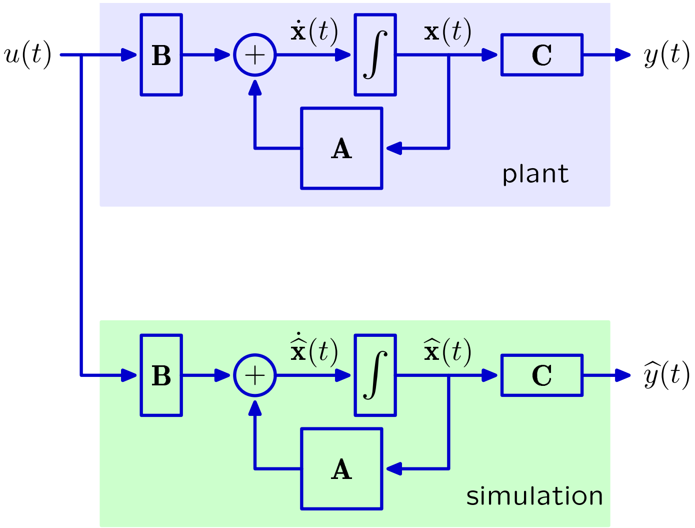
6.3100: Dynamic System Modeling and Control Design

Observer Design

November 25, 2024

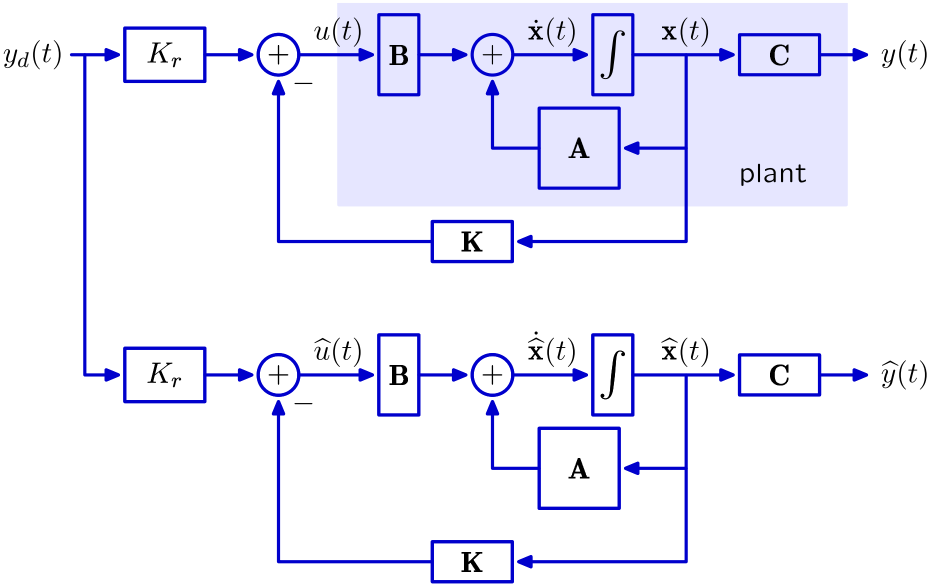
Observers

Last time, we introduced the notion of an **observer**, which is a **model** of a plant that is used to improve performance of a controller.



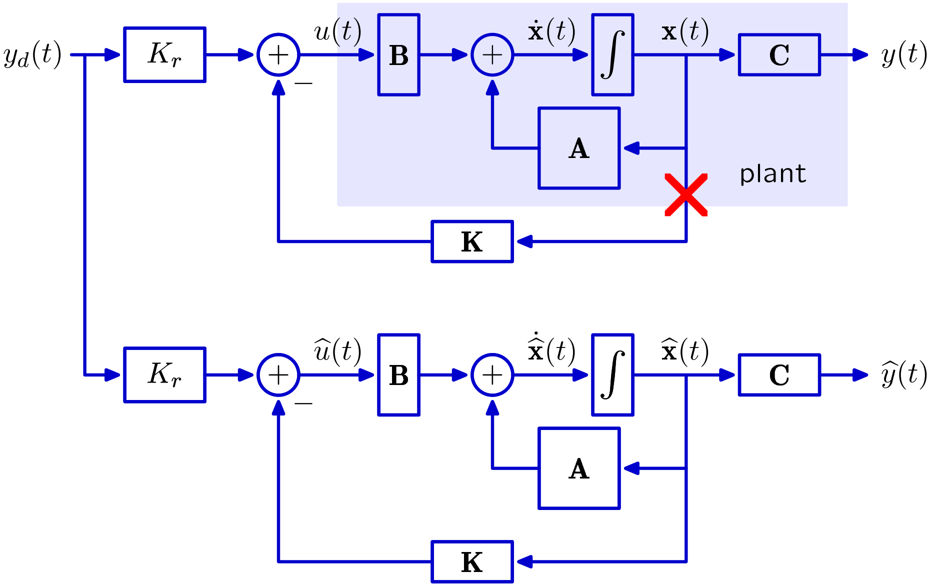
Observers

We embed both the plant and observer in state-space controllers, so that the simulation can provide estimates of the state ($\widehat{\mathbf{x}}(t)$) and input ($\widehat{u}(t)$).



