of a system is a scaled multiple of its input, we say that the input signal is an **eigenfunction** of the system.

Geometric Signals

When the inputs to adders, gains, and delays are proportional to z^n, their outputs are also proportional to z^n.



Similarly if the input to any combination of adders, gains, and delays is proportional to z^n, then the output is also proportional to z^n.

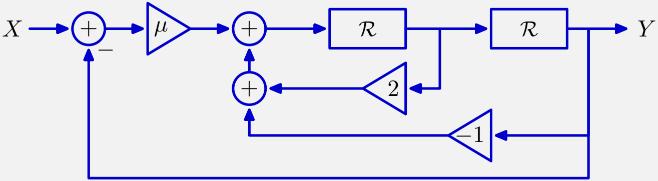
To find the constant of proportionality, simply substitute $\frac{1}{z}$ for $\mathcal{R}$ in the corresponding operator expression:

$$H(z) = \mathcal{H}(\mathcal{R})\Big|_{\mathcal{R}\to\frac{1}{z}}$$

$H(z)$ is called the **system function** or **transfer function**.

Check Yourself

Consider the following block diagram for the robotic steering problem.

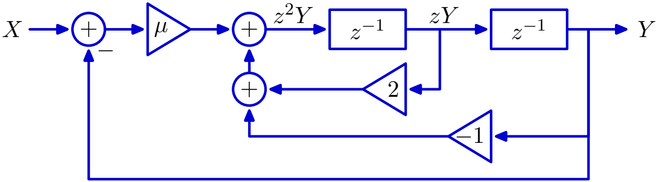


Which (if any) of the following expressions represent $H(z) = \frac{Y}{X}$?

1. $(1+\mu)z^2 - 2z + 1$
2. $\dfrac{1}{z^2 - 2z + 1 + \mu}$
3. $\dfrac{\mu}{z^2 - 2z + 1 + \mu}$
4. $\dfrac{z^2 - 2z + 1}{z^2 - 2z + 1 + \mu}$
5. none of the above

Check Yourself

Determine the system function for the robotic steering problem.



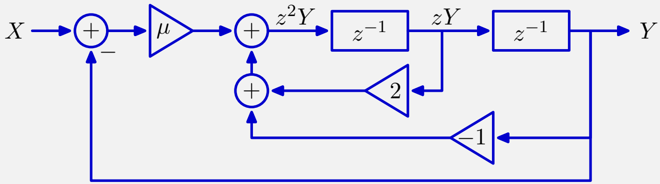
$$z^2Y = 2zY - Y + \mu(X - Y)$$

$$(z^2 - 2z + 1 + \mu)Y = \mu X$$

$$H(z) = \frac{Y}{X} = \frac{\mu}{z^2 - 2z + 1 + \mu}$$

Check Yourself

Consider the following block diagram for the robotic steering problem.



Which (if any) of the following expressions represent $H(z)$? 3

1. $(1+\mu)z^2 - 2z + 1$

2. $\dfrac{1}{z^2 - 2z + 1 + \mu}$

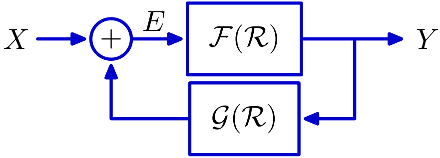
3. $\dfrac{\mu}{z^2 - 2z + 1 + \mu}$

4. $\dfrac{z^2 - 2z + 1}{z^2 - 2z + 1 + \mu}$

5. none of the above

Black's Equation

More generally, let $\mathcal{F}(\mathcal{R})$ represent the forward path and $\mathcal{G}(\mathcal{R})$ represent the feedback path for a feedback system.



$$Y = \mathcal{F}(\mathcal{R})E = \mathcal{F}(\mathcal{R})\Big(X + \mathcal{G}(\mathcal{R})Y\Big) = \mathcal{F}(\mathcal{R})X + \mathcal{F}(\mathcal{R})\mathcal{G}(\mathcal{R})Y$$

$$\Big(1-\mathcal{F}(\mathcal{R})\mathcal{G}(\mathcal{R})\Big)Y = \mathcal{F}(\mathcal{R})X$$

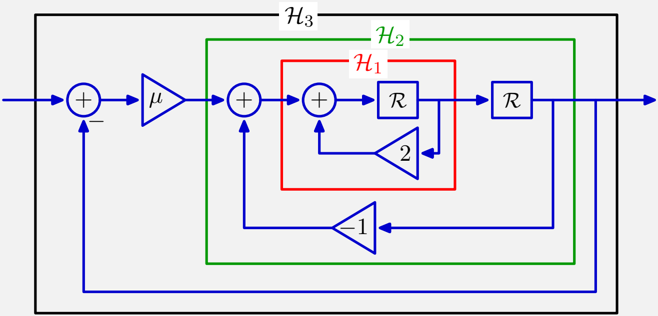
The transformation from X to Y is given by the operator expression

$$\mathcal{H}(\mathcal{R}) = \frac{Y}{X} = \frac{\mathcal{F}(\mathcal{R})}{1 - \mathcal{F}(\mathcal{R})\mathcal{G}(\mathcal{R})}$$

This equation is known as **Black's equation**.

Check Yourself

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ represent subsystems within the robotic steering block diagram $\mathcal{H}_3$.



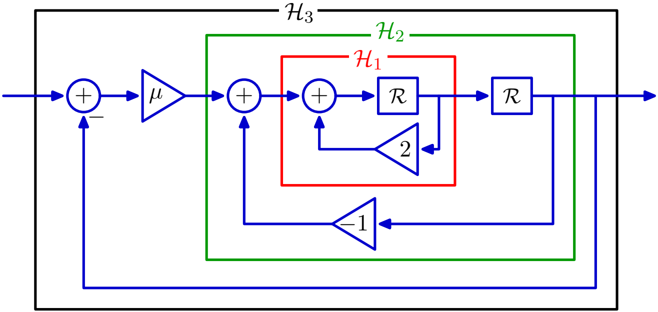
How many of the following expressions are true?

$$\mathcal{H}_1 = \frac{\mathcal{R}}{1 - 2\mathcal{R}} \qquad \mathcal{H}_2 = \frac{\mathcal{H}_1\mathcal{R}}{1 + \mathcal{H}_1\mathcal{R}} \qquad \mathcal{H}_2 = \frac{\mathcal{R}^2}{1 - 2\mathcal{R} + \mathcal{R}^2}$$

$$\mathcal{H}_3 = \frac{\mu\mathcal{H}_2}{1 + \mu\mathcal{H}_2} \qquad \mathcal{H}_3 = \frac{\mu\mathcal{R}^2}{1 - 2\mathcal{R} + (1 + \mu)\mathcal{R}^2}$$

Check Yourself

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ represent subsystems within the robotic steering block diagram $\mathcal{H}_3$.



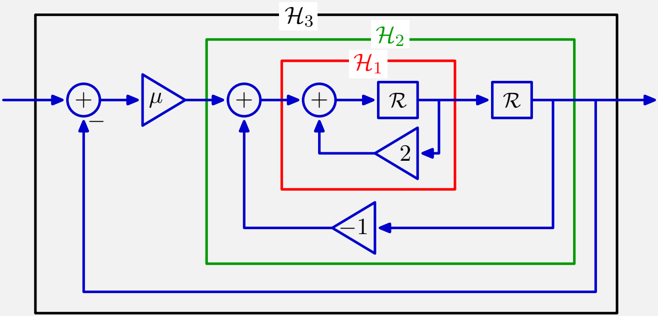
$$\mathcal{H}_1 = \frac{\mathcal{R}}{1 - 2\mathcal{R}} \quad \checkmark$$

$$\mathcal{H}_2 = \frac{\mathcal{H}_1\mathcal{R}}{1 + \mathcal{H}_1\mathcal{R}} \quad \checkmark \quad = \frac{\frac{\mathcal{R}^2}{1-2\mathcal{R}}}{1 + \frac{\mathcal{R}^2}{1-2\mathcal{R}}} = \frac{\mathcal{R}^2}{1 - 2\mathcal{R} + \mathcal{R}^2} \quad \checkmark$$

$$\mathcal{H}_3 = \frac{\mu\mathcal{H}_2}{1 + \mu\mathcal{H}_2} \quad \checkmark \quad = \frac{\frac{\mu\mathcal{R}^2}{1-2\mathcal{R}+\mathcal{R}^2}}{1 + \frac{\mu\mathcal{R}^2}{1-2\mathcal{R}+\mathcal{R}^2}} = \frac{\mu\mathcal{R}^2}{1 - 2\mathcal{R} + (1+\mu)\mathcal{R}^2} \quad \checkmark$$

Check Yourself

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ represent subsystems within the robotic steering block diagram $\mathcal{H}_3$.



How many of the following expressions are true? 5

$$\mathcal{H}_1 = \frac{\mathcal{R}}{1-2\mathcal{R}} \qquad \mathcal{H}_2 = \frac{\mathcal{H}_1\mathcal{R}}{1+\mathcal{H}_1\mathcal{R}} \qquad \mathcal{H}_2 = \frac{\mathcal{R}^2}{1-2\mathcal{R}+\mathcal{R}^2}$$

$$\mathcal{H}_3 = \frac{\mu\mathcal{H}_2}{1+\mu\mathcal{H}_2} \qquad \mathcal{H}_3 = \frac{\mu\mathcal{R}^2}{1-2\mathcal{R}+(1+\mu)\mathcal{R}^2}$$

Modularity

If a feed-forward system contains only adders, gains, and delays, then it's system function can be expressed as a polynomial in $\mathcal{R}$.

$$\mathcal{H}(\mathcal{R}) = b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots$$

Responses of such systems are transient in the sense that their outputs go to zero no later than N time steps after their input goes to zero, where N is the degree of $\mathcal{H}(\mathcal{R})$.

If both the forward and feedback paths through a system with feedback can be represented as polynomials in $\mathcal{R}$, then the system function can be expressed as a **rational polynomial** in $\mathcal{R}$.

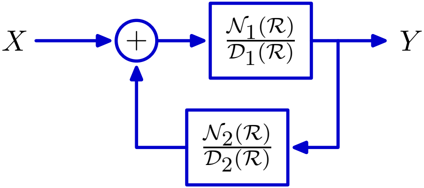
$$\mathcal{H}(\mathcal{R}) = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{1 + a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \cdots}$$

What happens if a system contains a feedback system within a feedback system?

Modularity

If the forward path through a feedback system contains feedback, then the forward path can be represented by a rational polynomial $\mathcal{N}_1(\mathcal{R})/\mathcal{D}_1(\mathcal{R})$.

If the feedback path through a feedback system contains feedback, then the feedback path can be represented by a rational polynomial $\mathcal{N}_2(\mathcal{R})/\mathcal{D}_2(\mathcal{R})$.



We can apply Black's formula to find the resulting system function:

$$\frac{Y}{X} = \frac{\frac{N_1(\mathcal{R})}{D_1(\mathcal{R})}}{1 - \frac{N_1(\mathcal{R})}{D_1(\mathcal{R})}\frac{N_2(\mathcal{R})}{D_2(\mathcal{R})}} = \frac{\mathcal{N}_1(\mathcal{R})\mathcal{D}_2(\mathcal{R})}{\mathcal{D}_1(\mathcal{R})\mathcal{D}_2(\mathcal{R}) - \mathcal{N}_1(\mathcal{R})\mathcal{N}_2(\mathcal{R})}$$

Since the product of polynomials is polynomial, it follows that the overall system function is a rational polynomial.

Partial Fractions

The natural frequencies of a system can be identified by expanding the system functional $\mathcal{H}$ in partial fractions.

$$\mathcal{H} = \frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{1 + a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \cdots}$$

Factor denominator:

$$\mathcal{H} = \frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{(1 - p_0\mathcal{R})(1 - p_1\mathcal{R})(1 - p_2\mathcal{R})(1 - p_3\mathcal{R})\cdots}$$

Partial fractions:

$$\mathcal{H} = \frac{Y}{X} = \frac{C_0}{1 - p_0\mathcal{R}} + \frac{C_1}{1 - p_1\mathcal{R}} + \frac{C_2}{1 - p_2\mathcal{R}} + \cdots + D_0 + D_1\mathcal{R} + D_2\mathcal{R}^2 + \cdots$$

One natural frequency (p_i^n) arises from each factor of the denominator.

The polynomial terms (D_i) represent transient response components.

Poles

The form of each persistent mode is geometric, and the bases p_i of the geometrics are called the **poles** of the system.

$$\mathcal{H}(\mathcal{R}) = \frac{Y}{X} = \frac{C_0}{1-p_0\mathcal{R}} + \frac{C_1}{1-p_1\mathcal{R}} + \frac{C_2}{1-p_2\mathcal{R}} + \cdots + D_0 + D_1\mathcal{R} + D_2\mathcal{R}^2 + \cdots$$

Poles can be found by factoring the system functional $\mathcal{H}(\mathcal{R})$ as shown above. But an easier way to find the poles is to solve for the roots of the denominator of the system function $H(z)$:

$$H(z) = \mathcal{H}(\mathcal{R})\Big|_{\mathcal{R}\rightarrow\frac{1}{z}}$$

as shown below.

$$\begin{aligned} H(z) &= \frac{C_0}{1-p_0z^{-1}} + \frac{C_1}{1-p_1z^{-1}} + \frac{C_2}{1-p_2z^{-1}} + \cdots + D_0 + D_1z^{-1} + D_2z^{-2} + \cdots \\ &= \frac{C_0z}{z-p_0} + \frac{C_1z}{z-p_1} + \frac{C_2z}{z-p_2} + \cdots + D_0 + D_1z^{-1} + D_2z^{-2} + \cdots \end{aligned}$$

Check Yourself

Consider the system described by

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

How many of the following are true?

1. The unit sample response converges to zero.
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$.
3. There is a pole at $z = \frac{1}{2}$.
4. There are two poles.
5. None of the above

Check Yourself

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

$$\left(1 + \frac{1}{4}\mathcal{R} - \frac{1}{8}\mathcal{R}^2\right) Y = \left(\mathcal{R} - \frac{1}{2}\mathcal{R}^2\right) X$$

$$H(\mathcal{R}) = \frac{Y}{X} = \frac{\mathcal{R} - \frac{1}{2}\mathcal{R}^2}{1 + \frac{1}{4}\mathcal{R} - \frac{1}{8}\mathcal{R}^2}$$

$$= \frac{\frac{1}{z} - \frac{1}{2}\frac{1}{z^2}}{1 + \frac{1}{4}\frac{1}{z} - \frac{1}{8}\frac{1}{z^2}} = \frac{z - \frac{1}{2}}{z^2 + \frac{1}{4}z - \frac{1}{8}} = \frac{z - \frac{1}{2}}{(z + \frac{1}{2})(z - \frac{1}{4})}$$

1. The unit sample response converges to zero.
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$.
3. There is a pole at $z = \frac{1}{2}$.
4. There are two poles.
5. None of the above

Check Yourself

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

$$\left(1 + \frac{1}{4}\mathcal{R} - \frac{1}{8}\mathcal{R}^2\right) Y = \left(\mathcal{R} - \frac{1}{2}\mathcal{R}^2\right) X$$

$$H(\mathcal{R}) = \frac{Y}{X} = \frac{\mathcal{R} - \frac{1}{2}\mathcal{R}^2}{1 + \frac{1}{4}\mathcal{R} - \frac{1}{8}\mathcal{R}^2}$$

$$= \frac{\frac{1}{z} - \frac{1}{2}\frac{1}{z^2}}{1 + \frac{1}{4}\frac{1}{z} - \frac{1}{8}\frac{1}{z^2}} = \frac{z - \frac{1}{2}}{z^2 + \frac{1}{4}z - \frac{1}{8}} = \frac{z - \frac{1}{2}}{(z + \frac{1}{2})(z - \frac{1}{4})}$$

1. The unit sample response converges to zero. ✓
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$. ✗
3. There is a pole at $z = \frac{1}{2}$. ✗
4. There are two poles. ✓
5. None of the above ✗

Check Yourself

Consider the system described by

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

How many of the following are true? 2

1. The unit sample response converges to zero.
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$.
3. There is a pole at $z = \frac{1}{2}$.
4. There are two poles.
5. None of the above

Fibonacci's Bunnies

Think about Fibonacci numbers as the output of a discrete-time system.

> "How many pairs of rabbits can be produced from a single pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"

Let $c[n]$ represent the number of pairs of children in generation n. Assume that children become adults in one generation, so the total number of pairs of adults in generation n is the sum of the number of pairs of adults in generation $n{-}1$ plus the number of pairs of children in generation $n{-}1$.

$$a[n] = a[n{-}1] + c[n{-}1]$$

Each pair of adults produces a new pair of children in each generation, which adds to the number of pairs of children added externally ($x[n]$):

$$c[n] = x[n] + a[n{-}1]$$

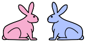
Difference equation model:

$$y[n] = a[n] + c[n] = y[n{-}1] + y[n{-}2] + x[n-1]$$

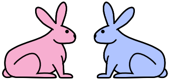
Start the population by adding one pair of children at $n{=}0$:

$$y[-1] = 0; \quad y[0] = 1$$

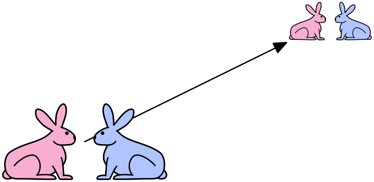
Population Growth



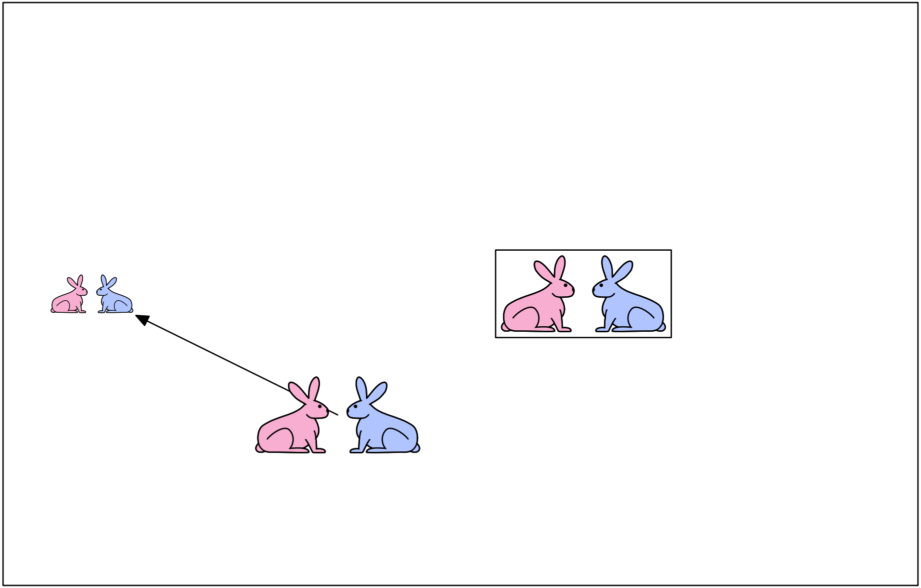
Population Growth



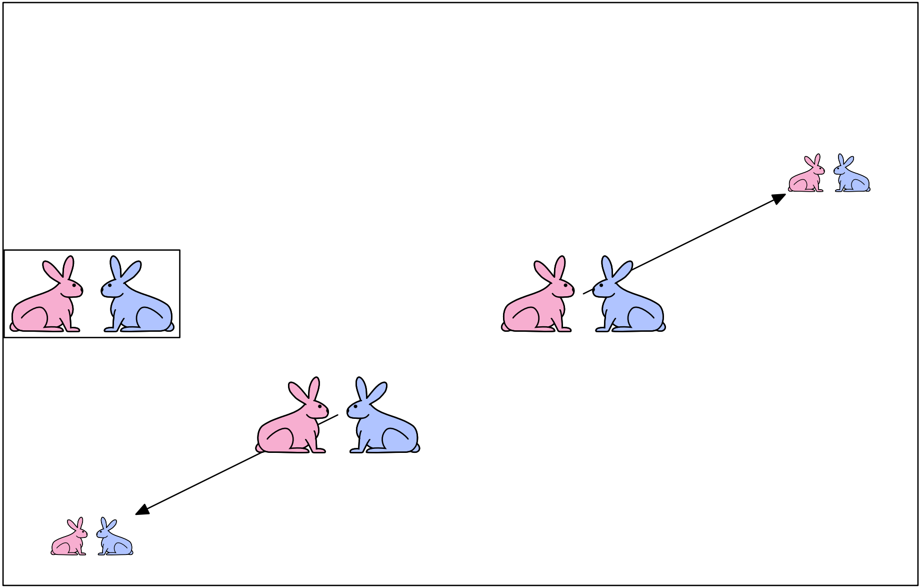
Population Growth



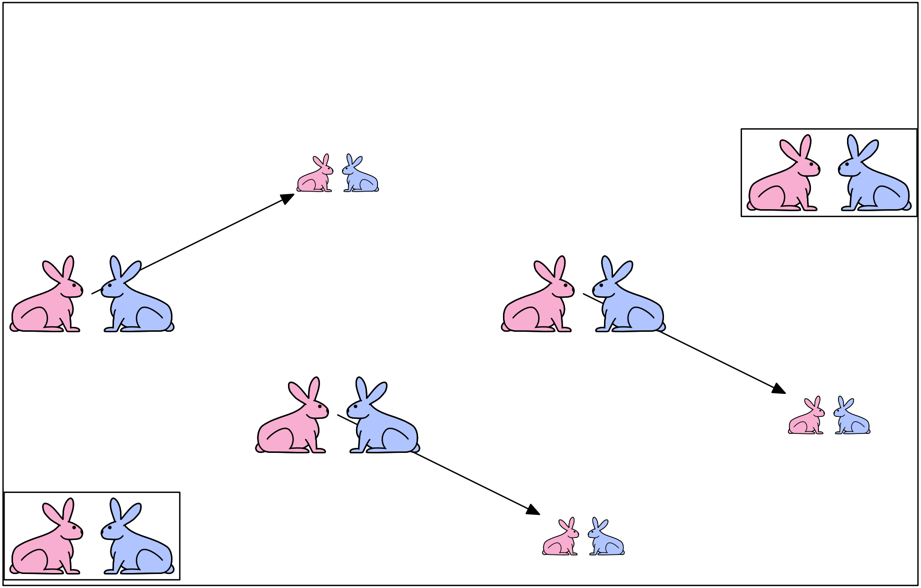
Population Growth



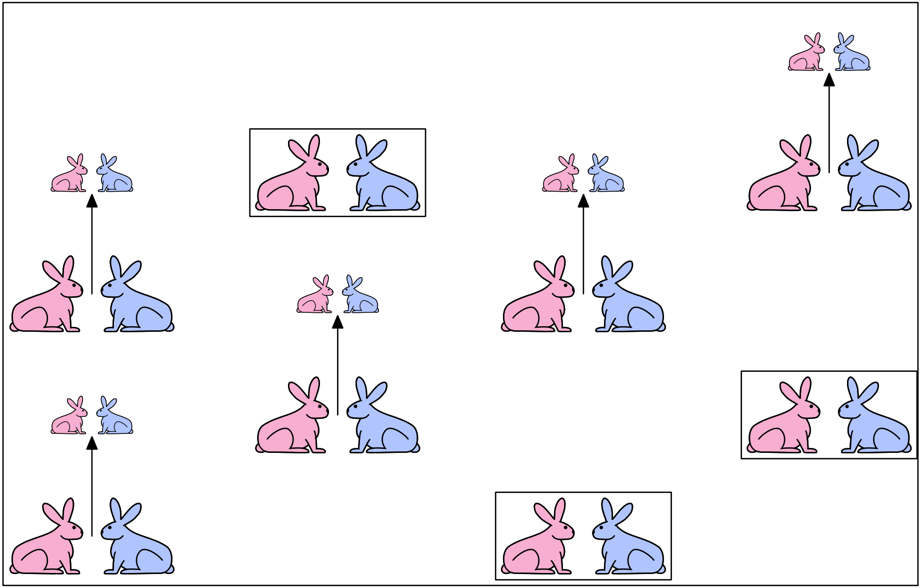
Population Growth



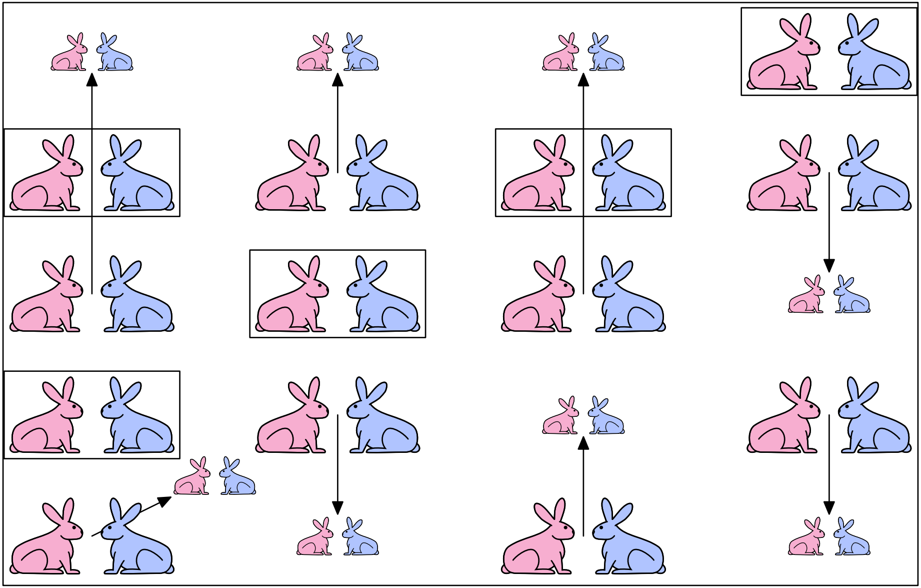
Population Growth



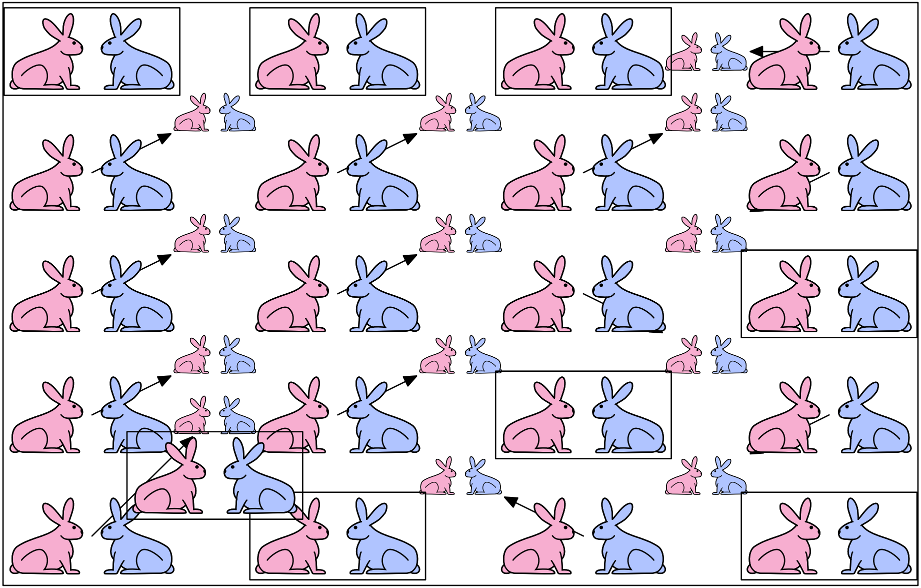
Population Growth



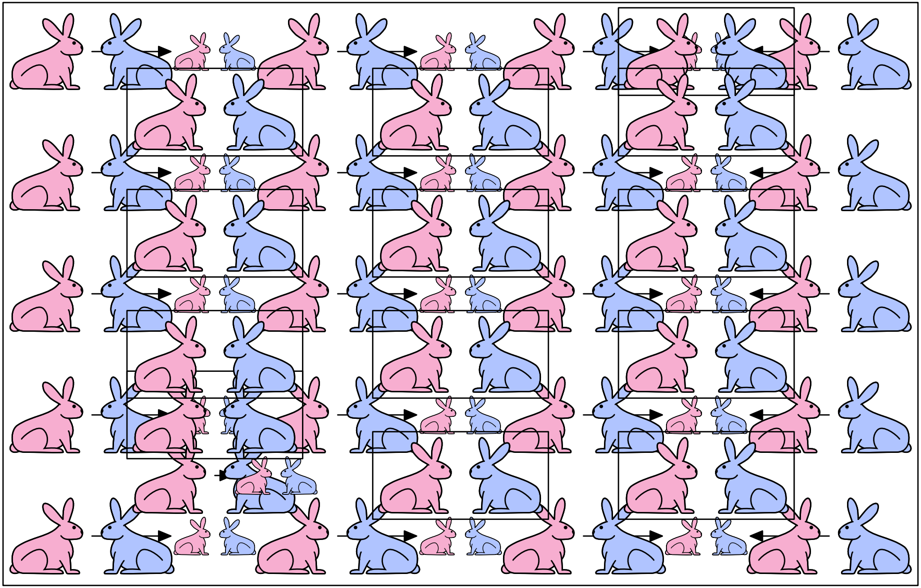
Population Growth



Population Growth



Population Growth



Check Yourself

Bunnie system:

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

What are the pole(s) of the bunnie system?

