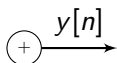**6.3100: Dynamic System Modeling and Control Design**

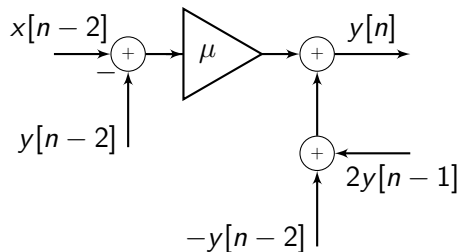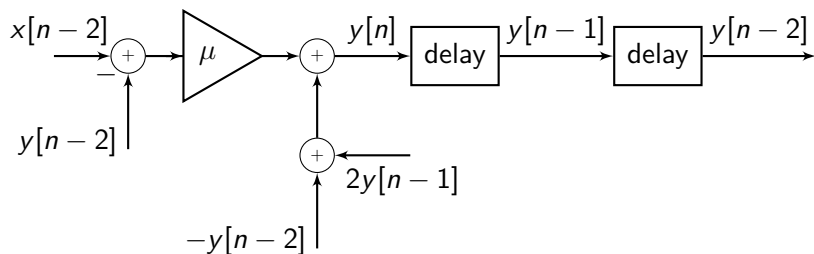# From DT to CT

October 02, 2024

Block Diagram Overview

$$y[n] = 2y[n-1] - y[n-2] + \mu(x[n-2] - y[n-2])$$

Block Diagram Overview

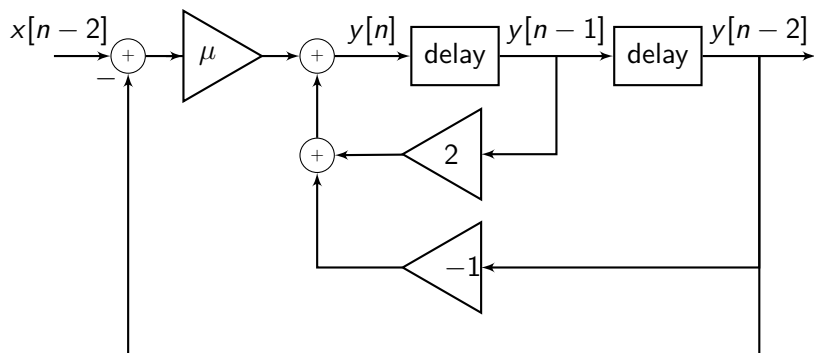$$y[n] = 2y[n-1] - y[n-2] + \mu(x[n-2] - y[n-2])$$

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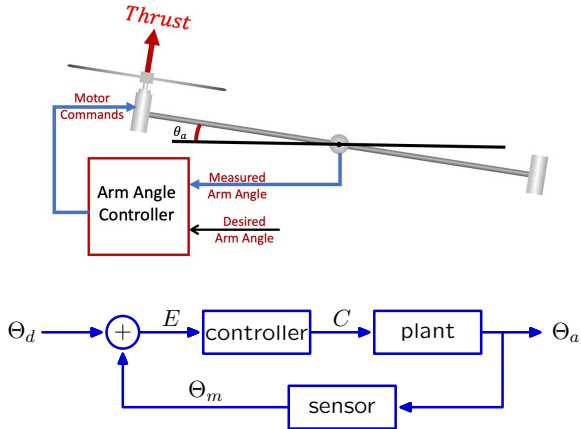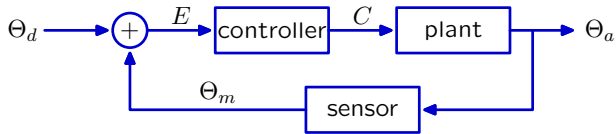
Discrete Time or Continuous Time?

Like many modern control systems, our propeller arm system combines parts that are most naturally described in continuous time (plant and sensor) with a controller that is unequivocally discrete time.