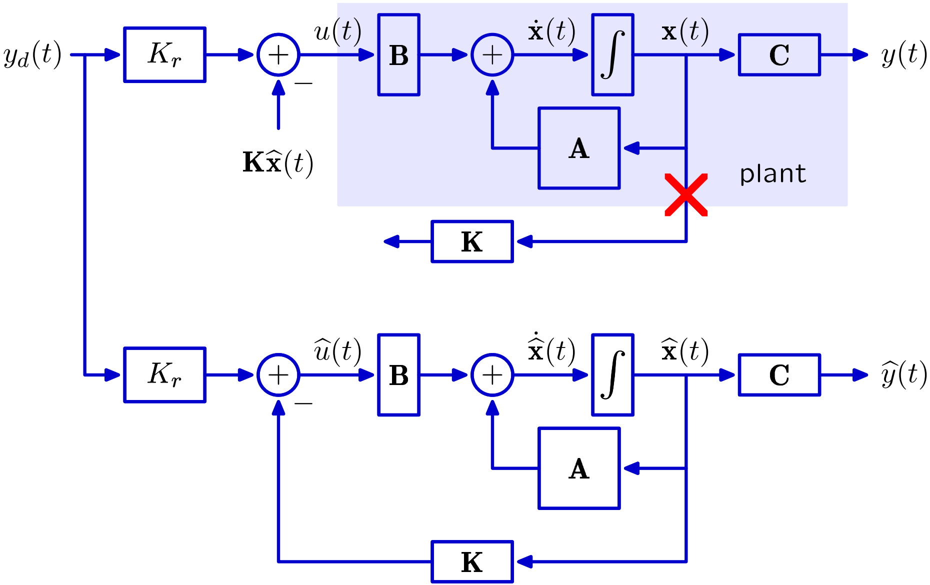
Observers

If feedback from all of the states in $\mathbf{x}(t)$ is not possible ...



Observers

... we can substitute the simulated states $\widehat{\mathbf{x}}(t)$ for the missing states $\mathbf{x}(t)$. Similarly, we can substitute $\widehat{u}(t)$ for $u(t)$ as well.



Observers

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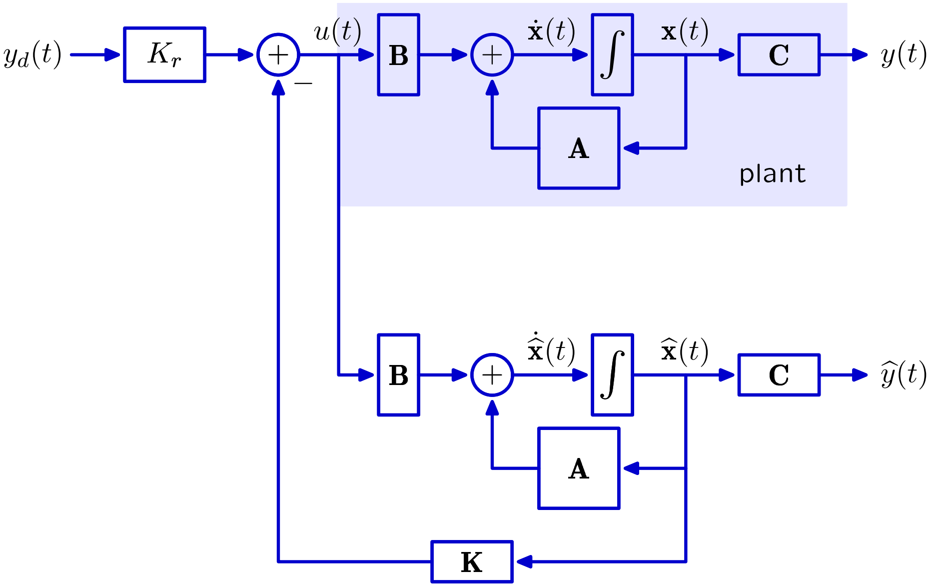
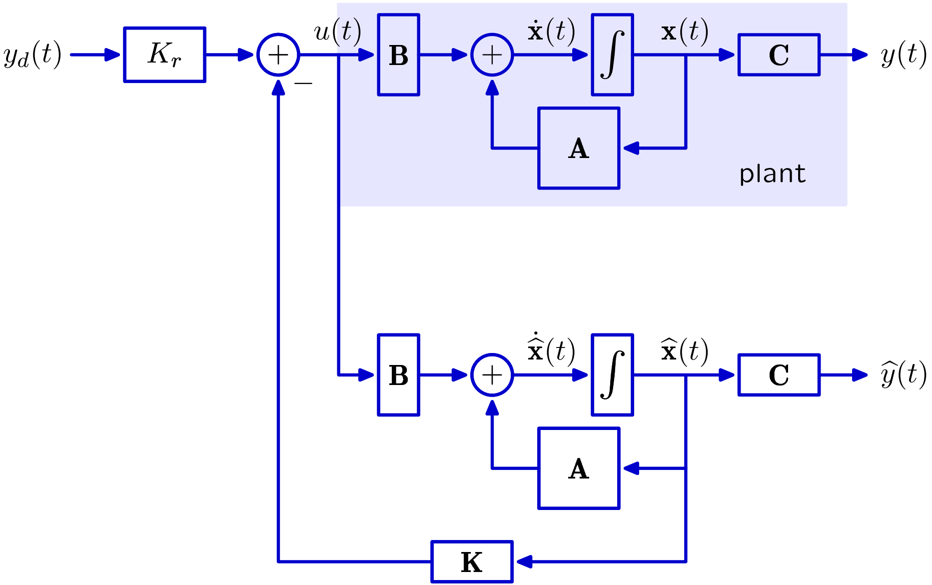


Observers

Unfortunately the resulting system is **feedforward**. It's utility depends critically on the accuracy of the match between the plant and observer.

