Dynamic System Modeling and Control Design

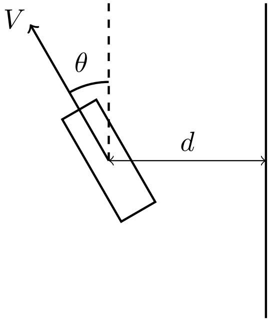
PD and PID Control

September 23, 2024

Outline

1. Recap of Second Order Systems and PD Control
2. Steady State Error with PD Control
3. Introduction to PID Control

Recap: Line Following Example



$$
\begin{aligned}
d_m[n] &= d_m[n-1] + \Delta T V \theta[n-1], \\
\theta[n] &= \theta[n-1] + \Delta T \omega[n-1], \\
\omega[n] &= \gamma u[n].
\end{aligned}
$$

Goal: control the angular velocity of the robot to follow the line.

- Assume we have an optical sensor to measure the distance, d_m.

Line Following System Equation

We arrived at the following system equation for our system:

$$d_m[n] - 2d_m[n-1] + d_m[n-2] = \Delta T^2 V \gamma u[n-2].$$

We analyzed the PD controller for our control system:

$$u[n] = K_p \left(d_d[n] - d_m[n]\right) + K_d \left(\frac{d_d[n] - d_d[n-1]}{\Delta T} - \frac{d_m[n] - d_m[n-1]}{\Delta T} \right).$$

Third Order Characteristic Equation

We obtain the following characteristic equation:

$$\boxed{\begin{aligned} & d_m[n] - 2d_m[n-1] + \left(1 + \Delta T^2 V\gamma K_p + K_d V\gamma\Delta T\right) d_m[n-2] - K_d V\gamma\Delta T d_m[n-3] \\ & \quad = \left(\Delta T^2 \gamma V K_p + K_d \Delta T V\gamma\right) d_d[n-2] - K_d \Delta T V\gamma d_d[n-3] \end{aligned}}$$

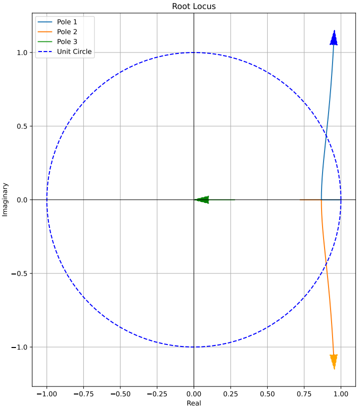
This is a third order difference equation, which has the following solution:

$$d_m[n] = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n.$$

Now there are 3 natural frequencies! Each are a function of K_p and K_d.

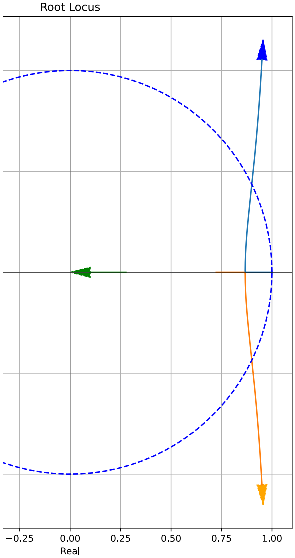
Root Locus Plot

With a non-zero K_d, we can see that now there is an optimal K_p value!




```
Kp_values = np.linspace(0, 10000, 50000)
Kd = 20
V = 1
gamma = 1
Delta_T = 0.01
```

Finding the Optimal Gain K_p with Fixed K_d?

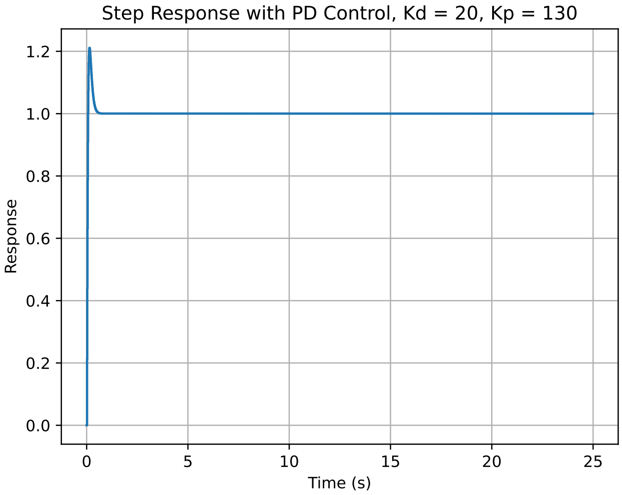


How do we find the optimal value of K_p?

```
# Get magnitude of natural frequencies
mag_poles = np.abs(poles)
# Get maximum natural frequency across all Kp's
row_max = np.max(mag_poles, axis=1)
# Get index of min-max natural frequency
min_max_idx = np.argmin(row_max)
print("Kp = ", Kp_values[min_max_idx])
```

```
Kp = 130.002
```

Fastest PD Step Response Plot



Steady-State Error?