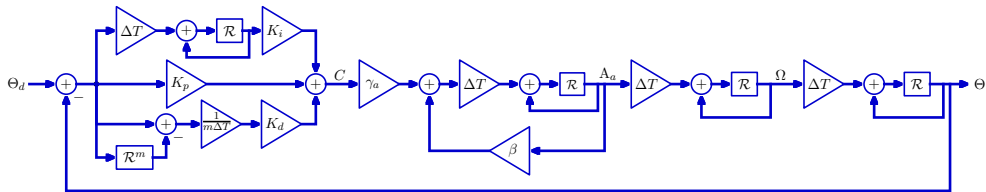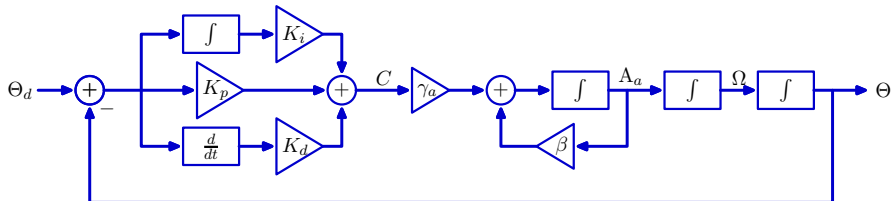What are the inputs and outputs of the plant / sensor / controller?
Are they best described as discrete-time or continuous time?

Two Descriptions

We modeled the propeller arm system in discrete time.

Alternatively, we could have modeled it in continuous time instead.

The former provides a more literal description of what the controller does.
The latter is a more literal description of what the plant and sensor do.

Two Descriptions

We modeled the propeller arm system in discrete time.
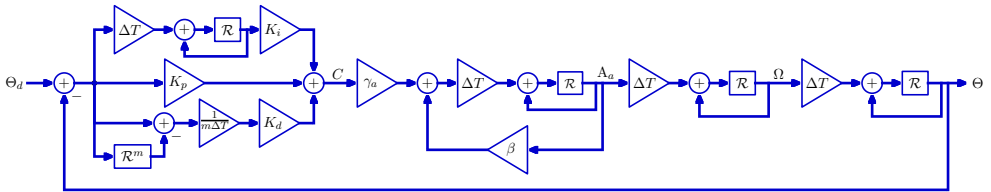
Alternatively, we could have modeled it in continuous time instead.

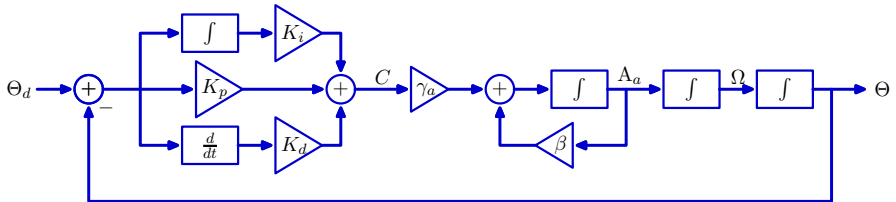

The former is readily analyzed using a **step-by-step** numerical method.
The latter provides new insights based on responses to **sinusoidal inputs**.

This week: New insights from continuous-time representations.

Difference Equations and Differential Equations

Discrete-time systems are readily described with **difference equations**.

Example:

$$y_d[n] = x_d[n] + py_d[n-1]$$

Continuous-time systems are readily described with **differential equations**.

Example:

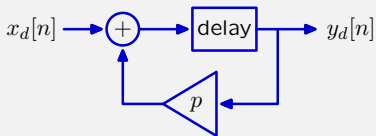$$\frac{dy_c(t)}{dt} = x_c(t) + py_c(t)$$

Check Yourself

The following difference equation

$$y_d[n] = x_d[n] + py_d[n-1]$$

is best represented by which of the following block diagrams?