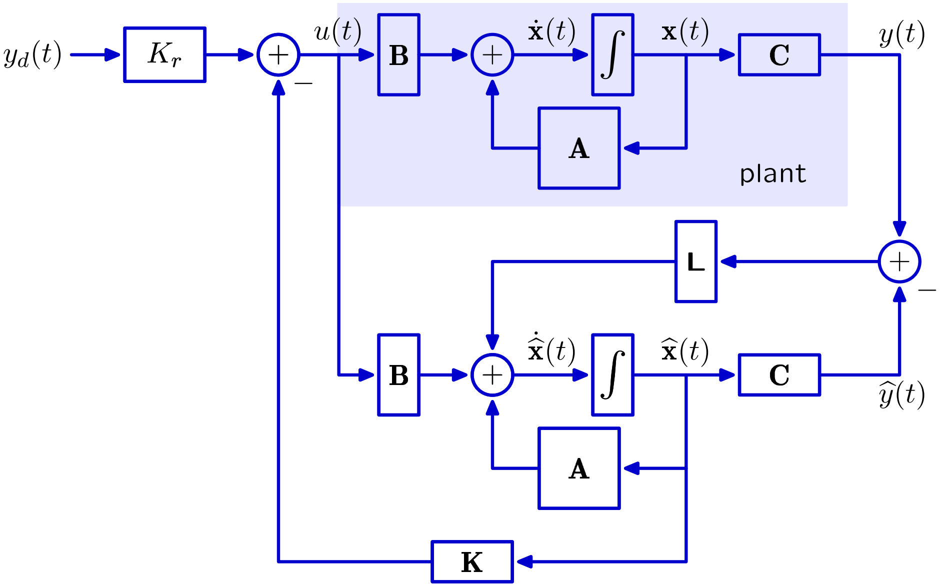


Observers

Fortunately, we can use **feedback** to measure and correct simulation errors!

Calculate the difference between $y(t)$ and $\widehat{y}(t)$.

Then use that signal (times $\mathbf{L}$) to correct $\dot{\widehat{\mathbf{x}}}(t)$.

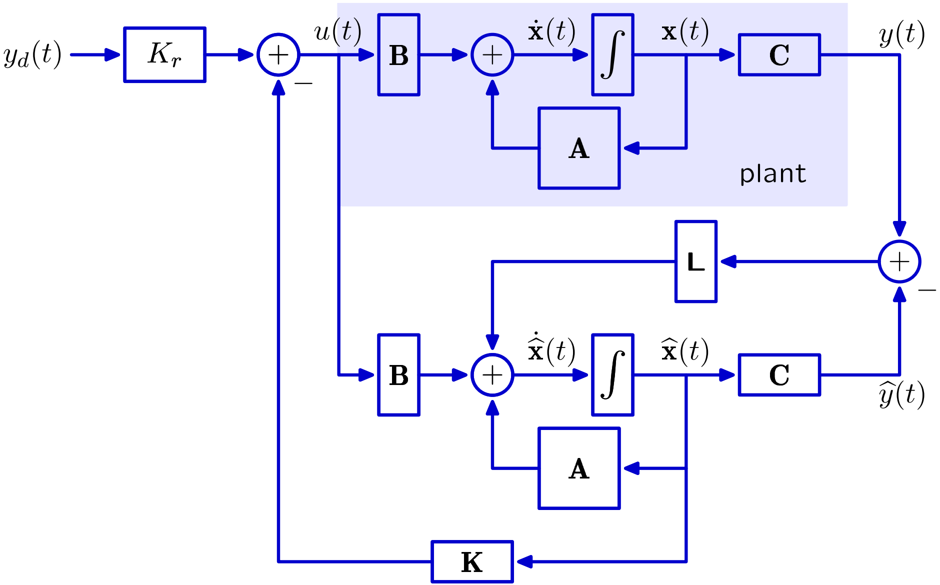


Observers

Analyze by finding matrix expressions for the derivatives of the states.

