6.3100: Dynamic System Modeling and Control Design

State-Space Approach

- State-Space Formulation
- Characteristic Equation and Natural Frequencies
- Relating State-Space and Transfer Function Representations

October 28, 2024

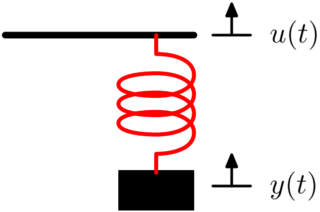
State-Space Approach

In the first half of this subject, we focused on **classical control**. Next, we introduce **state-space control** which is the basis of **modern control**.

We will use the state-space approach in Labs 5 and 6.

Today: Introduce state-space control by building on our classical model of a simple system containing a mass, spring, and dashpot.

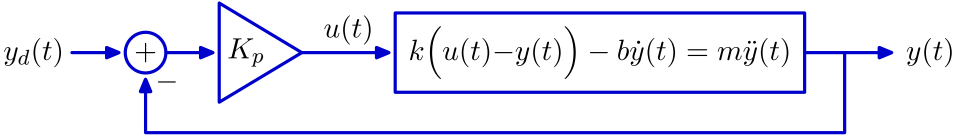
Classical Analysis



Start by describing the **plant** mathematically (Newton's law):

$$\underbrace{k\Big(u(t)-y(t)\Big) - b\dot{y}(t)}_{F} = \underbrace{m\ddot{y}(t)}_{ma}$$

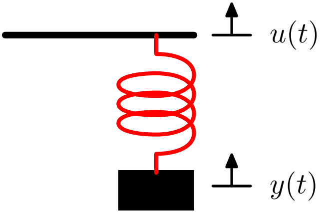
Use proportional control: $u(t) = K_p\Big(y_d(t)-y(t)\Big)$



Combine these equations into a single second-order system equation:

$$m\ddot{y}(t) + b\dot{y}(t) + k(1+K_p)y(t) = kK_p y_d(t)$$

State-Space Analysis



Start with the same dynamical description of the plant:

$$\underbrace{k\Big(u(t)-y(t)\Big) - b\dot{y}(t)}_{F} = \underbrace{m\ddot{y}(t)}_{ma}$$

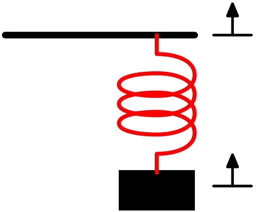
Identify the **state variables**:

→ The future of a system is fully described by the values of its state variables at time t_0 and the input to that system for $t \geq t_0$.

→ **No information about the system for $t < t_0$ is needed!**

The state variables for the mass-spring-dashpot system are $y(t)$ and $\dot{y}(t)$.

State-Space Analysis



Start with the same dynamical description of the plant:

$$\underbrace{k\Big(u(t) - y(t)\Big) - b\dot{y}(t)}_{F} = \underbrace{m\ddot{y}(t)}_{ma}$$

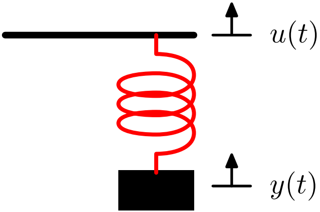
Rewrite dynamics as two **first-order** equations using state variables:

$$\underbrace{\frac{d}{dt}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}}_{\dot{\mathbf{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ k/m \end{bmatrix}}_{\mathbf{B}} u(t)$$

which can be expressed as a single, first-order **matrix** equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

State-Space Analysis



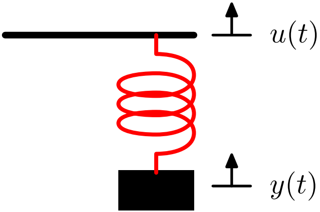
The output $y(t)$ is a weighted sum of states $\mathbf{x}(t)$ and input $u(t)$:

$$y(t) = \underbrace{[1 \quad 0]}_{\mathbf{C}} \underbrace{\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}}_{\mathbf{x}(t)} + Du(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + Du(t)$$

where D is often zero (as it is for the mass-spring-dashpot system).