1

$x_d[n] \rightarrow \boxed{+} \rightarrow \boxed{\text{delay}} \rightarrow y_d[n]$, with feedback through p

2

$x_d[n] \rightarrow \boxed{+} \rightarrow y_d[n]$, with feedback through delay then p

3

$x_d[n] \rightarrow \boxed{+} \rightarrow \boxed{\text{delay}} \rightarrow \boxed{p} \rightarrow y_d[n]$

4

$x_d[n] \rightarrow \boxed{+} \rightarrow \boxed{p} \rightarrow y_d[n]$, with feedback through delay
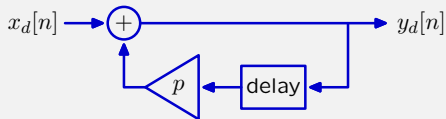
5. None of the above

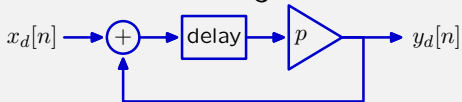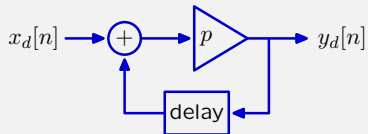## Check Yourself

The following differential equation
$$\frac{dy_c(t)}{dt} = x_c(t) + py_c(t)$$
is best represented by which of the following block diagrams?
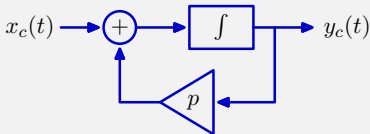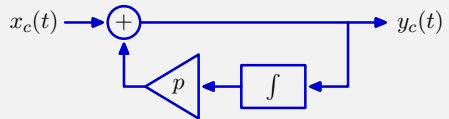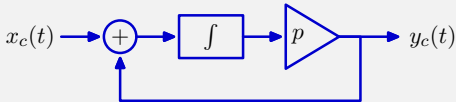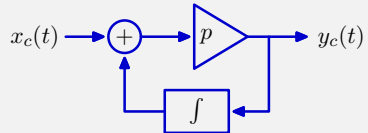
5. None of the above

Discrete-Time and Continuous-Time Systems

Discrete-time systems and continuous-time systems can be analyzed with **similar** but **not identical** methods.

Discrete-Time Example:

$$y_d[n] = x_d[n] + py_d[n-1]$$
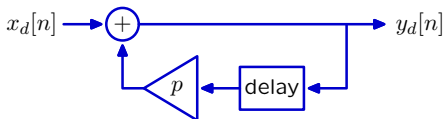
Continuous-Time Example:

$$\frac{dy_c(t)}{dt} = x_c(t) + py_c(t)$$
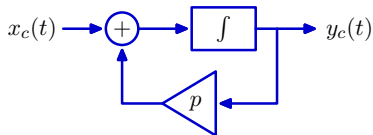
Discrete-Time Eigenfunctions

When the inputs to discrete-time adders, gains, and delays are proportional to z^n, their outputs are also proportional to z^n.
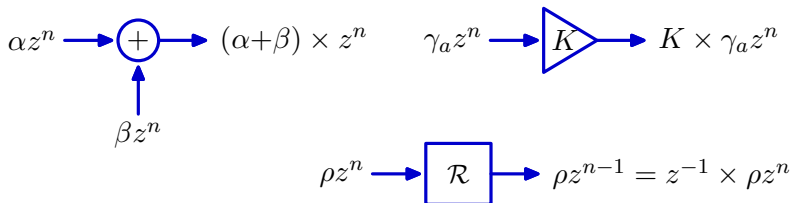
The eigenfunctions of discrete-time systems are **complex geometrics**.

Continuous-Time Eigenfunctions

When the inputs to continuous-time adders, gains, integrators, and differentiators are proportional to e^{st}, their outputs are proportional to e^{st}.
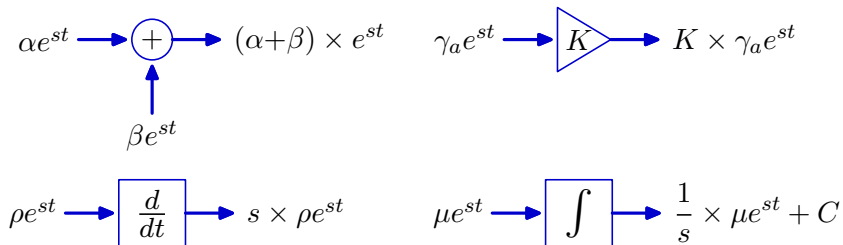
The eigenfunctions of continuous-time systems are **complex exponentials**.