Plant dynamics: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t)$

Simulation dynamics: $\dot{\widehat{\mathbf{x}}}(t) = \mathbf{A}\widehat{\mathbf{x}}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t) + \mathbf{L}(y(t) - \widehat{y}(t))$



Observers

Combined dynamics of the plant and observer.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t)$$
$$\dot{\widehat{\mathbf{x}}}(t) = \mathbf{A}\widehat{\mathbf{x}}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t) + \mathbf{L}\Big(y(t) - \widehat{y}(t)\Big)$$

Define $\mathbf{e}(t)$ to be the difference between the plant and simulation states:

$$\mathbf{e}(t) = \mathbf{x}(t) - \widehat{\mathbf{x}}(t)$$

Subtract $\dot{\widehat{\mathbf{x}}}(t)$ from $\dot{\mathbf{x}}(t)$ to find the derivative of $\mathbf{e}(t)$:

$$\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t) - \mathbf{L}\Big(y(t) - \widehat{y}(t)\Big) = \mathbf{A}\mathbf{e}(t) - \mathbf{L}\mathbf{C}\mathbf{e}(t)$$

Append the $\dot{\mathbf{x}}(t)$ and $\dot{\mathbf{e}}(t)$ to make a new **combined** state vector:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} K_r y_d(t)$$

Observers

Combined dynamics of the plant and observer.

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} K_r y_d(t)$$

The poles of this system are the roots of its characteristic equation:

$$\left| s\mathbf{I} - \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \right| = 0$$

Because the evolution matrix has block triangular form, the characteristic equation can be factored into two parts:

$$\left| s\mathbf{I} - \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \right| = \left| s\mathbf{I} - (\mathbf{A}-\mathbf{BK}) \right| \times \left| s\mathbf{I} - (\mathbf{A}-\mathbf{LC}) \right| = 0$$

and the poles of the augmented system are the union of the poles of the plant and simulation dynamics.

Furthermore, $\mathbf{K}$ can be chosen to optimize $\mathbf{A}-\mathbf{BK}$ and
$\mathbf{L}$ can be chosen to optimize $\mathbf{A}-\mathbf{LC}$.

We've seen the first part of this problem before!

Linear Quadratic Regulator (LQR) – Redux!

The LQR method minimizes a **cost function** J that weighs the squares of the state variables $\mathbf{x}(t)$ and input $\mathbf{u}(t)$.

$$J = \int_0^\infty \left(\mathbf{x}^\mathbf{T}(t)\,\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\mathbf{T}(t)\,\mathbf{R}\mathbf{u}(t)\right) dt$$

where $\mathbf{u}(t)$ and $\mathbf{x}(t)$ are related

- by the state transition equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and
- by the feedback constraint: $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$.

and $\mathbf{Q}$ and $\mathbf{R}$ represent weights.

The “optimal” $\mathbf{K}$ is given by

$$\mathbf{K} = \mathbf{R}^{-\mathbf{1}}\mathbf{B}^\mathbf{T}\mathbf{S}$$

where $\mathbf{S}$ is the symmetric $n \times n$ solution to the **algebraic Riccati equation**:

$$\mathbf{A}^\mathbf{T}\mathbf{S} + \mathbf{S}\mathbf{A} - \mathbf{S}\mathbf{B}\mathbf{R}^{-\mathbf{1}}\mathbf{B}^\mathbf{T}\mathbf{S} + \mathbf{Q} = \mathbf{0}$$

Linear Quadratic Regulator (LQR) – Double Redux!

The LQR method minimizes a **cost function** J' that weighs the squares of observer state errors $\widehat{\mathbf{e}}(t) = \mathbf{x}(t) - \widehat{\mathbf{x}}(t)$ and observer output errors $y(t) - \widehat{y}(t)$.

$$J' = \int_0^\infty \Big(\mathbf{e}(t)^T \mathbf{Q}\, \mathbf{e}(t) + (y(t) - \widehat{y}(t))^T\, \mathbf{R}\, (y(t) - \widehat{y}(t))\Big) dt$$

where $(y(t) - \widehat{y}(t))$ and $\mathbf{e}(t)$ are related

- by a state transition equation: $\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t) - \mathbf{L}(y(t) - \widehat{y}(t))$ and
- by the feedback constraint: $y(t) - \widehat{y}(t) = \mathbf{C}\mathbf{e}(t)$.

and $\mathbf{Q}$ and $\mathbf{R}$ represent weights.

The "optimal" $\mathbf{L}$ is given by

$$\mathbf{L}^{\mathbf{T}} = \mathbf{R}^{-1}\mathbf{C}\mathbf{S}$$

where $\mathbf{S}$ is the symmetric $n \times n$ solution to the **algebraic Riccati equation**:

$$\mathbf{A}\mathbf{S} + \mathbf{S}\mathbf{A}^{\mathbf{T}} - \mathbf{S}\mathbf{C}^{\mathbf{T}}\mathbf{R}^{-1}\mathbf{C}\mathbf{S} + \mathbf{Q} = \mathbf{0}$$

Choosing K and L

Since optimizing **K** and **L** can be cast into problems with the same form, the optimizations can be solved using the same methods.

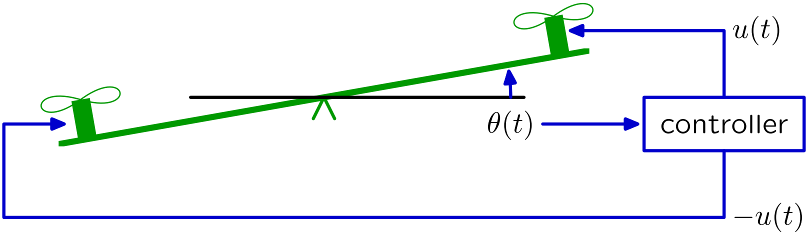
```
K = place(A,B,[poles])
L = transpose(place(transpose(A),transpose(C),[poles]))
```

or

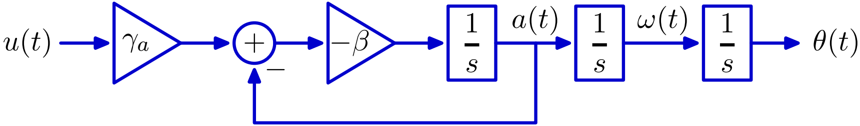
```
K = lqr(A,B,Qk,Rk)
L = transpose(lqr(transpose(A),transpose(C),Ql,Rl))
```

Observers

Example: propeller arm (lab 6)



Model of plant:



$$\gamma_a = 55; \quad \beta = -14$$

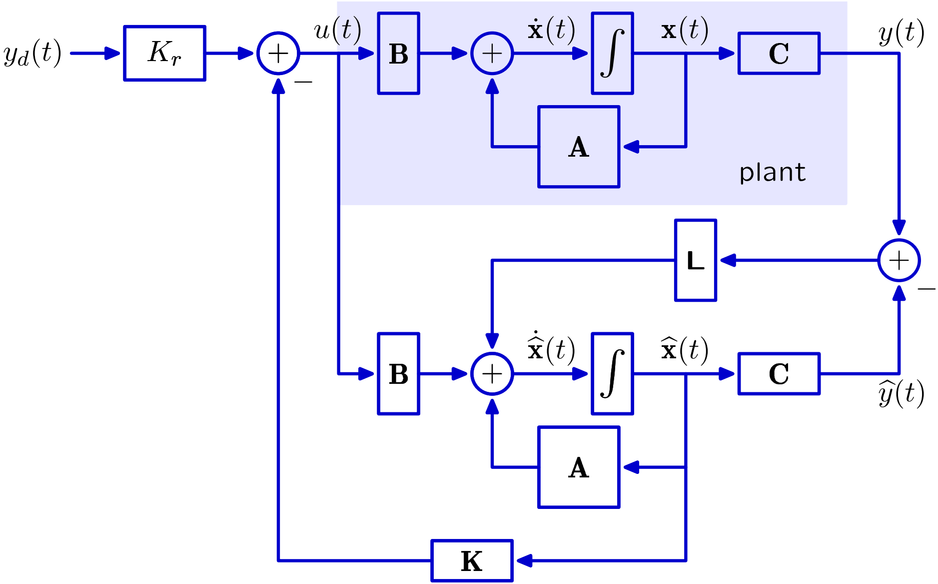
$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\\ a\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & \beta\end{bmatrix}}_{\mathbf{A}}\begin{bmatrix}\theta\\ \omega\\ a\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 0\\ -\beta\gamma_a\end{bmatrix}}_{\mathbf{B}} u(t); \qquad y(t) = \underbrace{\begin{bmatrix}1 & 0 & 0\end{bmatrix}}_{\mathbf{C}}\begin{bmatrix}\theta\\ \omega\\ a\end{bmatrix}$$

Observers

Substitute $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ into the observer framework.

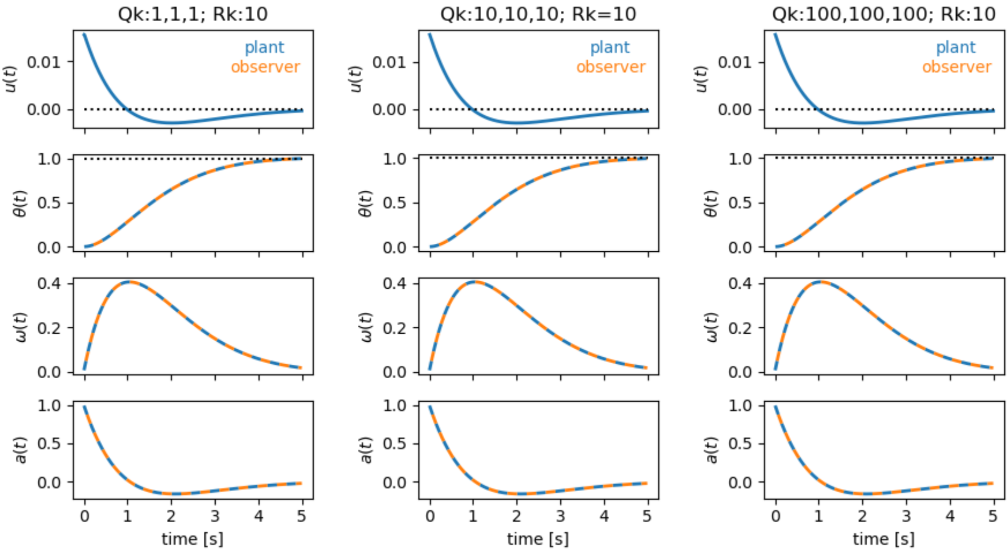
Plant dynamics: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t)$

Simulation dynamics: $\dot{\widehat{\mathbf{x}}}(t) = \mathbf{A}\widehat{\mathbf{x}}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t) + \mathbf{L}(y(t) - \widehat{y}(t))$



Choosing Gains

Effect of increasing Q_k's.

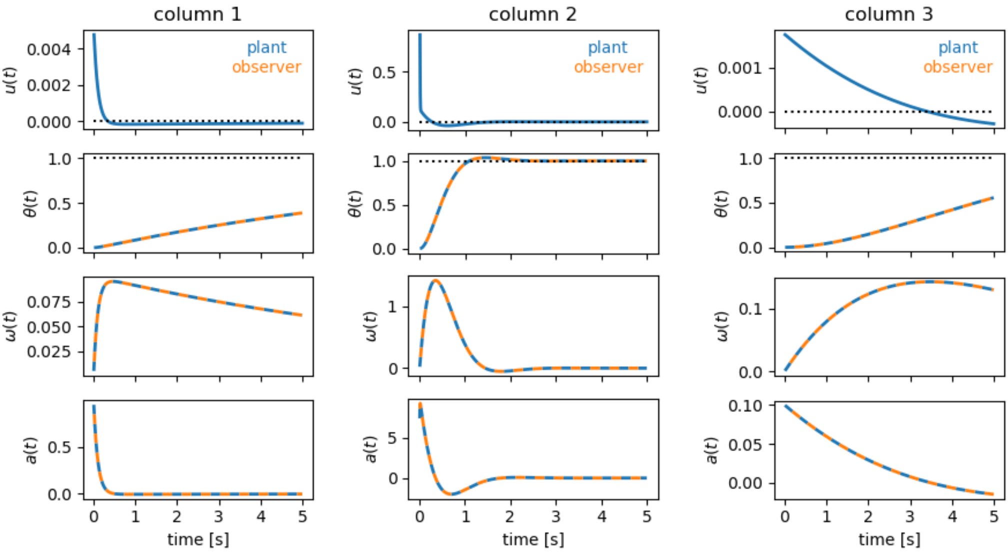


Responses of plant and observer match. Why?

Responses for different Q_k's match. Why?

Choosing Gains

Effect of increasing $Q_k[0]$, $Q_k[1]$, and $Q_k[2]$ – one at a time.

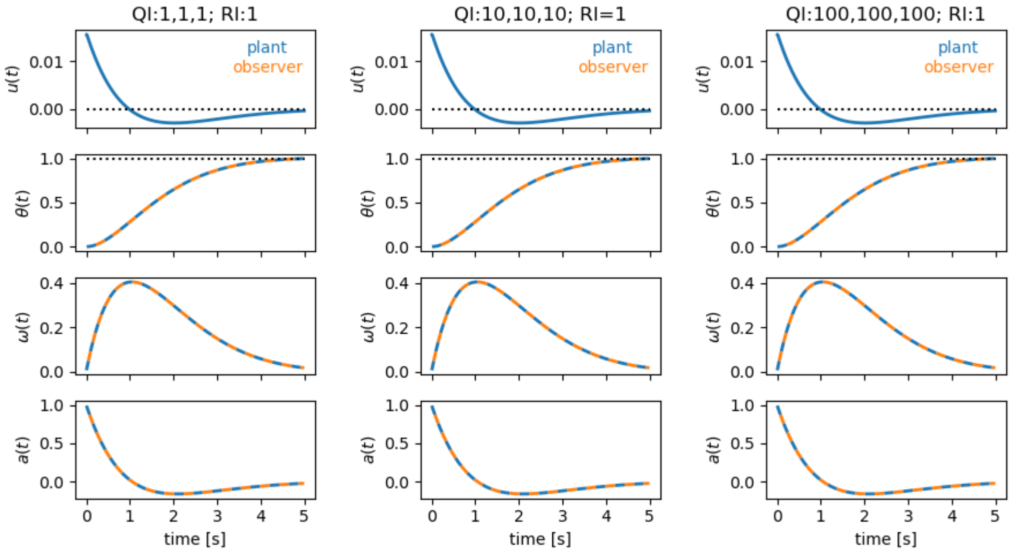


Which column shows

`Qk:100,1,1?` `Qk:1,100,1?` `Qk:1,1,100?`

Choosing Gains

Effect of increasing Q_l's.



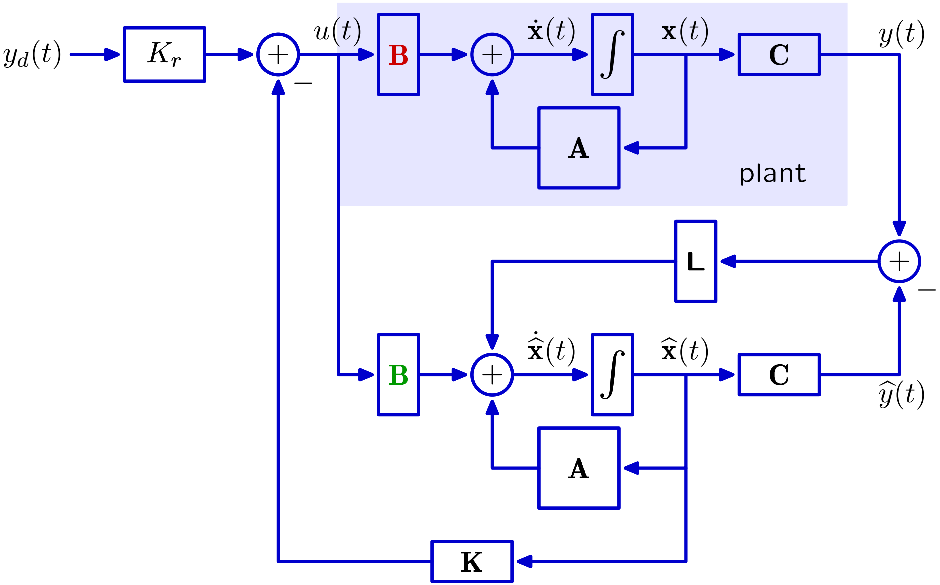
Responses of plant and observer match. Why?