State-Space Analysis



State-Space Model of Plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

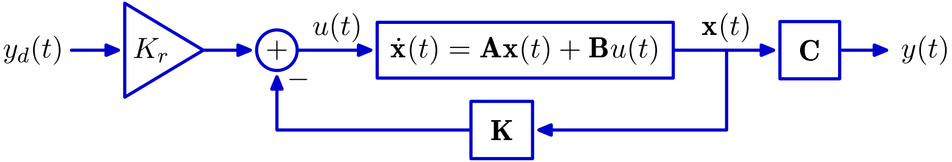
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

$\mathbf{A}$: system (or state transition) matrix
$\mathbf{B}$: input matrix
$\mathbf{C}$: output matrix
$\mathbf{D}$: feed-through (or feed-forward) matrix

We will focus on **single input / single output** (SISO) systems, but this structure readily generalizes to multiple input / multiple output systems.

Full-State Feedback

A useful feature of the state-space formulation is that we can easily incorporate feedback from the entire state, not just from the output.



The scalar input to the plant ($u(t)$) is the difference between a scaled version of the desired output $y_d(t)$ and a weighted sum of the states:

$$u(t) = K_r y_d(t) - \mathbf{K}\mathbf{x}(t)$$

Combine with the system equation for the plant:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\Big(K_r y_d(t) - \mathbf{K}\mathbf{x}(t)\Big) \\ &= \Big(\mathbf{A} - \mathbf{B}\mathbf{K}\Big)\mathbf{x}(t) + \mathbf{B}K_r y_d(t) \equiv \mathbf{A_c}\mathbf{x}(t) + \mathbf{B_c} y_d(t)\end{aligned}$$

$\mathbf{A_c} = (\mathbf{A} - \mathbf{B}\mathbf{K})$ is the **closed-loop system matrix.**

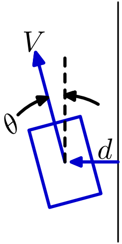
$\mathbf{B_c} = \mathbf{B}K_r$ is the **closed-loop input matrix.**

Check Yourself

Find a **state-space description**

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t)$$

of the robotic steering plant:



$$\dot{d}(t) = V\sin(\theta(t)) \approx V\theta(t)$$

$$\dot{\theta}(t) = \omega(t)$$

$$\omega(t) = \gamma u(t)$$

where the input is $u(t)$ and the output is the distance $d(t)$.

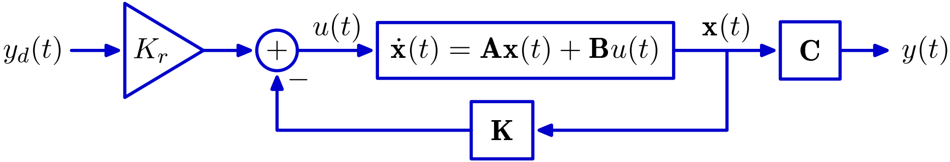
Which of the following matrices could be $\mathbf{A}$?

1. $\begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}$ 2. $\begin{bmatrix} 0 & -\gamma K_p \\ V & 0 \end{bmatrix}$ 3. $\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$ 4. $\begin{bmatrix} 0 & 0 \\ -\gamma K_p & 0 \end{bmatrix}$

5. none of the above

Add Proportional Feedback Control to Robotic Steering

State-space formulation of proportional feedback.



The scalar input to the plant ($u(t)$) is the difference between a scaled version of the desired output $y_d(t)$ and a weighted sum of the states:

$$u(t) = K_r y_d(t) - \mathbf{K}\mathbf{x}(t)$$

Combine with the system equation for the plant:

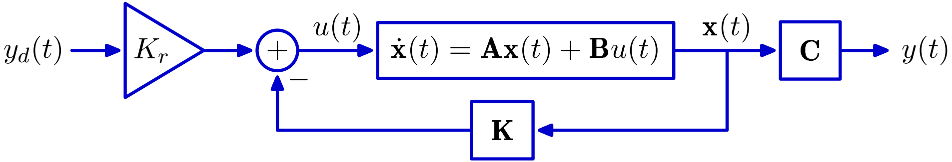
$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\Big(K_r y_d(t) - \mathbf{K}\mathbf{x}(t)\Big) \\ &= \Big(\mathbf{A} - \mathbf{B}\mathbf{K}\Big)\mathbf{x}(t) + \mathbf{B}K_r y_d(t) \equiv \mathbf{A_c}\mathbf{x}(t) + \mathbf{B_c} y_d(t)\end{aligned}$$

$\mathbf{A_c} = (\mathbf{A} - \mathbf{B}\mathbf{K})$ is the **closed-loop system matrix.**

$\mathbf{B_c} = \mathbf{B}K_r$ is the **closed-loop input matrix.**

Check Yourself

Proportional control of state-space model of robotic steering:



The closed-loop system can be represented as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A_c}\mathbf{x}(t) + \mathbf{B_c}y_d(t)$$

Which of the following matrices could be $\mathbf{A_c}$?

1. $\begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}$
2. $\begin{bmatrix} 0 & -\gamma K_p \\ V & 0 \end{bmatrix}$
3. $\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$
4. $\begin{bmatrix} 0 & 0 \\ -\gamma K_p & 0 \end{bmatrix}$
5. none of the above

Summary (so far)

Classical Approach

- Describe a system by an **ad hoc collection** of scalar variables.
- Describe dynamics of a system by 1+ possibly **high-order** diff. eqn's.

ad hoc collection of scalar variables $\rightarrow$ system of differential equations

State-Space Approach

- Describe a system by its **states**.
- Describe dynamics of a system by **first-order** relations among states.
- Collect the states and relations in a single first-order **matrix** equation.

first-order relations of state variables $\rightarrow$ first-order matrix equation

Next: Analyzing natural frequencies

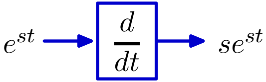
Classical Analysis of Natural Frequencies

Find the natural frequencies of the closed-loop spring system.

Start with the homogeneous equation:

$$m\ddot{y}(t) + b\dot{y}(t) + k(1{+}K_p)y(t) = 0$$

Use the eigenfunction property



to convert the differential equation to a difference equation as follows.

Let $y(t) = e^{st}$ then

$$\left(ms^2 + bs + k(1{+}K_p)\right)e^{st} = 0$$

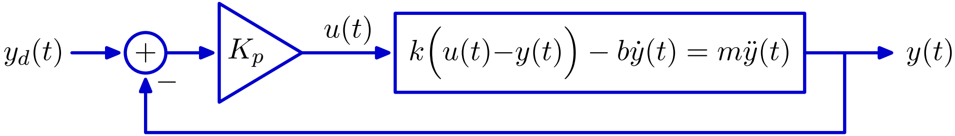
Since $e^{st}{\neq}0$, the parenthesized part must be zero:

$$ms^2 + bs + k(1{+}K_p) = 0$$

The roots of this **characteristic equation** are the natural frequencies of the closed-loop system.

Check Yourself

Consider proportional control of a mass-spring-dashpot system:



Characteristic equation:

$$ms^2 + bs + k(1+K_p) = 0$$

Which (if any) of the following sets of parameters gives rise to an **oscillatory** step response when $K_p = 1$?

	m	b	k	K_p
A:	1	3	1	1
B:	2	0	1	1
C:	1	2	1/2	1
D:	2	1	1	1

1. A 2. B&C 3. B&D 4. D 5. none of the above

State-Space Analysis of Natural Frequencies

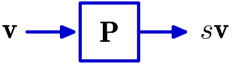
Find the natural frequencies of the **closed-loop** system.

Start with the homogeneous equation:

$$\dot{\mathbf{x}}(t) = \Big(\mathbf{A} - \mathbf{B}\mathbf{K}\Big)\mathbf{x}(t)$$

Use the eigenvector/eigenvalue property:

$$\mathbf{P}\mathbf{v} = s\mathbf{v} = s\mathbf{I}\mathbf{v}$$



$$(s\,\mathbf{I} - \mathbf{P})\mathbf{v} = 0$$

Either $\mathbf{v} = \mathbf{0}$ (trivial solution) or $(s\,\mathbf{I} - \mathbf{P})$ is singular (determinant is zero):

$$\Big|s\,\mathbf{I} - \mathbf{P}\Big| = 0$$

For the mass-spring-dashpot system,

$$\mathbf{P} = \mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} - \begin{bmatrix} 0 \\ k/m \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$|s\,\mathbf{I} - \mathbf{P}| = \left| \begin{bmatrix} s & -1 \\ k(1+K_1)/m & s+(b+kK_2)/m \end{bmatrix} \right| = 0$$

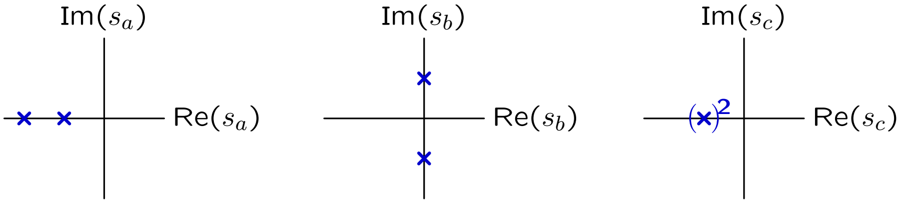
Characteristic equation: $s^2 + (b+kK_2)s/m + k(1+K_1)/m = 0 \quad \surd$

Check Yourself

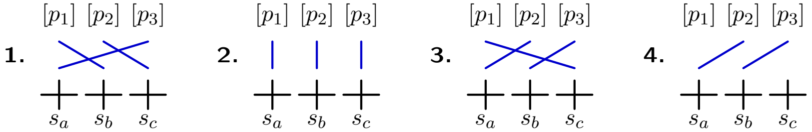
Match the following system matrices:

$$\mathbf{P_1} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \mathbf{P_2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{P_3} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

to their eigenvalues:



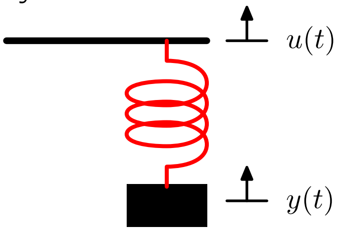
Which (if any) of the following maps is correct?



5. none

From State-Space to Transfer Function

The mass-spring-dashpot system



can be represented by the following state-space description:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ k/m \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Derive the transfer function representation:

$$H(s) = \frac{Y(s)}{Y_d(s)}$$

for this system when operated with proportional control:

$$u(t) = K_r y_d(t) - \mathbf{K}\mathbf{x}(t) = K_p y_d(t) - \begin{bmatrix} K_p & 0 \end{bmatrix} \mathbf{x}(t)$$

From State-Space to Transfer Function

Derive the transfer function for the following state-space system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

when operated with proportional control:

$$u(t) = K_r y_d(t) - \mathbf{K}\mathbf{x}(t) = K_p y_d(t) - [\, K_p \quad 0 \,]\, \mathbf{x}(t)$$

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\
&= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\Big(K_r y_d(t) - \mathbf{K}\mathbf{x}(t)\Big) \\
&= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}K_r y_d(t) \\
&= \mathbf{A_c}\mathbf{x}(t) + \mathbf{B_c} y_d(t)
\end{aligned}$$

$$(s\mathbf{I} - \mathbf{A_c})\mathbf{X}(s) = \mathbf{B_c} Y_d(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A_c})^{-1}\mathbf{B_c} Y_d(s)$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A_c})^{-1}\mathbf{B_c} Y_d(s)$$

$$H(s) = \frac{Y(s)}{Y_d(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A_c})^{-1}\mathbf{B_c}$$

From State-Space to Transfer Function

$$H(s) = \frac{Y(s)}{Y_d(s)} = \mathbf{C}(s\mathbf{I}-\mathbf{A_c})^{-1}\mathbf{B_c}$$

For the mass, spring, dashpot system:

$$H(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ (1+K_p)k/m & s+b/m \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ K_pk/m \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + sb/m + (1+K_p)k/m} \begin{bmatrix} s+b/m & 1 \\ -(1+K_p)k/m & s \end{bmatrix} \begin{bmatrix} 0 \\ K_pk/m \end{bmatrix}$$

$$= \frac{1}{s^2 + sb/m + (1+K_p)k/m} \begin{bmatrix} s+b/m & 1 \end{bmatrix} \begin{bmatrix} 0 \\ K_pk/m \end{bmatrix}$$

$$= \frac{K_pk/m}{s^2 + sb/m + (1+K_p)k/m} \qquad \surd$$

Check Yourself

Consider a plant described by the following differential equation:

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t)$$

Which of the following show $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ matrices for this plant.

1. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$
2. $\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}$
3. $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{C} = \begin{bmatrix} -1 & 1 \end{bmatrix}$
4. all of the above
5. none of the above

From Classical to Modern Control

New approach:

- replace the high-order differential equation in classical control with a set of first-order differential equations, each characterizing a single **state**.
- combine individual first-order states into a composite **state vector**.
- describe how states interact with each other with a **system matrix**.
- describe how the input(s) affect each state with an **input vector**.
- describe the output(s) as a weighted sum of states (and inputs).

Advantages:

- more powerful **full-state** feedback
- solutions in terms of standardized methods based on **linear algebra** instead of problem-specific differential equations.

Applications:

- finding characteristic equation and natural frequencies
- relating state-space and transfer function representations

Next time: step response and matrix exponentials