With our system equation with PD control:

$$
\begin{aligned}
d_m[n] - 2d_m[n-1] + \left(1 + \Delta T^2 V\gamma K_p + K_d V\gamma\Delta T\right) d_m[n-2] - K_d V\gamma\Delta T d_m[n-3] \\
= \left(\Delta T^2 \gamma V K_p + K_d \Delta T V\gamma\right) d_d[n-2] - K_d \Delta T V\gamma d_d[n-3],
\end{aligned}
$$

we can analyze the steady-state error $e[n] = d_d[n] - d_m[n]$:

$$
\begin{aligned}
d_m[n] - 2d_m[n-1] + \left(1 + \Delta T^2 V\gamma K_p + K_d V\gamma\Delta T\right) d_m[n-2] - K_d V\gamma\Delta T d_m[n-3] \\
= \left(\Delta T^2 \gamma V K_p + K_d \Delta T V\gamma\right) d_m[n-2] - K_d \Delta T V\gamma d_m[n-3] \\
+ \left(\Delta T^2 \gamma V K_p + K_d \Delta T V\gamma\right) e[n-2] - K_d \Delta T V\gamma e[n-3],
\end{aligned}
$$

Obtaining Steady-State Equation

Cancelling out some terms,

$$
\begin{aligned}
&d_m[n] - 2d_m[n-1] + \left(1 + \cancel{\Delta T^2 V \gamma K_p + K_d V \gamma \Delta T}\right) d_m[n-2] - \cancel{K_d V \gamma \Delta T d_m[n-3]} \\
&\quad = \cancel{\left(\Delta T^2 \gamma V K_p + K_d \Delta T V \gamma\right) d_m[n-2]} - \cancel{K_d \Delta T V \gamma d_m[n-3]} \\
&\quad + \left(\Delta T^2 \gamma V K_p + K_d \Delta T V \gamma\right) e[n-2] - K_d \Delta T V \gamma e[n-3],
\end{aligned}
$$

and simplifying,

$$
\begin{aligned}
&d_m[n] - 2d_m[n-1] + d_m[n-2] \\
&\qquad = \left(\Delta T^2 \gamma V K_p + K_d \Delta T V \gamma\right) e[n-2] - K_d \Delta T V \gamma e[n-3].
\end{aligned}
$$

Steady-State Error

Now, we want to analyze the steady-state error, i.e., $n \to \infty$:

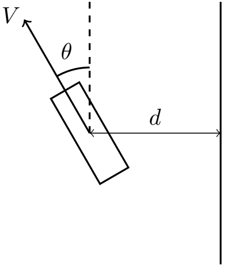
$$\begin{aligned} &d_m[\infty] - 2d_m[\infty] + d_m[\infty] \\ &\qquad = \left(\Delta T^2 \gamma V K_p + K_d \Delta T V \gamma\right) e[\infty] - K_d \Delta T V \gamma e[\infty]. \end{aligned}$$

We find that, as long as $K_p \neq 0$, the steady-state error,

$$\Delta T^2 \gamma V K_p e[\infty] = 0 \Rightarrow e[\infty] = 0,$$

is zero!

Line Following with Loss



$$
\begin{aligned}
d_m[n] &= d_m[n-1] - \beta \Delta T d_m[n-1] + \Delta T V \theta[n-1], \\
\theta[n] &= \theta[n-1] + \Delta T \omega[n-1], \\
\omega[n] &= \gamma u[n].
\end{aligned}
$$

What's the story when we include loss with $\beta \neq 0$?

$$
\begin{aligned}
& d_m[n] - (2+\beta\Delta T) d_m[n-1] + \left(1 + \Delta T^2 V \gamma K_p + K_d V \gamma \Delta T\right) d_m[n-2] - K_d V \gamma \Delta T d_m[n-3] \\
&\quad = \left(\Delta T^2 \gamma V K_p + K_d \Delta T V \gamma\right) d_d[n-2] - K_d \Delta T V \gamma d_d[n-3],
\end{aligned}
$$

Steady-State for Line Following with Loss

We can go through the same exercise and obtain,

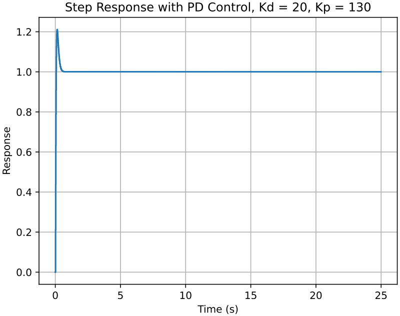
$$
\begin{aligned}
&d_m[n] - (2 + \beta \Delta T) d_m[n-1] + d_m[n-2] \\
&\qquad = \left(\Delta T^2 \gamma V K_p + K_d \Delta T V \gamma\right) e[n-2] - K_d \Delta T V \gamma e[n-3].
\end{aligned}
$$

This time, we obtain steady-state error of:

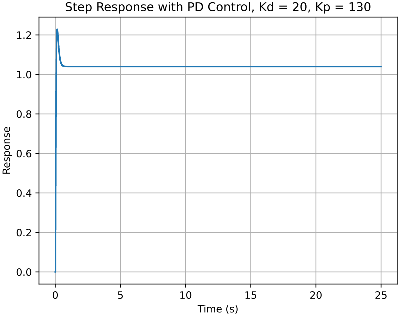
$$
e[\infty] = \frac{-\beta d_m[\infty]}{\Delta T \gamma V K_p} \neq 0.
$$

Visualizing PD Control Step Response with Error

$\beta = 0$ $\quad\quad$ $\beta = 0.05$



$\Rightarrow$



We never reach zero error; we'll need a new control signal.

PD Control Evoles into PID Control

With the error term defined as,

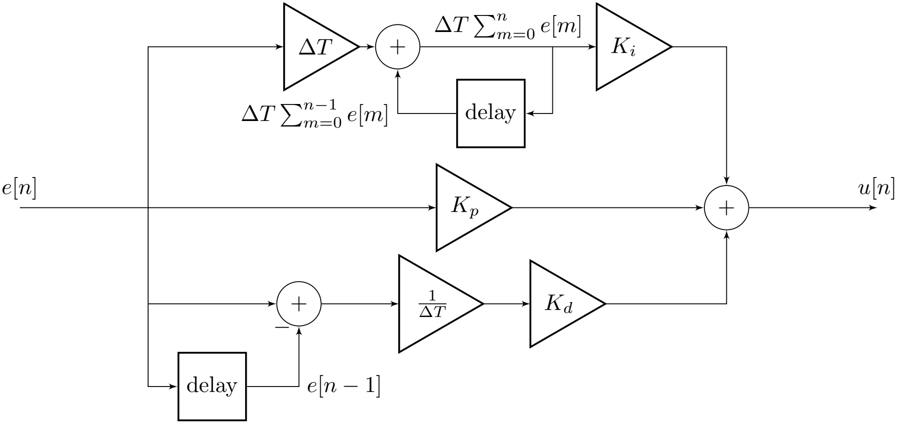
$$e[n] = d_d[n] - d_m[n],$$

we can define the Proportional-Integral-Derivative (PID) controller,

$$u[n] = K_p e[n] + K_i \Delta T \sum_{m=0}^{m=n} e[m] + K_d \frac{e[n] - e[n-1]}{\Delta T}.$$

The "integral" component accumulates all of the past errors.

PID Controller Block Diagram



Towards PID Analysis

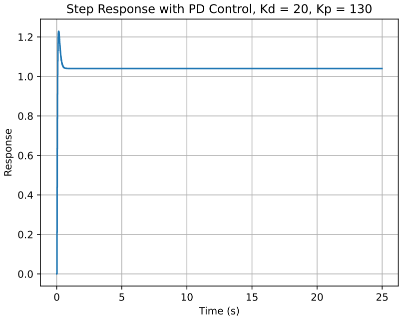
Our system equation with the "integral" component becomes,

$$
\begin{aligned}
&K_i \Delta T^3 V \gamma \sum_{m=0}^{n-2} d_m[m] + d_m[n] - (2 + \beta \Delta T) d_m[n-1] \\
&+ (1 + K_p \Delta T^2 \gamma V + K_d \Delta T V \gamma) d_m[n-2] - K_d V \gamma \Delta T d_m[n-3] \\
&= K_i \Delta T^3 V \gamma \sum_{m=0}^{n-2} d_d[m] + (K_p \Delta T^2 V \gamma + K_d \Delta T V \gamma) d_d[n-2] \\
&- K_d \Delta T V \gamma d_d[n-3].
\end{aligned}
$$

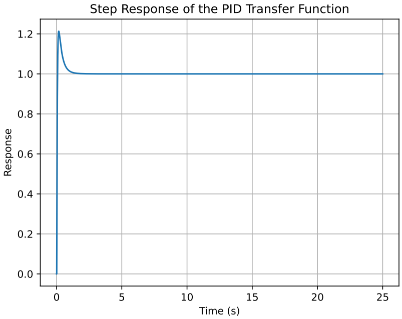
How do we handle these sums? ...It's tricky! We will see on Wednesday.

Visualizing Step Response with Error, $\beta = 0.05$

$K_i = 0$ $K_i = 116.8$



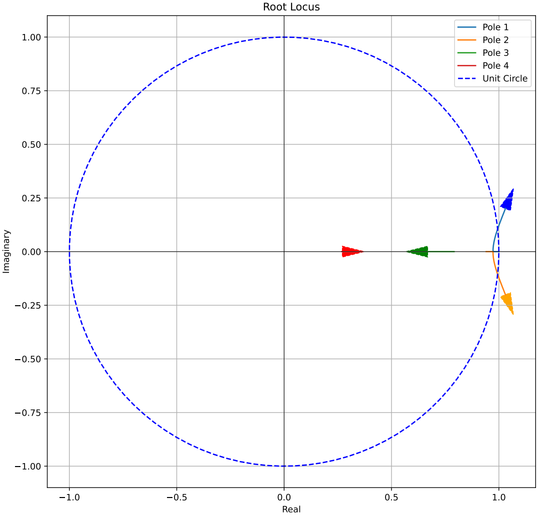
$\Rightarrow$



Colab for PID numerical simulation: (here).

PID Root Locus with $K_p = 100, K_d = 20$

$K_i = 116.8$ at optimal point, for these values of K_p, K_d.

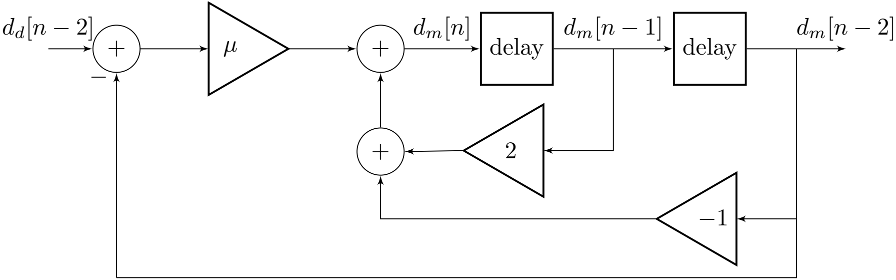


Insights from Difference Equations

Let's revisit our system equation with $u[n] = K_p(d_d[n] - d_m[n])$:

$$d_m[n] = 2d_m[n-1] - d_m[n-2] + \Delta T^2 V\gamma K_p(d_d[n-2] - d_m[n-2]).$$

We can visualize this equation with a block diagram:

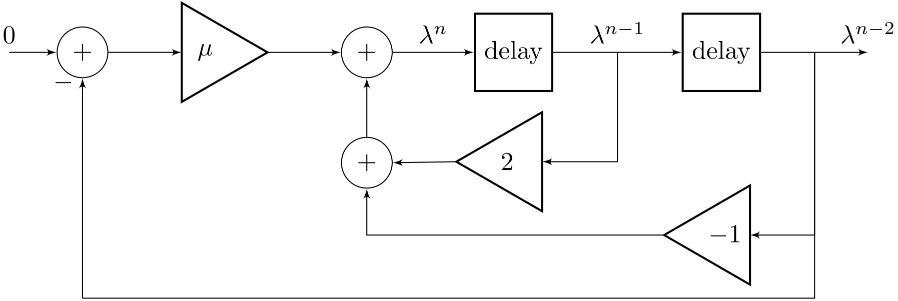


Insights from Difference Equations

Additionally, we found the homogeneous solution ($d_d[n] = 0$) by solving:

$$\lambda^n = 2\lambda^{n-1} - \lambda^{n-2} - \Delta T^2 V K_p \gamma \lambda^{n-2}$$

We can also visualize this equation with a block diagram:



Time and Frequency Domain

We can exploit relations between time and frequency domain formulations to simplify our work and deepen our understanding of control systems.

On Wednesday, we will begin by casting the two formulations into a common framework.