Responses for different Q_l's match. Why?

Observer Mismatch

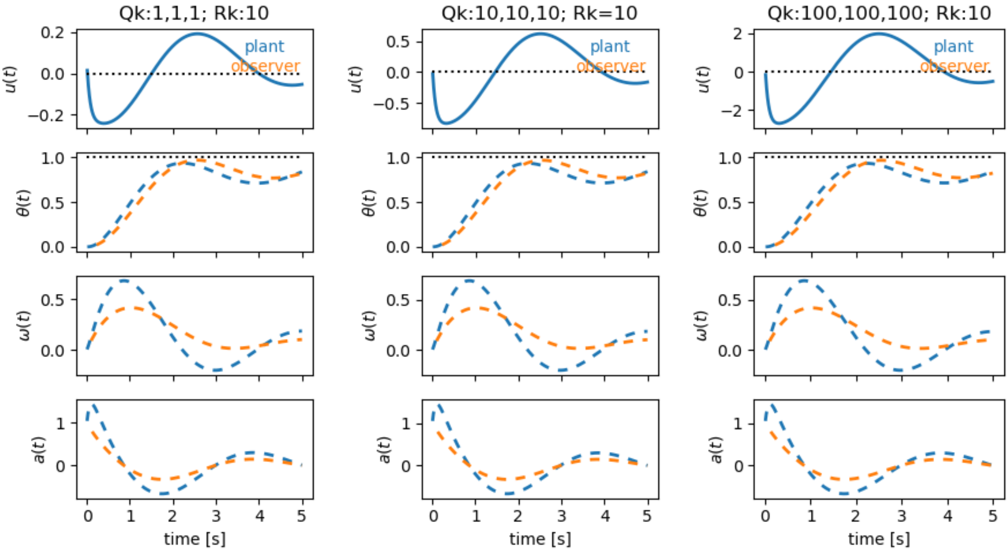
What if the observer does not match the plant?

What if the red $\mathbf{B}$ is different from the green $\mathbf{B}$?



Choosing Gains

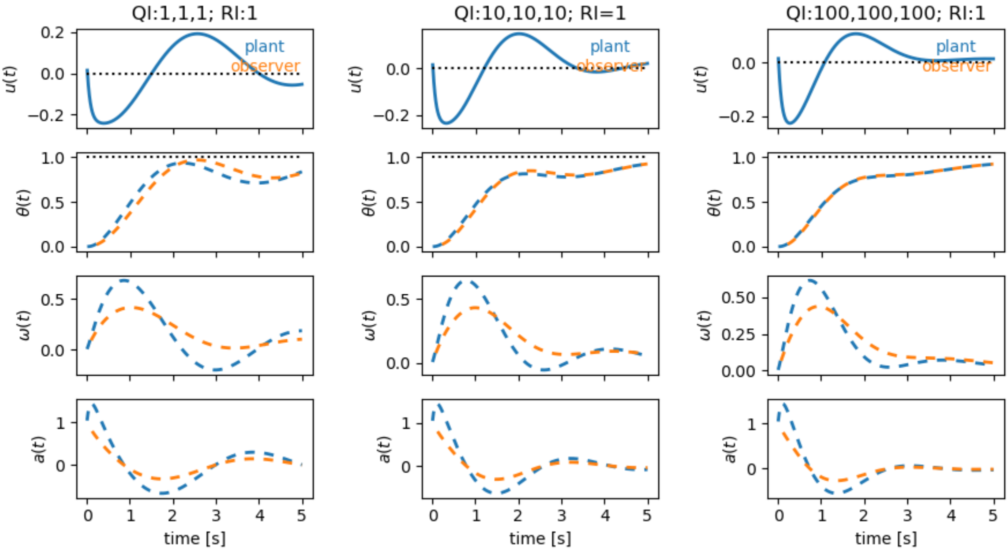
Effect of increasing Q_k's when γ_a for plant is half that for observer.



Increasing $\mathbf{K}$'s by increasing Q_k's has little effect on mismatch. Why?

Choosing Gains

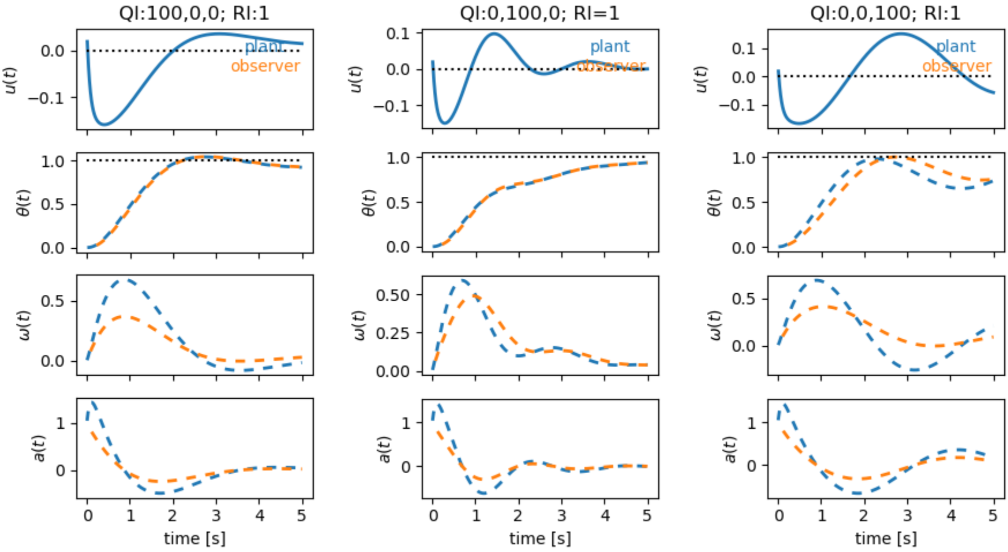
Effect of increasing Q_l's when γ_a for plant is half that for observer.



Increasing $\mathbf{L}$'s by increasing Q_l's improves match (especially for $\theta(t)$).

Choosing Gains

Effect of increasing Q_l's one at a time.



Better to assess $\mathbf{L}$ using differences between plant and observer variables.

Choosing Gains

To choose $\mathbf{K}$, we minimize a **cost function** J that weighs the squares of the state variables $\mathbf{x}(t)$ and input $\mathbf{u}(t)$:

$$J = \int_0^\infty \Big(\mathbf{x}^\mathbf{T}(t)\,\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\mathbf{T}(t)\,\mathbf{R}\mathbf{u}(t)\Big)dt$$

The most effective $\mathbf{K}$ is the one that minimizes the squared values of $\mathbf{x}(t)$ and $\mathbf{u}(t)$.

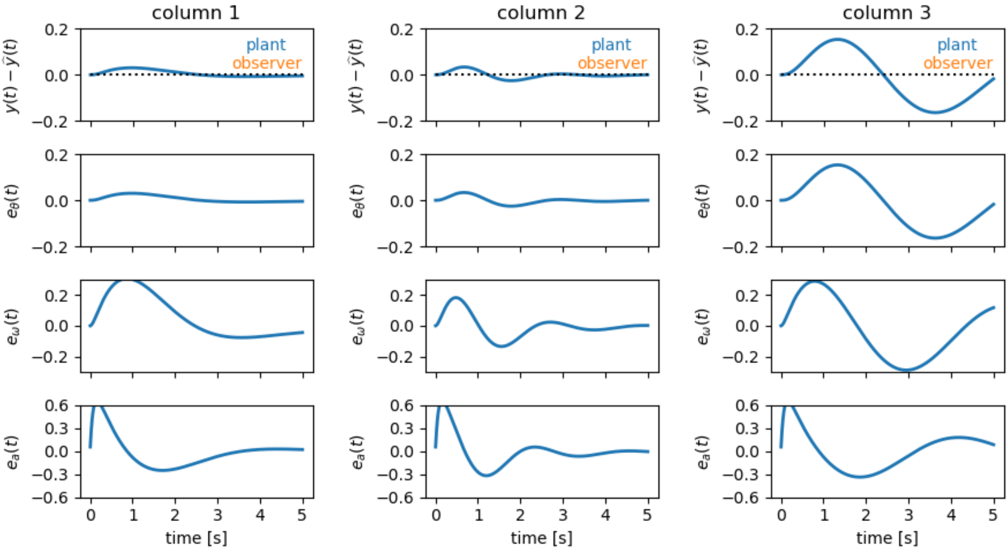
To choose $\mathbf{L}$, we minimize a **cost function** J' that weighs the squares of observer state errors $\mathbf{e}(t) = \mathbf{x}(t) - \widehat{\mathbf{x}}(t)$ and observer output errors $y(t) - \widehat{y}(t)$:

$$J' = \int_0^\infty \Big(\mathbf{e}(t)^T\mathbf{Q}\,\mathbf{e}(t) + (y(t) - \widehat{y}(t))^T\,\mathbf{R}\,(y(t) - \widehat{y}(t))\Big)dt$$

The most effective $\mathbf{L}$ is the one that minimizes the squared values of $\mathbf{e}(t)$ and $y(t) - \widehat{y}(t)$.

Choosing Gains

Effect of increasing Q_l's one at a time.



Which column shows

`Ql:100,0,0?` `Ql:0,100,0?` `Ql:0,0,100?`

Next Time

No lecture on Wednesday. No office hours on Wednesday.

Happy Thanksgiving!