Check Yourself

Determine the **natural frequencies** of these systems.

$$y_d[n] = x_d[n] + py_d[n-1]$$

$$\frac{dy_c(t)}{dt} = x_c(t) + py_c(t)$$

Which of the following statements about the natural frequencies of these systems are true?

1. The natural freq's are the same b/c eigenfunctions are the same.
2. The natural freq's are the same b/c block diagrams are the same.
3. Different b/c first depends on n and second depends on t.
4. Both natural frequencies are equal to p.
5. None of the above

Functional Characterization of DT Systems

DT systems are **stable** (i.e., their zero-input responses converge to zero) when their natural frequencies are inside the unit circle.

$$\lim_{n\to\infty} y_d[n] = \lim_{n\to\infty} \lambda^n = \lim_{n\to\infty} \left(|\lambda|e^{j\angle\lambda}\right)^n = \lim_{n\to\infty} |\lambda|^n e^{jn\angle\lambda} = 0 \text{ if } |\lambda| < 1$$
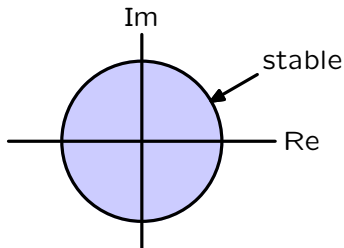
Responses of DT systems are
- **oscillatory** when $0 < \text{abs}(\angle(\lambda)) < \pi$,
- **monotonic** when $\angle(\lambda) = 0$, and
- **alternating** when $\angle(\lambda) = \pi$.

Functional Characterization of CT Systems

CT systems are **stable** (i.e., their zero-input responses converge to zero) when their natural frequencies are in the left half-plane.

$$\lim_{t \to \infty} y_c(t) = \lim_{t \to \infty} e^{st} = \lim_{t \to \infty} e^{(a+jb)t} = \lim_{t \to \infty} e^{at} e^{jbt} = 0 \text{ if } a < 0$$
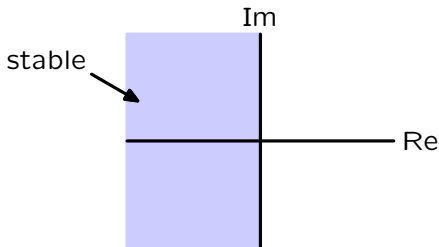
Responses of CT systems are

- stable when $\mathrm{Re}\,(s) < 0$.
- oscillatory when $\mathrm{Im}\,(s) \neq 0$.
- eternal sinusoids when $\mathrm{Re}\,(s) = 0$.

Check Yourself

The following plots show regions of the s-plane or z-plane in red.

A Im / Re

B Im / Re

C Im / Re

D Im / Re

E Im / Re

F Im / Re

G Im / Re

H Im / Re

I Im / Re

J Im / Re

K Im / Re

L Im / Re
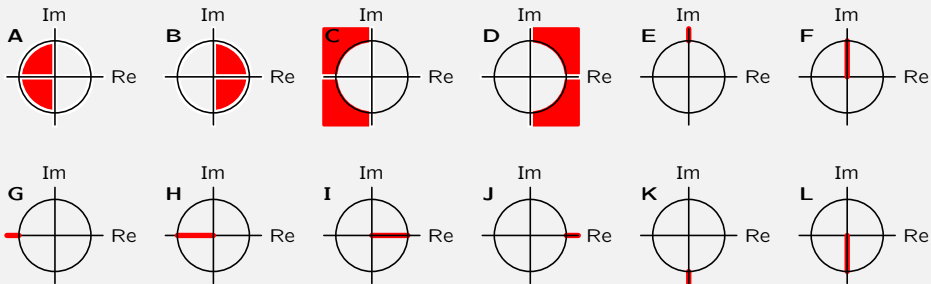
For which frequencies z are DT systems stable with ringing?

For which frequencies s are CT systems stable with ringing?

