6.3100: Dynamic System Modeling and Control Design
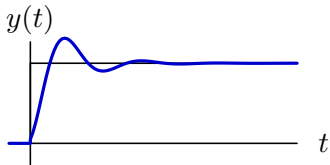
Frequency Response

October 07, 2024

From Transients to Frequency Responses

To date, we have described systems primarily by their responses to sudden changes in their input.

Example: step response

Today we will look at a powerful (and mathematically equivalent) characterization based on sinusoids – the **frequency response**.
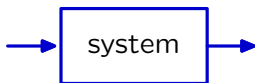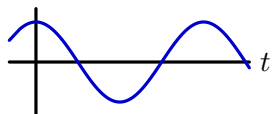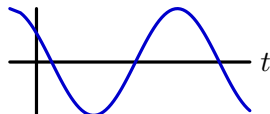
Frequency Response Preview

If a CT system contains only adders, gains, differentiators, and integrators, then its response to a sinusoidal input will be a sinusoidal output with

- the same frequency,
- possibly different amplitude, and
- possibly different phase angle.

$x(t) = \cos(\omega t)$

$y(t) = M \cos(\omega t + \phi)$

These properties of sinusoids result from the CT eigenfunctions e^{st}.

Euler's Formula

Euler's formula asserts a powerful (and surprising!) relation between real and complex numbers (Leonhard Euler, 1748):

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Richard Feynman called this "the most remarkable formula in mathematics." The special case $\theta = \pi$ leads to Euler's Identity:

$$e^{j\pi} + 1 = 0$$

which relates five fundamental constants

- e: the exponential base
- j: the complex base
- π: the circle number
- 0: the additive identity
- 1: the multiplicative identity

in a single equation.

Where Does Euler's Formula Come From?

Euler started with the presumption that a constant j whose square is -1 exists and can be manipulated as an ordinary algebraic constant.

Euler showed the relation between complex exponentials and sinusoids by solving the following differential equation two ways.

$$\frac{d^2 f(\theta)}{d\theta^2} + f(\theta) = 0$$

Let $f_1(\theta) = A\cos(\alpha\theta) + B\sin(\beta\theta)$

$$\frac{df_1(\theta)}{d\theta} = -\alpha A \sin(\alpha\theta) + \beta B \cos(\beta\theta)$$

$$\frac{d^2 f_1(\theta)}{d\theta^2} = -\alpha^2 A \cos(\alpha\theta) - \beta^2 B \sin(\beta\theta)$$

Let $\alpha = \beta = 1$

$$f_1(\theta) = A\cos\theta + B\sin\theta$$

Let $f_2(\theta) = Ce^{\gamma\theta}$

$$\frac{df_2(\theta)}{d\theta} = \gamma Ce^{\gamma\theta}$$

$$\frac{d^2 f_2(\theta)}{d\theta^2} = \gamma^2 Ce^{\gamma\theta}$$

Let $\gamma^2 = -1$

$$f_2(\theta) = Ce^{\pm j\theta}$$

If we arbitrarily take $f_2(\theta) = e^{j\theta}$, then $f_2(0) = 1$ and $f_2'(0) = j$.
To make $f_1(\theta) = f_2(\theta)$, A must be 1 and B must be j:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Where Does Euler's Formula Come From?

Euler's formula also follows from Maclaurin expansion of the exponential function, assuming the j behaves like any other algebraic constant.

Start with the expansion of the real-valued function:

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \frac{\theta^6}{6!} + \frac{\theta^7}{7!} + \cdots$$

Assume that the same expansion holds for complex-valued arguments:

$$e^{j\theta} = 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \frac{j^5\theta^5}{5!} + \frac{j^6\theta^6}{6!} + \frac{j^7\theta^7}{7!} + \cdots$$

$$= 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{j\theta^7}{7!} + \cdots$$

$$= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + j\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}_{\sin\theta}$$

Euler's formula results by splitting the even and odd powers of θ.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Geometric Interpretation

In 1799, Caspar Wessel was the first to describe complex numbers as points in the complex plane. Imaginary numbers had been in use since the 1500's.

$$c = a + jb$$
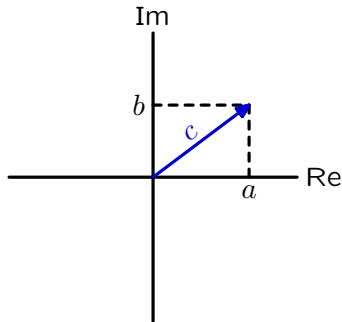
Complex numbers are fundamentally two dimensional. Unlike other constants (such as π), $j = \sqrt{-1}$ defines an entirely new (imaginary) dimension — and a new way to think about operations that involve complex numbers.

Euler's Formula

Euler's formula equates polar and rectangular descriptions of a unit vector at angle θ.

$$e^{j\theta} = \cos\theta + j\sin\theta$$
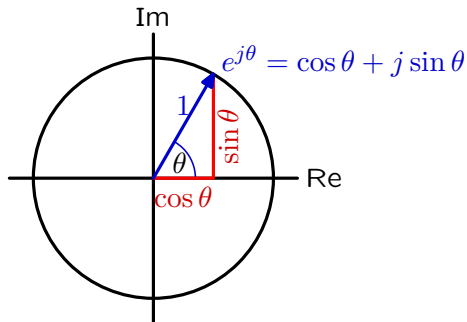
This construction provides a vector interpretation of complex addition and a rotation interpretation to complex multiplication.

Geometric Addition

Rules for adding complex numbers are same as those for adding vectors.

Let c_1 and c_2 represent complex numbers:

$$c_1 = a_1 + jb_1$$
$$c_2 = a_2 + jb_2$$

Then

$$c_1 + c_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

Geometric Multiplication

The two-dimensional view of complex numbers allows us to think about multiplication by an imaginary number as a **rotation**.

Multiplying by j
- **rotates** 1 **to** j,
- rotates j to -1,
- rotates -1 to $-j$, and
- rotates $-j$ to 1.

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Geometric Multiplication

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Multiplying by j

- rotates 1 to j,
- rotates j to -1,
- rotates -1 to $-j$, and
- rotates $-j$ to 1.

Multiplying by j rotates a vector by $\pi/2$.

Multiplying by $j^2 = -1$ rotates a vector by π.

Euler's Formula

Complex numbers are central to our understanding of Euler's formula.

$$e^{j\theta} = \cos\theta + j\sin\theta$$
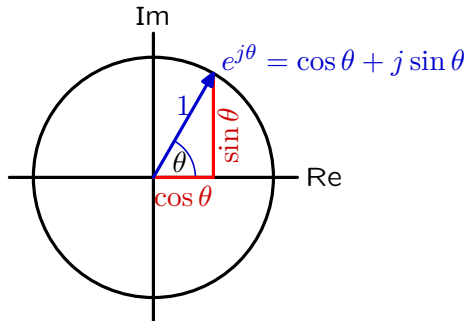
Check Yourself

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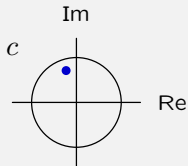
Which if any of the following figures shows the value of jc?
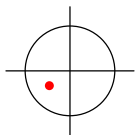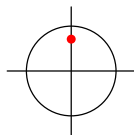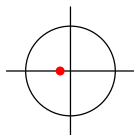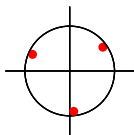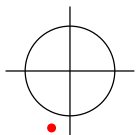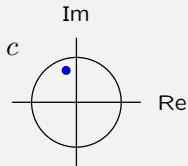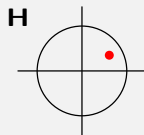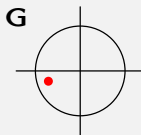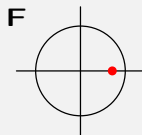Which if any of the following figures shows the value of $\text{Im}(c)$?
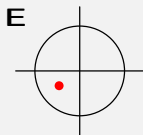Which if any of the following figures shows the value of $1/c$?
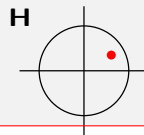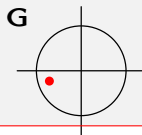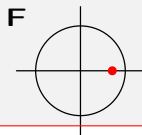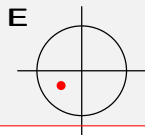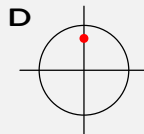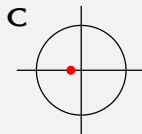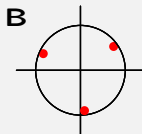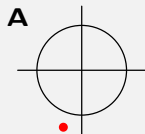
A

B

C

D

E

F

G

H

Check Yourself

Let c represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.

Which if any of the following figures shows the value of jc?

