Dynamic System Modeling and Control Design

General Solutions to First-Order DT systems, Stability and Convergence

Sept. 9, 2024

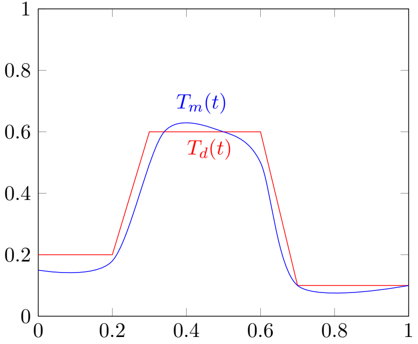
Outline

1. Prop. Control for First Order DT Systems
2. Solutions to First Order DT Systems
3. Choosing K_p for First Order DT Systems

Recap: Our First System

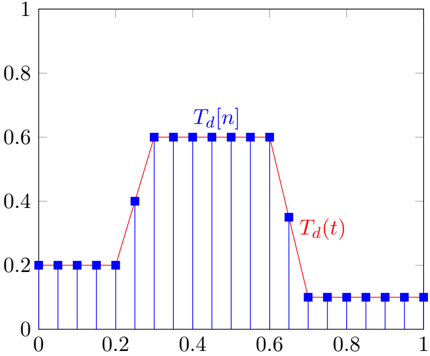
Physical systems operate in *continuous time* (CT). For example, suppose we want to operate a system at a desired temperature. We can then measure the actual temperature.

- $T_d(t)$: desired temperature
- $T_m(t)$: measured temperature

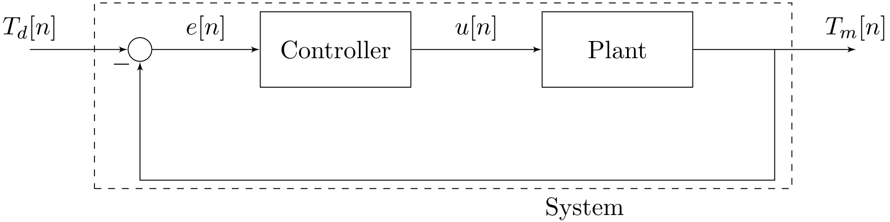


Recap: From Continuous to Discrete Time

Systems controlled by microcontrollers operate at a fixed rate, i.e., in *discrete time* (DT).



Recap: Closed Loop Feedback System



Recall our definition of a simple first order DT system and the proportional controller:

$$\text{Prop. controller:} \quad u[n] = K_p(T_d[n] - T_m[n]),$$

$$\text{Plant:} \quad \frac{T_m[n] - T_m[n-1]}{\Delta T} = \gamma u[n-1].$$

Proportional Control for First-Order DT System

From our proportional controller,

$$\text{Prop. controller:} \quad u[n] = K_p(T_d[n] - T_m[n]), \tag{1}$$

$$\text{Plant:} \quad \frac{T_m[n] - T_m[n-1]}{\Delta T} = \gamma u[n-1], \tag{2}$$

we can substitute (1) into (2) to obtain:

$$\frac{T_m[n] - T_m[n-1]}{\Delta T} = \gamma K_p(T_d[n-1] - T_m[n-1]).$$

Proportional Control for First-Order DT System

From before,

$$\frac{T_m[n] - T_m[n-1]}{\Delta T} = \gamma K_p(T_d[n-1] - T_m[n-1]).$$

Simplifying this equation and collecting terms, we obtain:

$$T_m[n] = (1 - \gamma \Delta T K_p) T_m[n-1] + \gamma \Delta T K_p T_d[n-1].$$

This equation has the form of a first-order DT system:

$$y[n] = \lambda y[n-1] + bx[n-1] \qquad (\#1)$$

General Form of First Order System

The general form of a first order DT system:

$$y[n] = \lambda y[n-1] + bx[n-1] \qquad (\#1)$$

Notes on the general form:

- Our goal is to solve for $y[n]$
- $x[n]$ is the input or driving function we set
- λ is the natural frequency
- b is a multiplicative constant

Case 1: Zero-Input Response (ZIR)

First, we can study the very simple case when $x[n] = 0$ for all n. The equation simplifies to,

$$y[n] = \lambda y[n-1].$$

The solution is given by:

$$y[n] = \lambda^n y[0]$$

The steady state solution depends on the value of λ:

- If $|\lambda| < 1$, then $\lim_{n\to\infty} y[n] = y[\infty] = 0$.
- If $\lambda = 1$, then $y[\infty] = y[0]$.
- If $\lambda = -1$, then $y[n] = (-1)^n y[0]$. The solutions does not converge.
- If $|\lambda| > 1$, then $|y[\infty]| \to \infty$. The solution does not converge.

Case 2: Zero-State Response (ZSR)

Next, we can study the case when $x[n] = 1$ for all n and $y[0] = 0$. In this case, equation (# 1) becomes,

$$y[n] = \lambda y[n-1] + b.$$

First, assuming that the solution converges, let $y[\infty] = \lim_{n\to\infty} y[n]$.

$$y[\infty] = \lambda y[\infty] + b,$$
$$y[\infty] = \frac{b}{1-\lambda}.$$

ZSR of First-Order DT System: Finding $y[n]$

$$y[n] = \lambda y[n-1] + b, \quad y[\infty] = \frac{b}{1-\lambda}$$

We can find $y[n]$ iteratively, as:

$$y[0] = 0, \tag{3}$$
$$y[1] = \lambda y[0] + b = b, \tag{4}$$
$$y[2] = \lambda y[1] + b = \lambda b + b, \tag{5}$$
$$y[3] = \lambda y[2] + b = \lambda^2 b + \lambda b + b. \tag{6}$$

Following this pattern, we get:

$$y[n] = \sum_{m=0}^{n-1} \lambda^m b, \quad y[\infty] = \sum_{m=0}^{\infty} \lambda^m b.$$

ZSR of First-Order DT System: Finding $y[n]$

$$y[n] = \sum_{m=0}^{n-1} \lambda^m b, \quad y[\infty] = \sum_{m=0}^{\infty} \lambda^m b.$$

With the above we can now find $y[n]$:

$$\begin{aligned} y[n] &= y[\infty] - \sum_{m=n}^{\infty} \lambda^m b = y[\infty] - \lambda^n \sum_{m=0}^{\infty} \lambda^m b \\ &= y[\infty] - \lambda^n y[\infty] = y[\infty](1 - \lambda^n) \end{aligned}$$

Thus, $y[n] = \frac{b}{1-\lambda}(1 - \lambda^n)$.

Check Yourself: Steady-State Solutions for ZSR of First-Order DT System

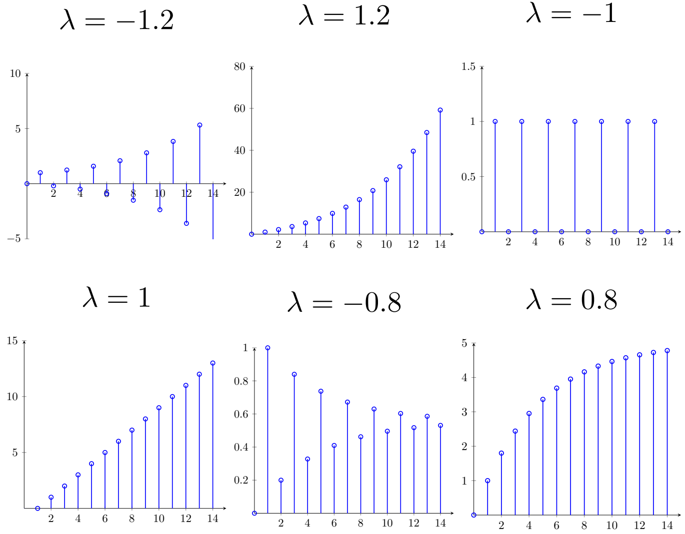
Our Zero-State Response output is defined as,

$$y[n] = \frac{b}{1-\lambda}(1-\lambda^n)$$

Assume that $b = 1$. Determine if the steady state solution converges of diverges for the six different scenarios of λ:

- $\lambda > 1$.
- $\lambda < -1$.
- $\lambda = -1$.
- $\lambda = 1$.
- $0 < \lambda < 1$.
- $-1 < \lambda < 0$.

Effect of λ on Steady State



Returning to Our Original System

Recall our original system equation:

$$T_m[n] = (1 - \gamma \Delta T K_p) T_m[n-1] + \gamma \Delta T K_p T_d[n-1].$$

Assume the desired temperature is **constant**. Comparing with (#1),

$$y[n] = \lambda y[n-1] + bx[n-1],$$

we can see that,

$$\lambda = 1 - \gamma \Delta T K_p, \quad b = \gamma \Delta T K_p T_d[n].$$

Let's consider the stability, steady-state error, and convergence rate.

Stability

$$T_m[n] = \underbrace{(1 - \gamma \Delta T K_p)}_{\lambda} T_m[n-1] + \underbrace{\gamma \Delta T K_p T_d[n-1]}_{b}.$$

Recall that for stability, we must have $-1 < \lambda < 1$. Therefore,

$$\begin{aligned} -1 < \lambda < 1, \\ -1 < 1 - \gamma \Delta T K_p < 1, \\ \frac{2}{\gamma \Delta T} > K_p > 0. \end{aligned}$$

K_p must be chosen in this range to guarantee $T_m[\infty]$ converges to a finite number.

Steady-State Error

We can evaluate the steady-state solution,

$$y[\infty] = \frac{b}{1-\lambda},$$

to find that,

$$T_m[\infty] = \frac{\gamma \Delta T K_p T_d[\infty]}{1-(1-\gamma \Delta T K_p)} = T_d[\infty].$$

In this particular problem, $T_m[\infty] = T_d[\infty]$. As long as we operate in a stable regime, there is no steady-state error. (Not true in general!)

Convergence Rate

Thus far, we have found a valid range of $K_p \in (0, \frac{2}{\gamma \Delta T})$. What is the optimal K_p? Recall (#1), what if we set $\lambda = 0$?

$$\lambda = 1 - \gamma \Delta T K_p = 0 \Rightarrow \gamma \Delta T K_p = 1.$$

$$y[n] = \frac{b}{1-\lambda}(1 - \lambda^n) = b = T_d[n] \Rightarrow T_m[n] = T_d[n].$$

A nice result! The temperature approaches the desired value in 1 step.

However, is this a *realistic* controller?