Black's Equation

Black's equation for CT systems is similar to that for DT systems.
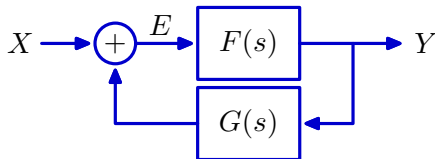
$$Y = F(s)E = F(s)\Big(X + G(s)Y\Big) = F(s)X + F(s)G(s)Y$$

$$\Big(1 - F(s)G(s)\Big)Y = F(s)X$$

The transformation from X to Y is given by following:

$$H(s) = \frac{Y}{X} = \frac{F(s)}{1 - F(s)G(s)}$$

which is known as **Black's equation**.

Modularity

Feed-forward systems that contain only adders, gains, differentiators, and integrators can be represented as a rational polynomial of the following form:

$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots}$$

If both the forward and feedback paths through a system with feedback can be represented as rational polynomials in s, then the system function can be expressed as a rational polynomial of the same form:

$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots}$$

Modularity

If the forward path through a feedback system contains feedback, then the forward path can be represented by a rational polynomial $N_1(s)/D_1(s)$.

If the feedback path through a feedback system contains feedback, then the feedback path can be represented by a rational polynomial $N_2(s)/D_2(s)$.
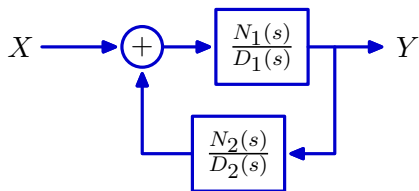
We can apply Black's formula to find the resulting system function:

$$\frac{Y}{X} = \frac{\frac{N_1(s)}{D_1(s)}}{1 - \frac{N_1(s)}{D_1(s)}\frac{N_2(s)}{D_2(s)}} = \frac{N_1(s)D_2(s)}{D_1(s)D_2(s) - N_1(s)N_2(s)}$$

Since the product of polynomials is polynomial, it follows that the overall system function is a rational polynomial.

Partial Fractions

The natural frequencies of a system can be identified by expanding the system function H in partial fractions.

$$H = \frac{Y}{X} = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots}{1 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots}$$

Factor denominator:

$$H = \frac{Y}{X} = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots}{(s - p_0)(s - p_1)(s - p_2)(s - p_3) \cdots}$$

We call the roots of the denominator the **poles** of the system.

Partial fractions:

$$H = \frac{Y}{X} = \frac{C_0}{s - p_0} + \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \cdots + D_0 + D_1 s + D_2 s^2 + \cdots$$

One **natural frequency** arises from each factor of the denominator.

Check Yourself

Consider the system described by
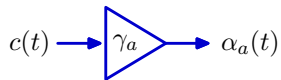$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$
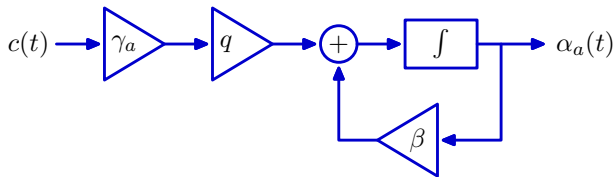
How many of the following are true?

1. If $x(t) = 0$ then $\lim_{t\to\infty} y(t) = 0$.
2. There are two poles.
3. There are poles at $s = 1$ and $s = 2$.
4. The step response of this system is oscillatory.
5. None of the above

Motor Model

The **instant** model of the motor in "Code of Arms" has the following form

$$c(t) \longrightarrow \boxed{\gamma_a} \longrightarrow \alpha_a(t)$$

and the **non-instant** model has the following form.

In both models, the motor takes an input command $c(t)$ and generates angular acceleration $\alpha_a(t)$ of the arm.

Motor Model

The **instant** model of the motor in "Code of Arms" has the following form

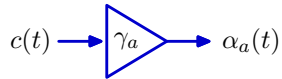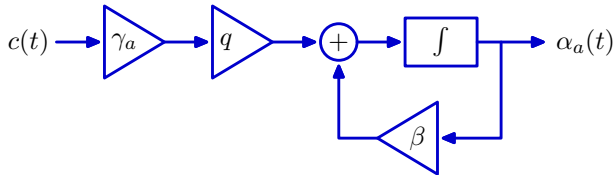$$c(t) \longrightarrow \boxed{\gamma_a} \longrightarrow \alpha_a(t)$$

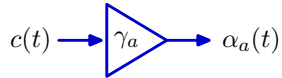and the **non-instant** model has the following form.