A

B

C

D

E

F

G

H
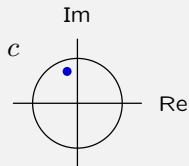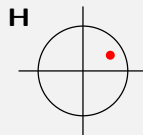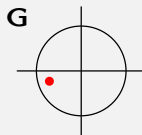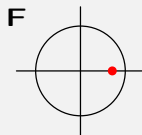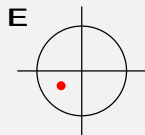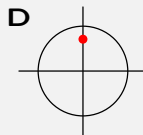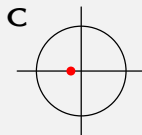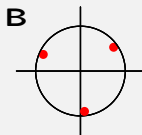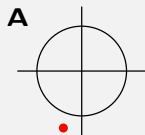
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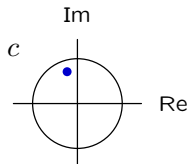
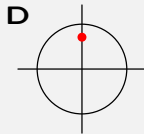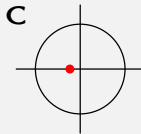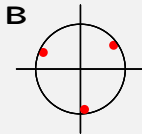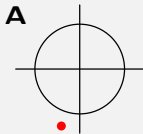Which if any of the following figures shows the value of jc? **G**

G
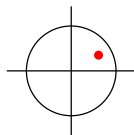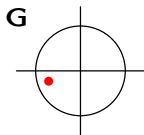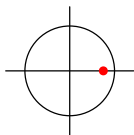
$|jc| = |c|$ and $\angle(jc) = \angle c + \pi/2$

Check Yourself

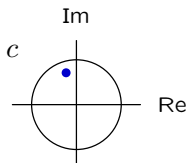Let c represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.

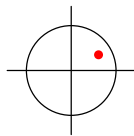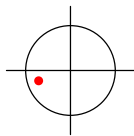Which if any of the following figures shows the value of $\text{Im}(c)$?

A
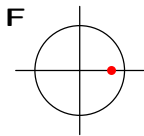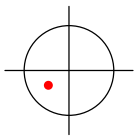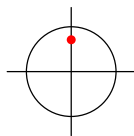

B

C

D
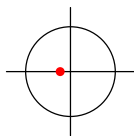

E

F

G

H

Check Yourself

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Which if any of the following figures shows the value of $\text{Im}(c)$? **F**

F

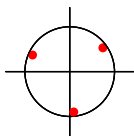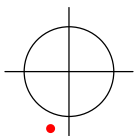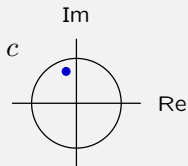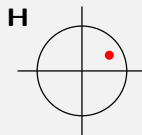$\text{Im}(c) = \frac{c - c^*}{j2}$, which is a real number.

Check Yourself

Let c represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.

Which if any of the following figures shows the value of $1/c$?

A

B

C

D

E

F

G

H
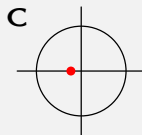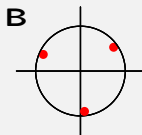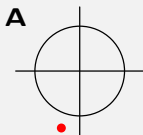
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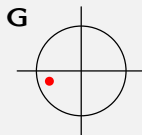Which if any of the following figures shows the value of $1/c$? **A**

A
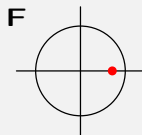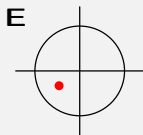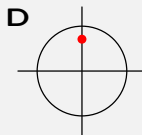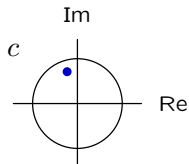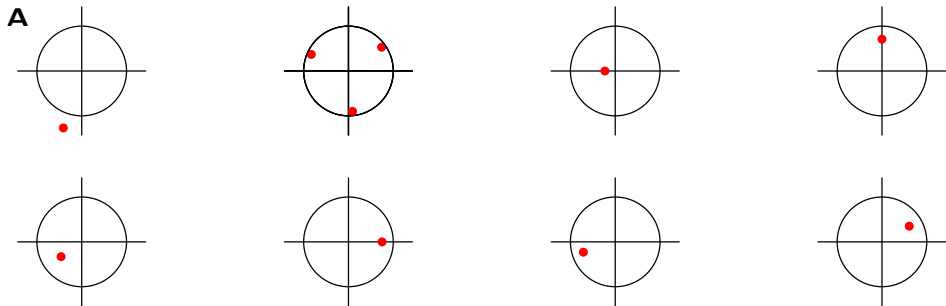
$|1/c| = 1/|c|$ and $\angle(1/c) = -\angle c$

Frequency Response

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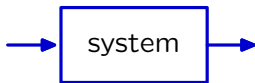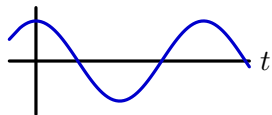
$y(t) = M \cos(\omega t + \phi)$
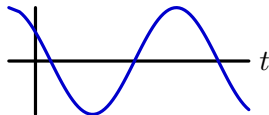
The **frequency response** of a system is a plot of the magnitude M and angle ϕ as a function of $\omega = 2\pi f$ where f is the frequency in hertz.

Frequency Response

The **frequency response** is a plot of the magnitude M and angle ϕ as a function of $\omega = 2\pi f$ where f is the frequency in hertz.

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + \frac{37}{4}y(t) = 5x(t)$$

$$x(t) = \cos(\omega t) \quad \rightarrow \quad y(t) = M(\omega)\cos(\omega t + \phi(\omega))$$
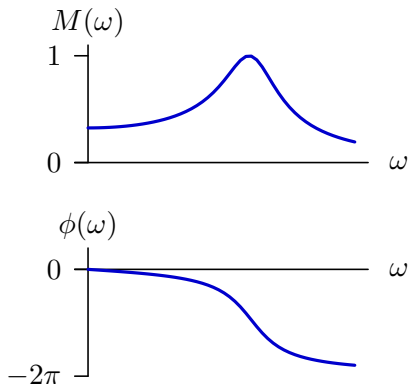
The frequency response is a natural way to describe many systems.

Example: Mass and Spring
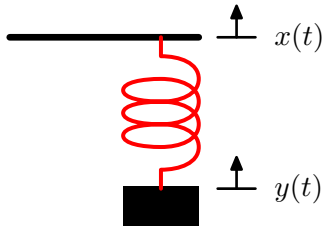
At low frequencies, the output is approximately equal to the input.

At middle frequencies, the output can get very large. There is a **resonance**.

At high frequencies, the output is small.

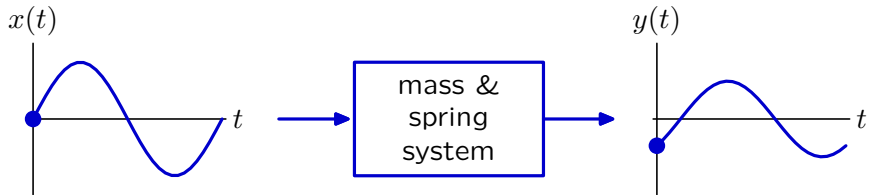## Frequency Response Calculation

A straightforward way to compute a frequency response is to substitute

$$x(t) = \cos(\omega t)$$

into the system's differential equation and solve for the response $y(t)$.

But there are much easier methods based on our work with eigenfunctions and system (transfer) functions.

System Function Approach

Start with the definition of the system function as the (complex-valued) eigenvalue associated with the eigenfunction e^{st}.

$$e^{st} \longrightarrow \boxed{H(s)} \longrightarrow H(s)e^{st}$$

Since s represents an arbitrary complex number, we can subsitute $j\omega$ for s:

$$e^{j\omega t} \longrightarrow \boxed{H(s)} \longrightarrow H(j\omega)e^{j\omega t}$$

We can similarly substitute $-j\omega$ for s:

$$e^{-j\omega t} \longrightarrow \boxed{H(s)} \longrightarrow H(-j\omega)e^{-j\omega t}$$

and then use Euler's formula to determine the response to a cosine:

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow \frac{1}{2}\Big(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\Big)$$

Real-Valued System Functions

If a system contains only adders, gains, differentiators, and integrators, then the system function is the ratio of polynomials in s with real-valued coefficients.

$$H(s) = \frac{\sum_k a_k s^k}{\sum_k b_k s^k}$$

$$H(j\omega) = \frac{\sum_k a_k (j\omega)^k}{\sum_k b_k (j\omega)^k}$$

$$H(-j\omega) = \frac{\sum_k a_k (-j\omega)^k}{\sum_k b_k (-j\omega)^k} = H^*(j\omega)$$

System Function Approach

Simplifying the expression for the response to a cosine input.

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow \frac{1}{2}\left(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right)$$

$$
\begin{aligned}
y(t) &= \frac{1}{2}\left(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right) \\
&= \mathrm{Re}\left\{H(j\omega)e^{j\omega t}\right\} \\
&= \mathrm{Re}\left\{|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t}\right\} \\
&= |H(j\omega)|\mathrm{Re}\left\{e^{j\omega t + j\angle H(j\omega)}\right\} \\
y(t) &= |H(j\omega)|\cos\left(\omega t + \angle H(j\omega)\right)
\end{aligned}
$$

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)|\cos\left(\omega t + \angle H(j\omega)\right)$$
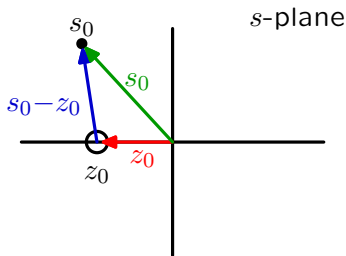
$$M(\omega) = |H(j\omega)|$$
$$\phi(\omega) = \angle H(j\omega)$$

Vector Diagrams

The value of $H(s)$ at a point $s=s_0$ can be determined graphically using vectorial analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

$$H(s_0) = K \frac{(s_0-z_0)(s_0-z_1)(s_0-z_2)\cdots}{(s_0-p_0)(s_0-p_1)(s_0-p_2)\cdots}$$

Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here z_0) to s_0, the point of interest in the s-plane.

Vector Diagrams

The value of $H(s)$ at a point $s=s_0$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(s_0) = K \frac{(s_0-z_0)(s_0-z_1)(s_0-z_2)\cdots}{(s_0-p_0)(s_0-p_1)(s_0-p_2)\cdots}$$

The magnitude is determined by the product of the magnitudes.

$$|H(s_0)| = |K| \frac{|(s_0-z_0)||(s_0-z_1)||(s_0-z_2)|\cdots}{|(s_0-p_0)||(s_0-p_1)||(s_0-p_2)|\cdots}$$

The angle is determined by the sum of the angles.

$$\angle H(s_0) = \angle K + \angle(s_0-z_0) + \angle(s_0-z_1) + \cdots - \angle(s_0-p_0) - \angle(s_0-p_1) - \cdots$$

Vector Diagrams

The frequency response is equal to $H(s)$ at $s=j\omega$.

The value of $H(s)$ at a point $s=j\omega$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(j\omega) = K\frac{(j\omega-z_0)(j\omega-z_1)(j\omega-z_2)\cdots}{(j\omega-p_0)(j\omega-p_1)(j\omega-p_2)\cdots}$$

The magnitude is determined by the product of the magnitudes.

$$|H(j\omega)| = |K|\frac{|(j\omega-z_0)||(j\omega-z_1)||(j\omega-z_2)|\cdots}{|(j\omega-p_0)||(j\omega-p_1)||(j\omega-p_2)|\cdots}$$

The angle is determined by the sum of the angles.

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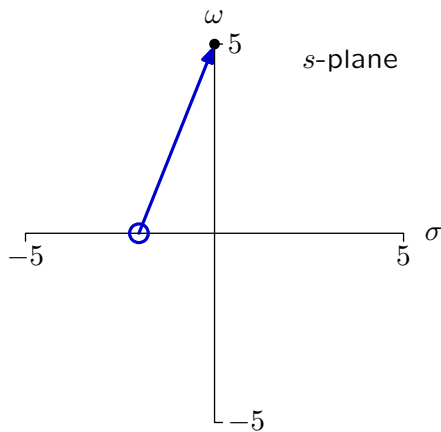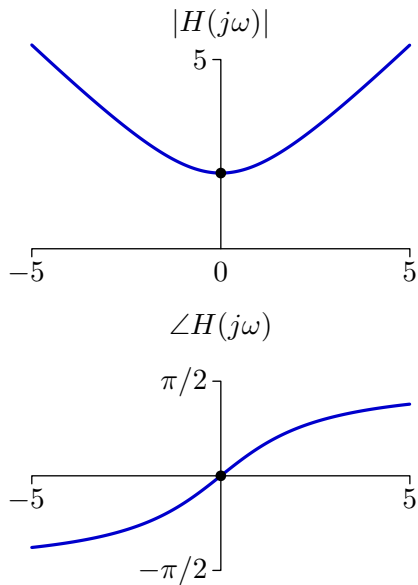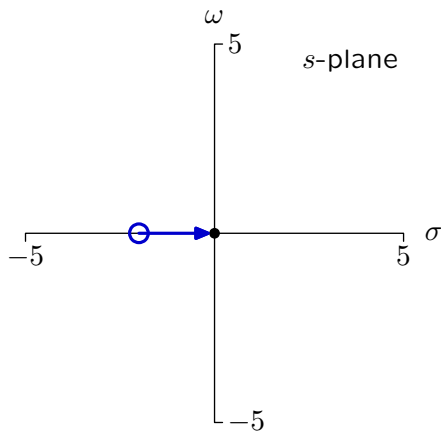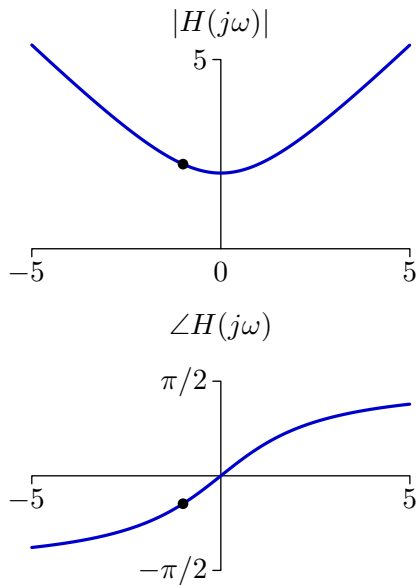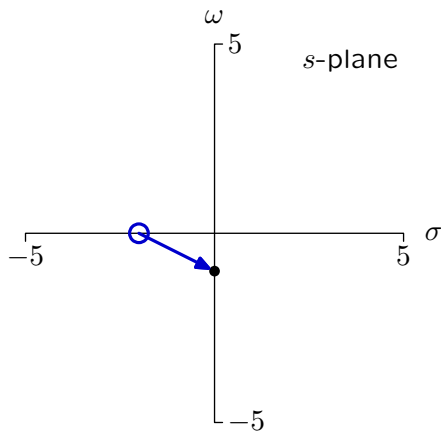## Vector Diagrams
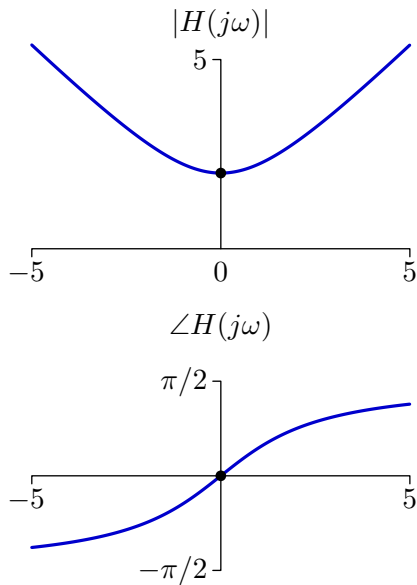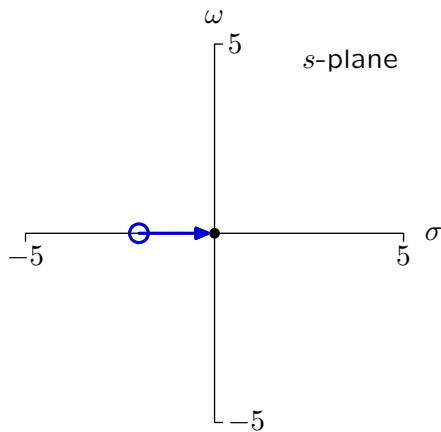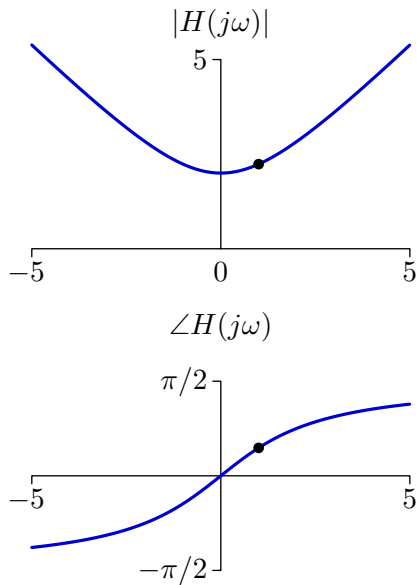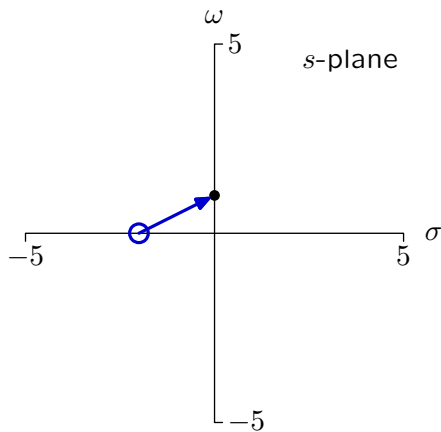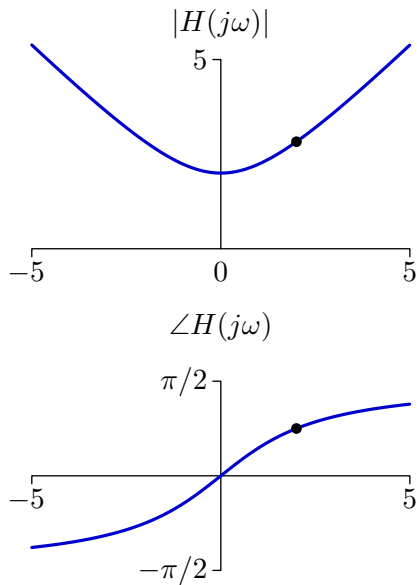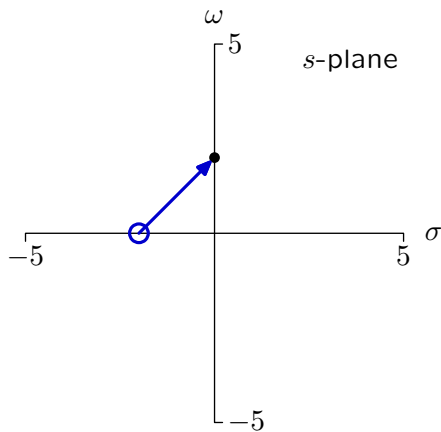
$$H(s) = s - z_1$$

Vector Diagrams

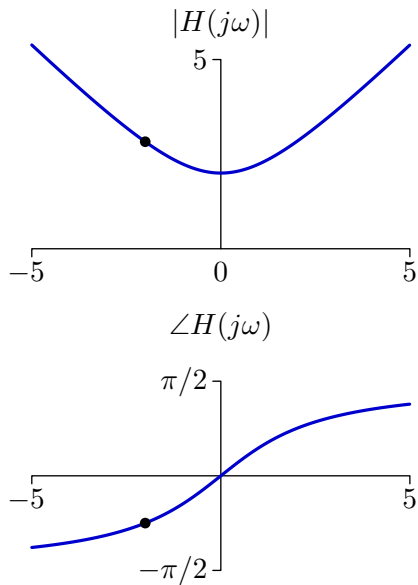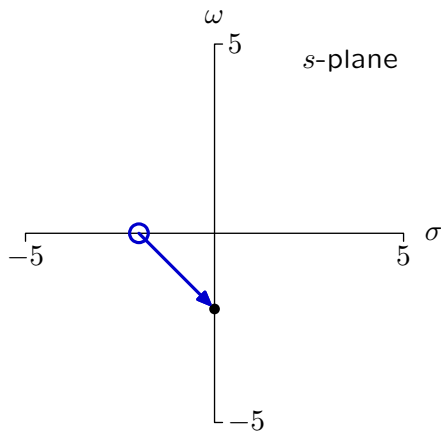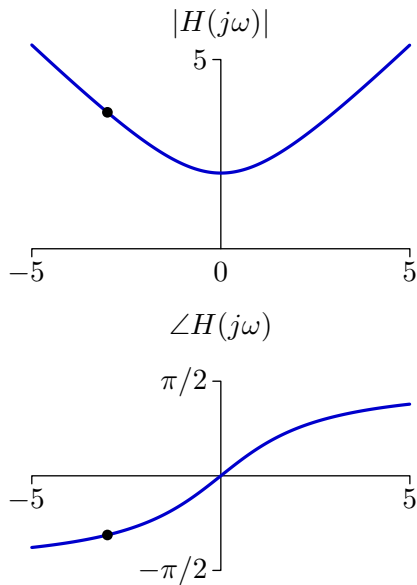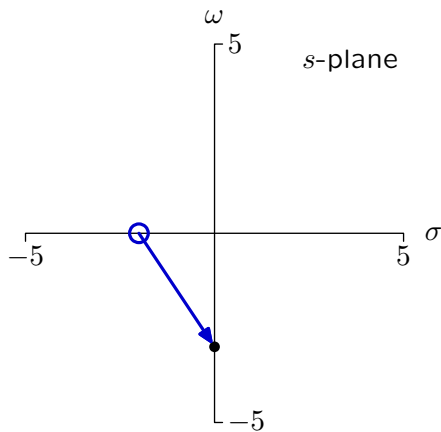$$H(s) = s - z_1$$

Vector Diagrams

$$H(s) = s - z_1$$

Vector Diagrams

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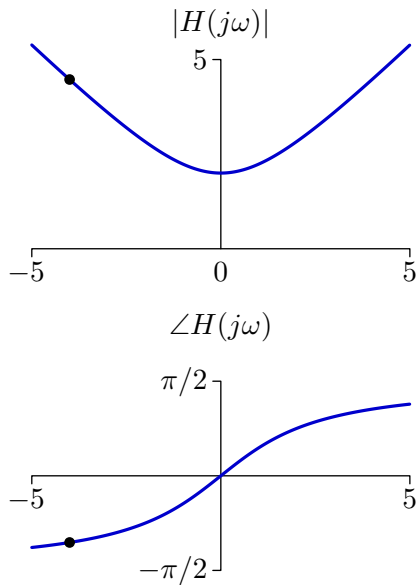
s-plane

$|H(j\omega)|$

$\angle H(j\omega)$

Vector Diagrams

$$H(s) = s - z_1$$

Vector Diagrams

$$H(s) = s - z_1$$
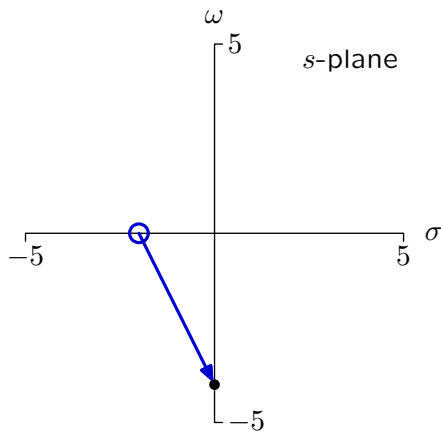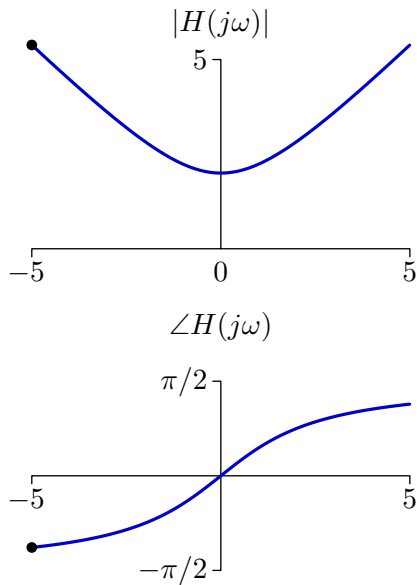

ω

s-plane

σ

$|H(j\omega)|$

$\angle H(j\omega)$

Vector Diagrams

$$H(s) = s - z_1$$

Vector Diagrams

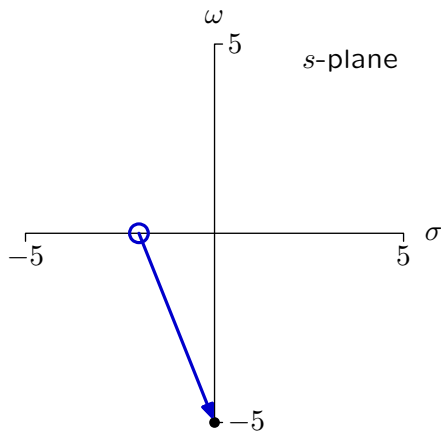$$H(s) = s - z_1$$

s-plane

$|H(j\omega)|$

$\angle H(j\omega)$

Vector Diagrams

$$H(s) = s - z_1$$

$$|H(j\omega)|$$

$$\angle H(j\omega)$$
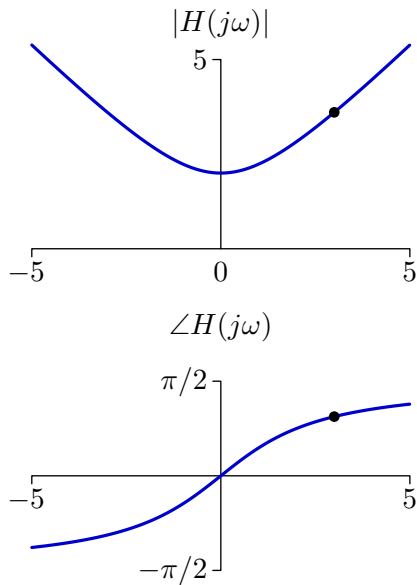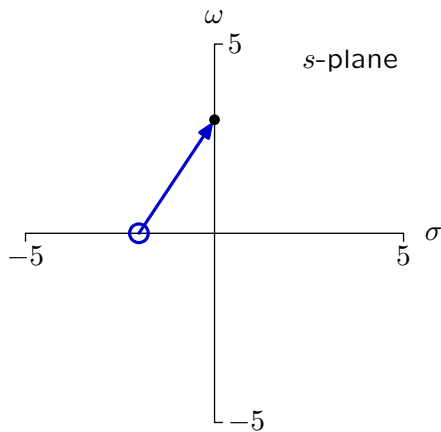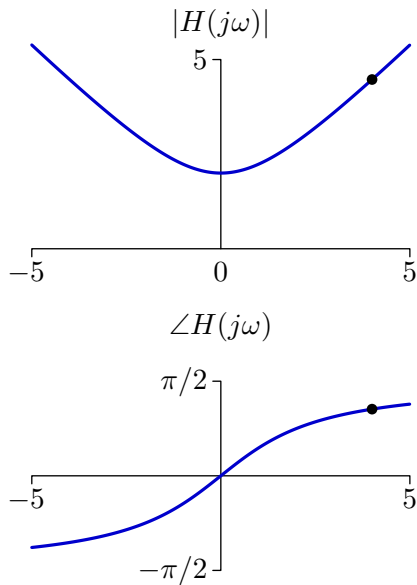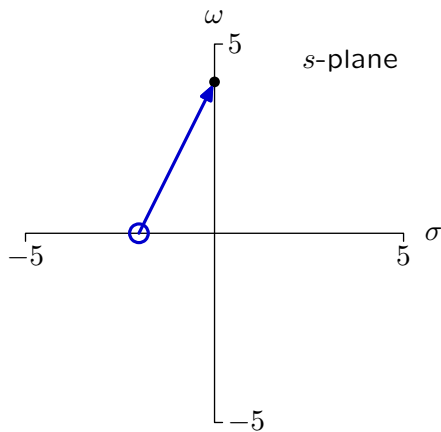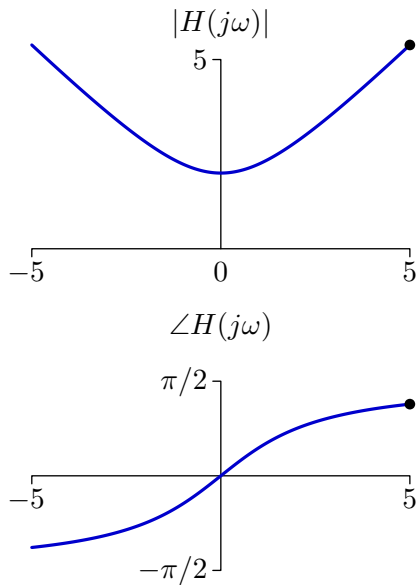
Vector Diagrams

$$H(s) = s - z_1$$

What is Negative Frequency?

Negative frequencies are needed to construct sinusoids from complex exponentials.
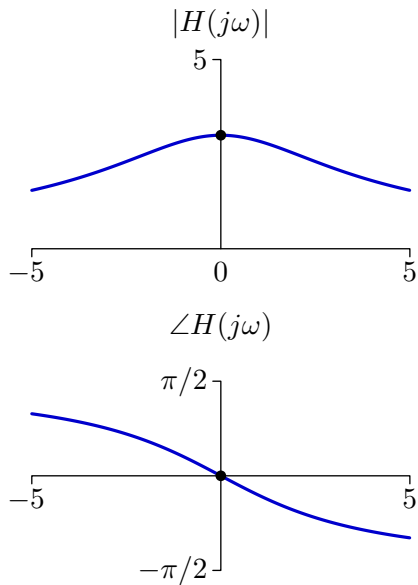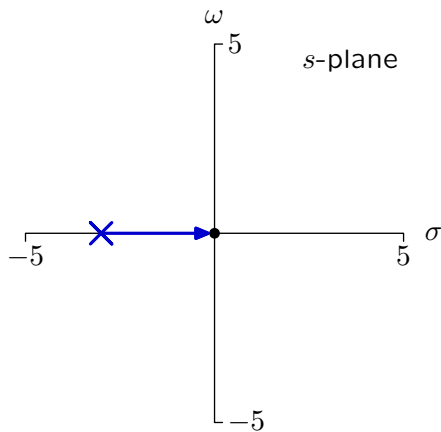
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

Negative frequencies are not physical. They are mathematical constructs to facilitate the use of complex exponentials.

$$H(s) = \frac{9}{s - p_1}$$

Vector Diagrams

$$H(s) = \frac{9}{s - p_1}$$

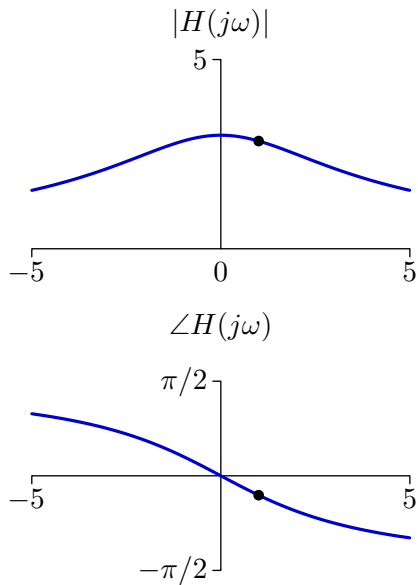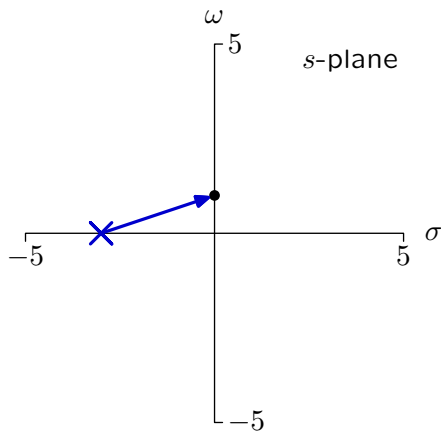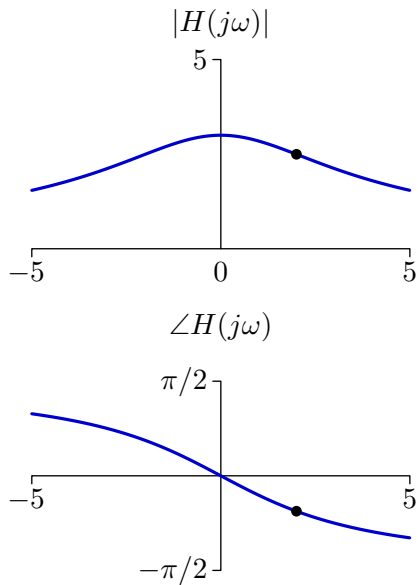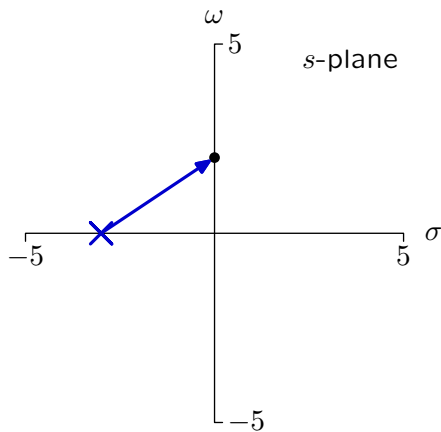

s-plane

$|H(j\omega)|$

$\angle H(j\omega)$

Vector Diagrams
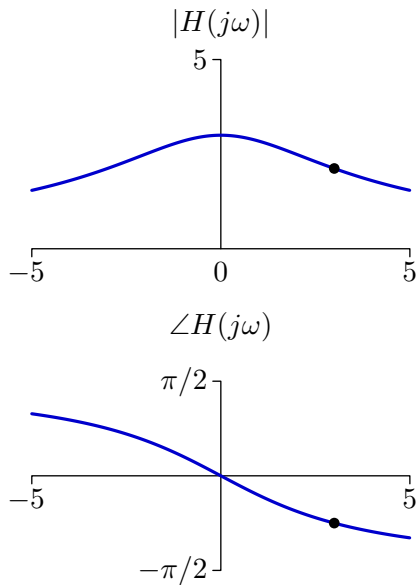
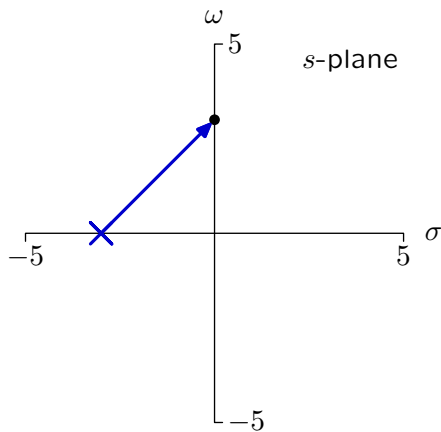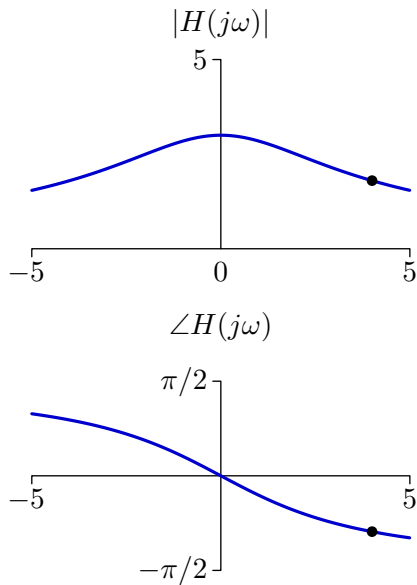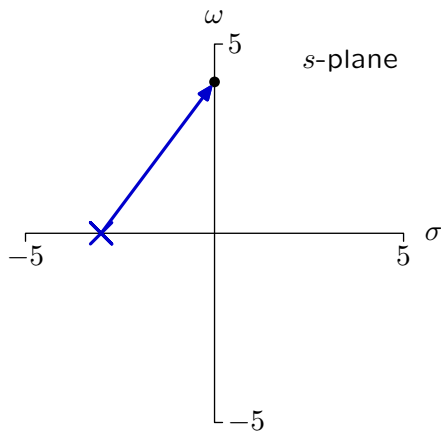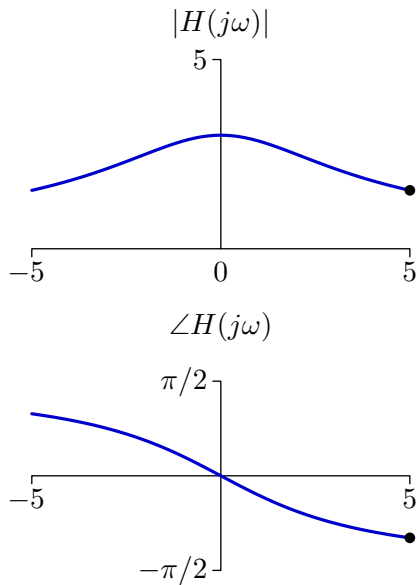$$H(s) = \frac{9}{s - p_1}$$

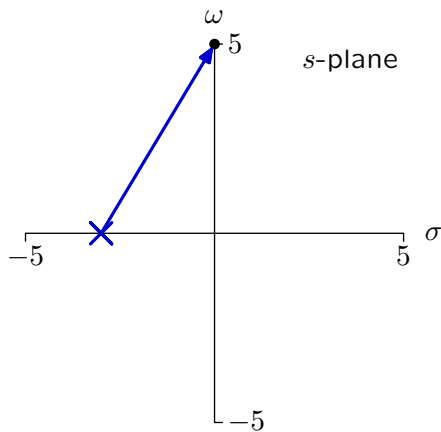## Vector Diagrams

$$H(s) = \frac{9}{s - p_1}$$
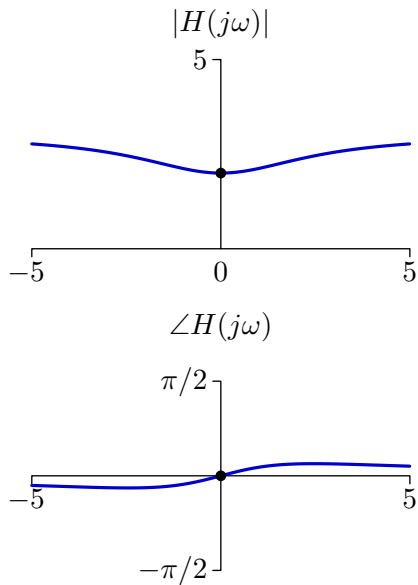
$$H(s) = \frac{9}{s - p_1}$$

Vector Diagrams

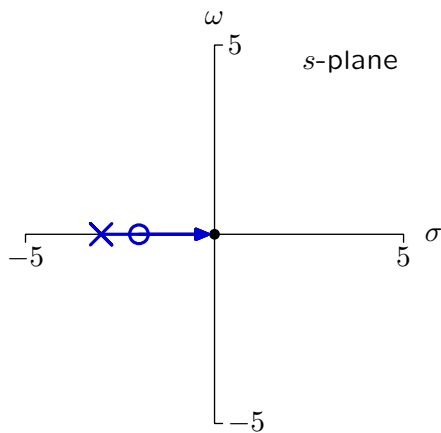$$H(s) = \frac{9}{s - p_1}$$

Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p_1}$$

s-plane

$|H(j\omega)|$

$\angle H(j\omega)$

Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p_1}$$

s-plane

$|H(j\omega)|$

$\angle H(j\omega)$

Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p_1}$$

$|H(j\omega)|$

$\angle H(j\omega)$

Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p_1}$$

$|H(j\omega)|$

$\angle H(j\omega)$

Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p_1}$$

ω

s-plane

σ

$|H(j\omega)|$

$\angle H(j\omega)$
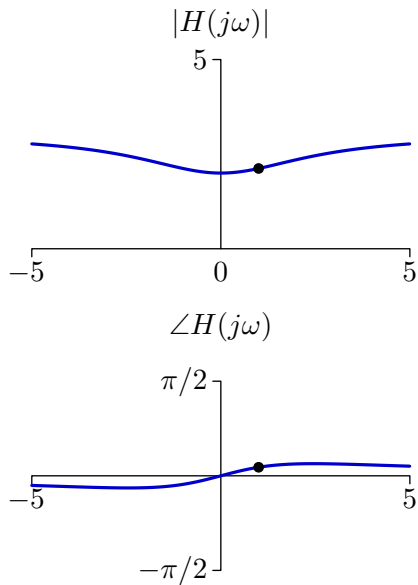
Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p_1}$$

s-plane

$|H(j\omega)|$

$\angle H(j\omega)$

Check Yourself

Sketch the magnitude and angle of the frequency response of the mass, spring, and dashpot system.
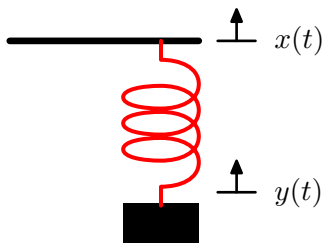
$$F = Ma = M\ddot{y}(t) = K(x(t) - y(t)) - B\dot{y}(t)$$

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = Kx(t)$$

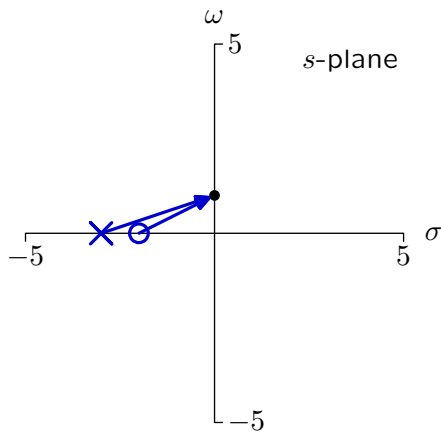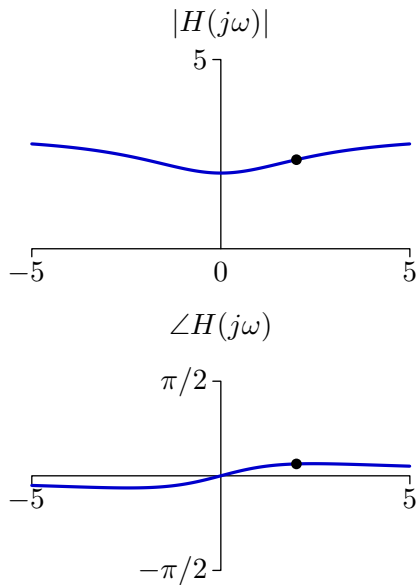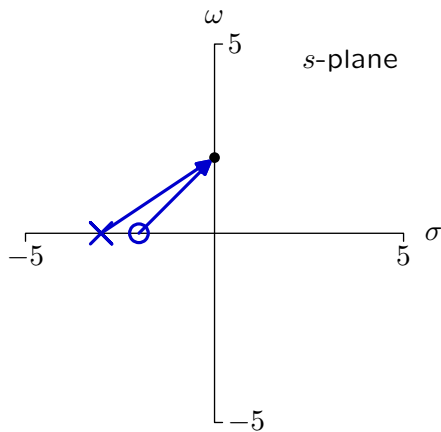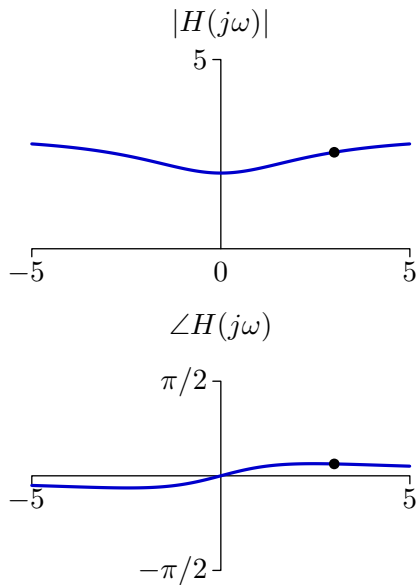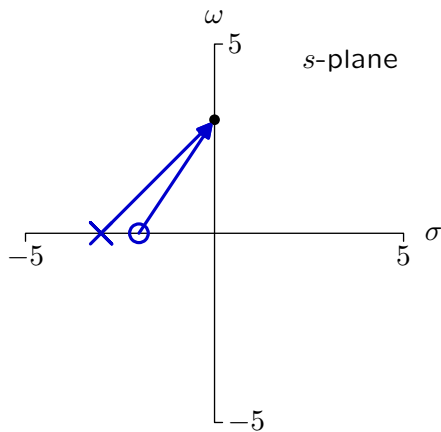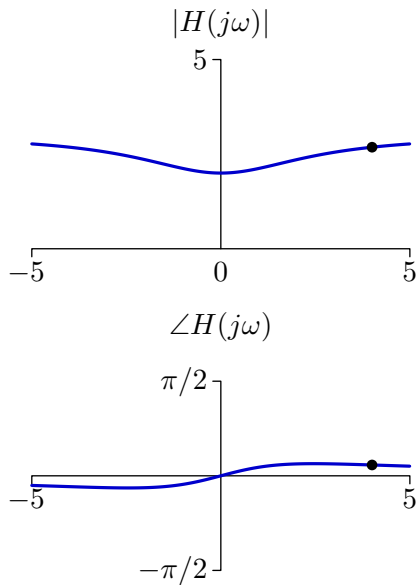$$(s^2M + sB + K)\ Y(s) = KX(s)$$

$$H(s) = \frac{K}{s^2M + sB + K}$$

Vector Diagrams

$$H(s) = \frac{15}{(s - p_1)(s - p_2)}$$
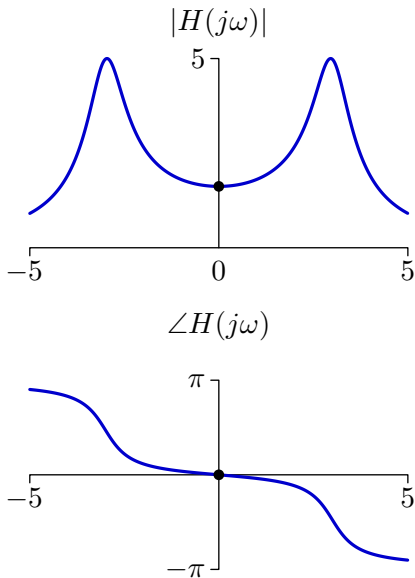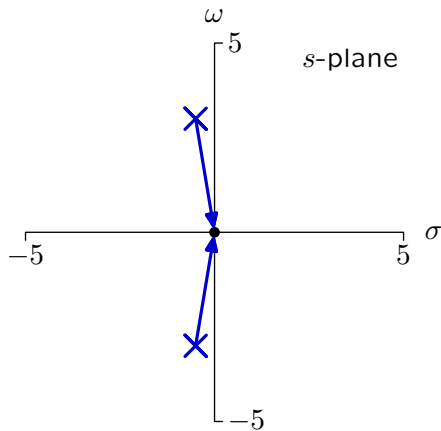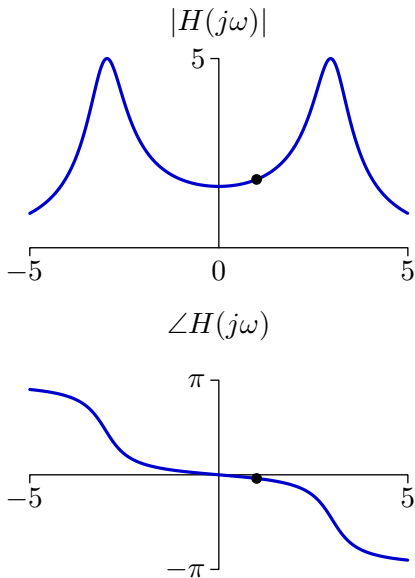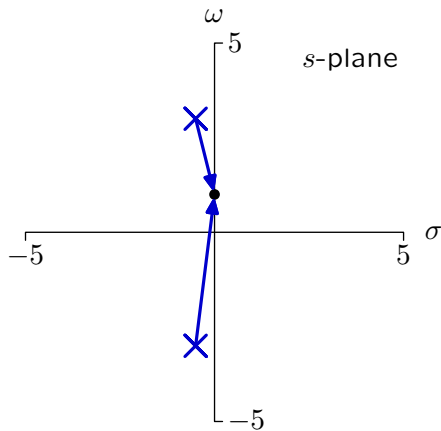
Vector Diagrams

$$H(s) = \frac{15}{(s - p_1)(s - p_2)}$$

Vector Diagrams

$$H(s) = \frac{15}{(s - p_1)(s - p_2)}$$

Vector Diagrams

$$H(s) = \frac{15}{(s - p_1)(s - p_2)}$$
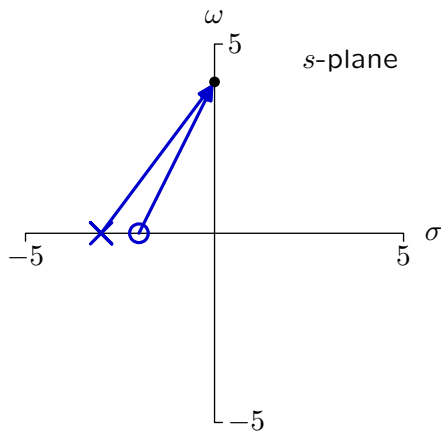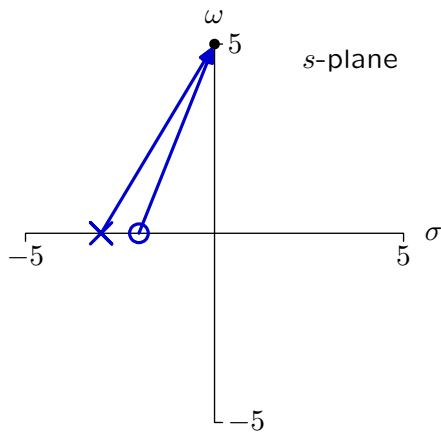
$$H(s) = \frac{15}{(s - p_1)(s - p_2)}$$

s-plane

$|H(j\omega)|$

$\angle H(j\omega)$
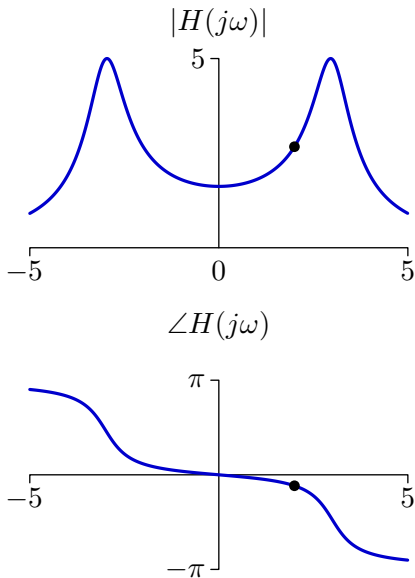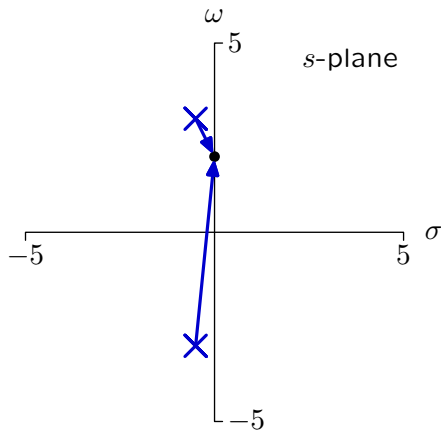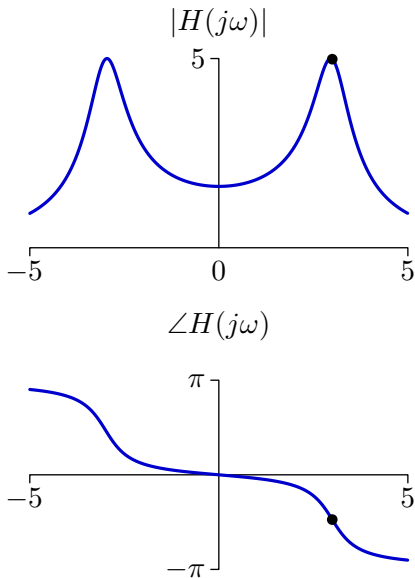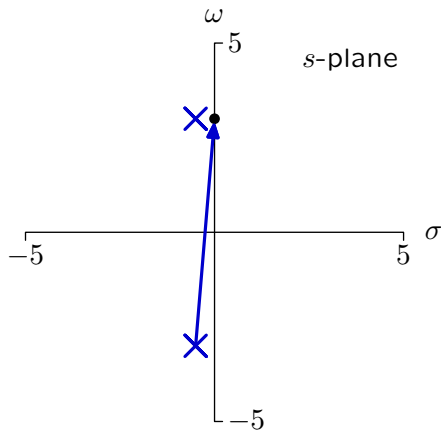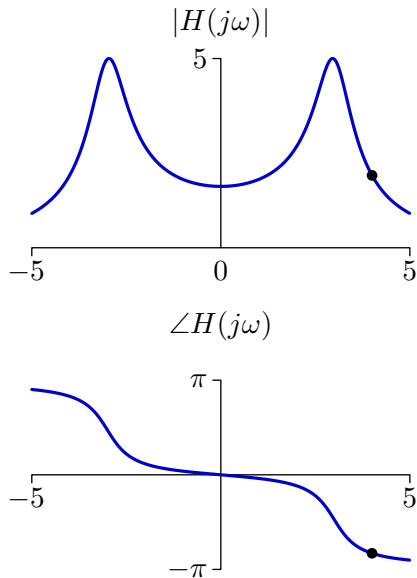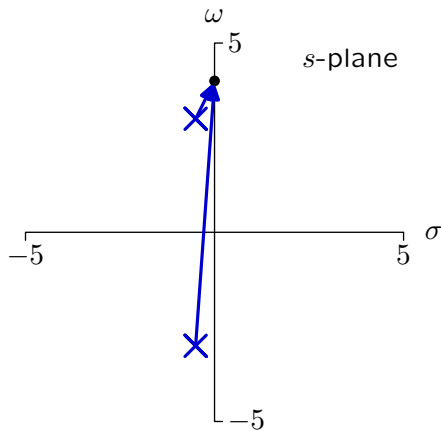
Vector Diagrams

$$H(s) = \frac{15}{(s - p_1)(s - p_2)}$$

Compare two systems that each have poles as $\frac{1+j}{2}$ and $\frac{1-j}{2}$:

$$H(z) = \frac{1}{z^2 - z - \frac{1}{2}} \quad \text{and} \quad H(s) = \frac{1}{s^2 - s - \frac{1}{2}}$$

Which (if any) of the following are true?

1. the homogeneous solutions for both systems are oscillatory
2. both systems are stable
3. the homogeneous solutions for both systems converge to 0
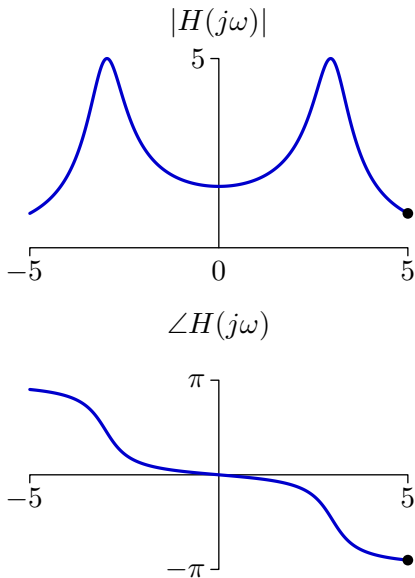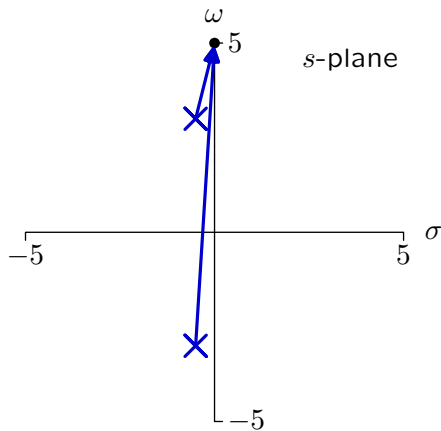
Check Yourself

Compare two systems that each have poles as $\frac{1+j}{2}$ and $\frac{1-j}{2}$:

$$H(z) = \frac{1}{z^2 - z - \frac{1}{2}} \quad \text{and} \quad H(s) = \frac{1}{s^2 - s - \frac{1}{2}}$$

The response of the discrete system (p^n) is a decaying oscillation.

The response of the continuous system (e^{pt}) is a growing oscillation.

The responses of the discrete and continuous systems are different because the functional dependence on the poles is different.

Compare two systems that each have poles as $\frac{1+j}{2}$ and $\frac{1-j}{2}$:

$$H(z) = \frac{1}{z^2 - z - \frac{1}{2}} \quad \text{and} \quad H(s) = \frac{1}{s^2 - s - \frac{1}{2}}$$
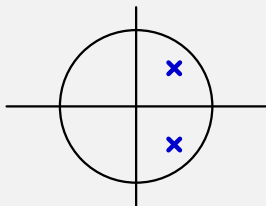
Which (if any) of the following are true?   1

1. the homogeneous solutions for both systems are oscillatory ✓
2. both systems are stable ✗
3. the homogeneous solutions for both systems converge to 0 ✗
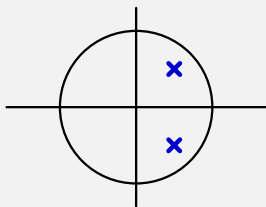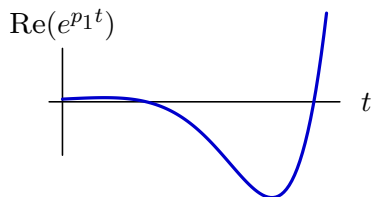
Check Yourself

Today we studied the **frequency response** of a CT system.

Our most important result is that the frequency response is easily determined from the system function.

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)| \cos\left(\omega t + \angle H(j\omega)\right)$$

The frequency response is equal to the **magnitude and angle** of the system function $H(s)$ evaluated at $s = j\omega$: $H(s)\Big|_{s=j\omega}$

What is the analogous statement for a DT system?

Check Yourself

Today we studied the **frequency response** of a CT system.

Our most important result is that the frequency response is easily determined from the system function.

$$\cos(\omega t) \longrightarrow \boxed{H_{ct}(s)} \longrightarrow |H_{ct}(j\omega)| \cos\left(\omega t + \angle H_{ct}(j\omega)\right)$$

The frequency response is equal to the **magnitude and angle** of the system function $H_{ct}(s)$ evaluated at $s = j\omega$: $H_{ct}(s)\Big|_{s=j\omega}$

For DT systems

$$\cos(\Omega n) \longrightarrow \boxed{H_{dt}(z)} \longrightarrow |H_{dt}(e^{j\Omega})| \cos\left(\Omega n + \angle H_{dt}(e^{j\Omega})\right)$$

The frequency response is equal to the **magnitude and angle** of the system function $H_{dt}(z)$ evaluated at $z = e^{j\Omega}$: $H_{dt}(z)\Big|_{z=e^{j\Omega}}$

The frequency response of a CT system is on the $j\omega$-axis.

The frequency response of a DT system is on the unit circle.

Summary

Today we developed the idea of a frequency response as an alternative way to describe the behavior of a system.

Next week we will see that the frequency responses provide a new way to think about the design of control systems.