1. 1
2. 1 and -1
3. -1 and -2
4. $1.618\ldots$ and $-0.618\ldots$
5. none of the above

Check Yourself

Bunnie system:

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

What are the pole(s) of the bunnie system?

System functional:

$$\mathcal{F}(\mathcal{R}) = \frac{Y}{X} = \frac{\mathcal{R}}{1 - \mathcal{R} - \mathcal{R}^2}$$

System function:

$$H(z) = \frac{z}{z^2 - z - 1}$$

The denominator of the system function is second order $\rightarrow$ 2 poles.

The poles are at $z_1 = \frac{1+\sqrt{5}}{2} = 1.618$ and $z_2 = \frac{1-\sqrt{5}}{2} = -0.618$.

Check Yourself

What are the pole(s) of the bunnie system? 4

1. 1
2. 1 and -1
3. -1 and -2
4. $1.618\ldots$ and $-0.618\ldots$
5. none of the above

Example: Fibonacci's Bunnies

Each pole corresponds to a natural frequency.

$$z_1 \approx 1.618 \quad \text{and} \quad z_2 \approx -0.618$$

One mode diverges, one mode oscillates!

Summary

Today we characterized fundamental differences between feedforward and feedback systems.

- Feedforward systems can be characterized by a sum of components that are each characterized by an aggregate gain and delay.
- Feedback systems can be characterized by a ratio of polynomials in $\mathcal{R}$ or equivalently by a ratio of polynomials in z.
- The natural frequencies of a feedback system are given by its **poles**, which are the roots of the denominator of the system function.