In both models, the motor takes an input command $c(t)$ and generates angular acceleration $\alpha_a(t)$ of the arm.
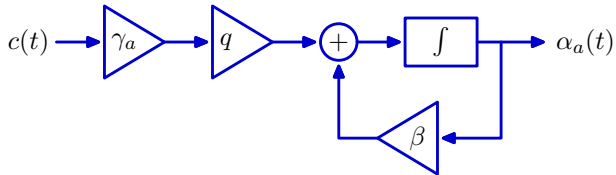
Find q so that the steady-state output $\alpha_a(\infty)$ is the same for both models when the input $c(t)$ is a constant c_0.

Motor Model

The **instant** model of the motor in "Code of Arms" has the following form

$$c(t) \longrightarrow \boxed{\gamma_a} \longrightarrow \alpha_a(t)$$

and the **non-instant** model has the following form.

If the output is constant, then the input to the integrator must be zero.

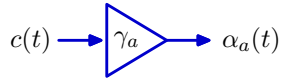If the input to the integrator is zero, then the two inputs to the adder must sum to zero:

$$\beta \alpha_a(\infty) + q \gamma_a c_0 = 0$$

But $\gamma_a c_0 = \alpha_a(\infty)$ so

$$q = -\beta$$

Motor Model

The **instant** model of the motor in "Code of Arms" has the following form

$$c(t) \longrightarrow \boxed{\gamma_a} \longrightarrow \alpha_a(t)$$

and the **non-instant** model has the following form.
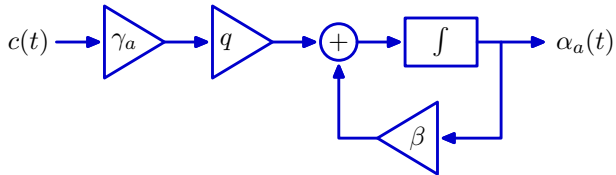
Determine the step response of the non-instant model.

Motor Model

Start by determining the differential equation that relates $c(t)$ and $\alpha_a(t)$.

$$\frac{d\alpha_a(t)}{dt} = \beta\alpha_a(t) - \beta\gamma_a c(t)$$

Assume the motor speed is initially zero and the input $c(t) = 1$ for $t > 0$.

Output $\alpha_a(t)$ can be written as a sum of homogeneous and particular parts.

The particular part $\alpha_p(t)$ has the form of the input signal: a constant B.

Substituting $\alpha_p(t) = B$ into the differential equations shows that $B = \gamma_a$.

The homogeneous part can be found by setting the input $c(t)$ to 0.

$$\frac{d\alpha_h(t)}{dt} = \beta\alpha_h(t)$$

The resulting equation defines an eigenfunction, so $\alpha_h(t) = Ae^{st}$ for $t > 0$.

$$sAe^{st} = \beta Ae^{st}$$
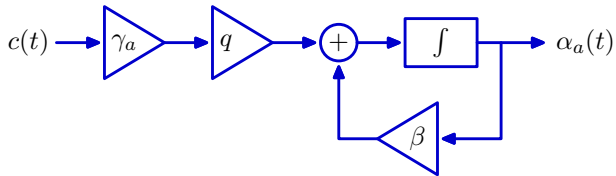
The only non-trival solution is $s = \beta$.

$$\alpha_a(t) = \alpha_h(t) + \alpha_p(t) = Ae^{\beta t} + \gamma_a$$

Setting $\alpha_a(0)$ to 0 shows that $A = -\gamma_a$ so the final answer is

$$\alpha_a(t) = \gamma_a(1 - e^{\beta t})$$

Step Response

Non-instant motor model:

Step response of motor model

$$\alpha_a(t) = \gamma_a(1 - e^{\beta t})$$

Result when $\beta = -20$: