6.3100: Dynamic System Modeling and Control Design

State-Space Characterization

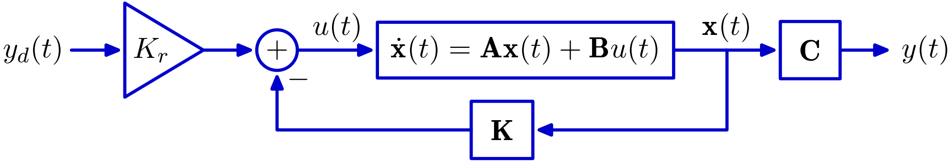
- Transfer Functions
- Step Responses
- Matrix Exponentials

October 30, 2024

State-Space Approach

Last time, we introduced the **State-Space** approach to control:

- Describe a system by its **states**.
- Describe dynamics of a system by **first-order** relations among states.
- Collect the states and relations in a single first-order **matrix** equation.



Plant: state matrix $\mathbf{A}$, input vector $\mathbf{B}$, and output vector $\mathbf{C}$:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t)$$

Feedback is characterized by a feedback vector $\mathbf{K}$ and input scaler K_r:

$$u(t) = K_r y_d(t) - \mathbf{K}\mathbf{x}(t)$$

Combine to obtain **closed-loop** characterization:

$$\dot{\mathbf{x}}(t) = \Big(\mathbf{A} - \mathbf{B}\mathbf{K}\Big)\mathbf{x}(t) + \mathbf{B}\mathbf{K_r} y_d(t) \equiv \mathbf{A_c}\mathbf{x}(t) + \mathbf{B_c} y_d(t)$$

State-Space Analysis of Natural Frequencies

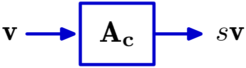
Find the natural frequencies of the **closed-loop** system.

Start with the homogeneous equation:

$$\dot{\mathbf{x}}(t) = \Big(\mathbf{A} - \mathbf{B}\mathbf{K}\Big)\mathbf{x}(t) = \mathbf{A_c}\mathbf{x}(t)$$

Use the eigenvector/eigenvalue property:

$$\mathbf{A_c}\mathbf{v} = s\mathbf{v} = s\mathbf{I}\mathbf{v}$$



$$(s\,\mathbf{I} - \mathbf{A_c})\mathbf{v} = 0$$

Either $\mathbf{v} = \mathbf{0}$ (trivial solution) or $(s\,\mathbf{I} - \mathbf{A_c})$ is singular (determinant is zero):

$$\Big|s\,\mathbf{I} - \mathbf{A_c}\Big| = 0$$

Example: mass-spring-dashpot system:

$$\mathbf{A_c} = \mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} - \begin{bmatrix} 0 \\ k/m \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

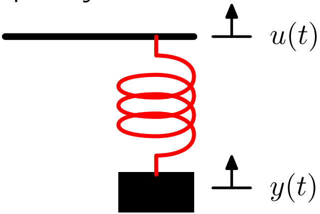
$$|s\,\mathbf{I} - \mathbf{A_c}| = \left| \begin{bmatrix} s & -1 \\ k(1+K_1)/m & s+(b+kK_2)/m \end{bmatrix} \right| = 0$$

Characteristic equation: $s^2 + (b+kK_2)s/m + k(1+K_1)/m = 0$ $\quad\checkmark$

From State-Space to Transfer Function

Find the transfer function representation from the state-space description.

Example: mass-spring-dashpot system:



Given the state-space representation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ k/m \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

find transfer function:

$$H(s) = \frac{Y(s)}{U(s)}$$

From State-Space to Transfer Function

Find the transfer function representation from the state-space description.

Start with the state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

Convert to frequency domain:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

From State-Space to Transfer Function

Find the transfer function representation from the state-space description.

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

For the mass, spring, dashpot system:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ k/m \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$H(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ k/m & s+b/m \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ k/m \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + sb/m + k/m} \begin{bmatrix} s+b/m & 1 \\ -k/m & s \end{bmatrix} \begin{bmatrix} 0 \\ k/m \end{bmatrix}$$

$$= \frac{1}{s^2 + sb/m + k/m} \begin{bmatrix} s+b/m & 1 \end{bmatrix} \begin{bmatrix} 0 \\ k/m \end{bmatrix}$$

$$= \frac{k/m}{s^2 + sb/m + k/m} \qquad \surd$$

Check Yourself

Consider a plant described by the following differential equation:

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t)$$

Which of the following show $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ matrices for this plant.

1. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$
2. $\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}$
3. $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{C} = \begin{bmatrix} -1 & 1 \end{bmatrix}$
4. all of the above
5. none of the above

Step Response

Find the step response $\mathbf{x_s}(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

where $\mathbf{P}$ and $\mathbf{Q}$ can represent open-loop matrices $\mathbf{A}$ and $\mathbf{B}$ or closed-loop matrices $\mathbf{A_c}$ and $\mathbf{B_c}$.

Start by finding the step response of the scalar version of this system:

$$\dot{x}(t) = px(t) + qu(t)$$

Step Response

Find the step response $x_s(t)$ of the following scalar system equation:

$$\dot{x}(t) = px(t) + qu(t)$$

Assume the initial value of the step response $x_s(0) = 0$, and $u(t)$=1 for t>0.

Homogeneous equation: $\dot{x}_h(t) = px_h(t)$

$$x_h(t) = \alpha e^{\beta t}$$

$$\dot{x}_h(t) = \beta\alpha e^{\beta t} = px_h(t) = p\alpha e^{\beta t} \quad \rightarrow \beta = p$$

$$x_h(t) = \alpha e^{pt}$$

Particular solution: $x_p(t) = \gamma$

$$\dot{x}_p(t) = 0 = p\gamma + q$$

$$\gamma = -q/p \quad \text{(provided that } p \neq 0\text{)}$$

Initial condition: $x_s(0) = \alpha - q/p = 0$

$$\alpha = q/p$$

Final solution:

$$x_s(t) = q(e^{pt}-1)/p$$

Step Response

Find the step response $\mathbf{x_s}(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

The step response $x_s(t)$ of the scalar version of this system:

$$\dot{x}(t) = px(t) + qu(t)$$

is

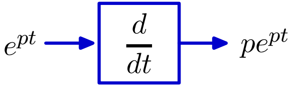
$$x_s(t) = q(e^{pt} - 1)/p$$

provided $p \neq 0$.

What's the matrix equivalent of the exponential function e^{pt}?

Scalar Exponential Function

Exponential functions are eigenfunctions of the derivative operator:



Express the exponential function as a power series:

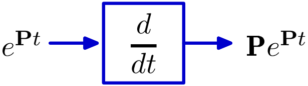
$$e^{pt} = 1 + \frac{pt}{1!} + \frac{p^2t^2}{2!} + \frac{p^3t^3}{3!} + \cdots$$

Differentiate term-by-term:

$$\begin{aligned}
\frac{d}{dt}e^{pt} &= 0 + \frac{p}{1!} + \frac{2p^2t}{2!} + \frac{3p^3t^2}{3!} + \cdots \\
&= p + p\frac{pt}{1!} + p\frac{p^2t^2}{2!} + p\frac{p^3t^3}{3!} + \cdots \\
&= p\left(1 + \frac{pt}{1!} + \frac{p^2t^2}{2!} + \frac{p^3t^3}{3!} + \cdots\right) \\
&= pe^{pt} \quad \checkmark
\end{aligned}$$

Matrix Exponential Function

Matrix exponentials are eigenfunctions of the matrix derivative operator:



The matrix exponential function can also be expanded as a power series:

$$e^{\mathbf{P}t} = \mathbf{I} + \frac{\mathbf{P}t}{1!} + \frac{\mathbf{P}^2t^2}{2!} + \frac{\mathbf{P}^3t^3}{3!} + \cdots$$

Differentiate term-by-term:

$$\begin{aligned}
\frac{d}{dt}e^{\mathbf{P}t} &= 0 + \frac{\mathbf{P}}{1!} + \frac{2\mathbf{P}^2t}{2!} + \frac{3\mathbf{P}^3t^2}{3!} + \cdots \\
&= \mathbf{P} + \mathbf{P}\frac{\mathbf{P}t}{1!} + \mathbf{P}\frac{\mathbf{P}^2t^2}{2!} + \mathbf{P}\frac{\mathbf{P}^3t^3}{3!} + \cdots \\
&= \mathbf{P}\left(\mathbf{I} + \frac{\mathbf{P}t}{1!} + \frac{\mathbf{P}^2t^2}{2!} + \frac{\mathbf{P}^3t^3}{3!} + \cdots\right) \\
&= \mathbf{P}e^{\mathbf{P}t} = e^{\mathbf{P}t}\mathbf{P} \quad \checkmark
\end{aligned}$$

Step Response

Find the step response $\mathbf{x_s}(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

Assume the initial value of the step response $\mathbf{x_s}(0) = \mathbf{0}$, and $u(t){=}1$ for $t{>}0$.

Homogeneous equation: $\dot{\mathbf{x}}_\mathbf{h}(t) = \mathbf{P}\mathbf{x}_\mathbf{h}(t)$

$$\mathbf{x}_\mathbf{h}(t) = e^{\mathbf{P}t}\boldsymbol{\Psi}$$

Particular solution: $\mathbf{x}_\mathbf{p}(t) = \boldsymbol{\Phi}$

$$\dot{\mathbf{x}}_\mathbf{p}(t) = \mathbf{0} = \mathbf{P}\boldsymbol{\Phi} + \mathbf{Q}$$

$$\boldsymbol{\Phi} = -\mathbf{P}^{-1}\mathbf{Q} \quad \text{(provided that } \mathbf{P} \text{ is not singular)}$$

Initial condition: $\mathbf{x}(0) = \boldsymbol{\Psi} - \mathbf{P}^{-1}\mathbf{Q} = \mathbf{0}$

$$\boldsymbol{\Psi} = \mathbf{P}^{-1}\mathbf{Q}$$

Step response:

$$\begin{aligned}\mathbf{x_s}(t) &= (e^{\mathbf{P}t} - \mathbf{I})\mathbf{P}^{-1}\mathbf{Q} \\ &= \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}\end{aligned}$$

Step Response

Find the step response $\mathbf{x_s}(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

Assume the initial value of the step response $\mathbf{x_s}(0) = \mathbf{0}$, and $u(t)=1$ for $t>0$.

Then the solution is

$$\mathbf{x_s}(t) = \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

provided $\mathbf{P}$ is not singular.

Notice that this solution to the matrix problem matches the scalar solution if the matrix problem is first order.

$$x_s(t) = q(e^{pt}-1)/p$$

But unlike the scalar approach (homogeneous and particular solutions, initial conditions, etc.), **the matrix solution works for any order**.

Step Response: Second-Order Example

Find the step response of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x_s}(t) = \mathbf{P^{-1}}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

Find $\mathbf{P^{-1}}$: (easy because $\mathbf{P}$ is diagonal)

$$\mathbf{P^{-1}} = \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

Step Response: Second-Order Example

Find the step response of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x_s}(t) = \mathbf{P^{-1}}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

Find the matrix exponential:

$$\begin{aligned}
e^{\mathbf{P}t} &= \mathbf{I} + \mathbf{P}t + \mathbf{P^2}t^2/2! + \mathbf{P^3}t^3/3! + \mathbf{P^4}t^4/4! + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} t + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^2 \frac{t^2}{2!} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^3 \frac{t^3}{3!} + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} t + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} \frac{t^3}{3!} + \cdots \\
&= \begin{bmatrix} 1 - t + t^2/2! - t^3/3! + \cdots & 0 \\ 0 & 1 - 2t + (2t)^2/2! - (2t)^3/3! + \cdots \end{bmatrix} \\
&= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}
\end{aligned}$$

Step Response: Second-Order Example

Find the step response of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x_s}(t) = \mathbf{P^{-1}}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

Substitute into the general expression for the step response of the state:

$$\begin{aligned}
\mathbf{x_s}(t) &= \mathbf{P^{-1}}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q} \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix} \left(\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} e^{-t} - 1 & 0 \\ 0 & e^{-2t} - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 - e^{-t} & 0 \\ 0 & (1 - e^{-2t})/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 - e^{-t} \\ (1 - e^{-2t})/2 \end{bmatrix}
\end{aligned}$$

Step Response: Second-Order Example

Find the step response of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x_s}(t) = \mathbf{P^{-1}}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

Much of the ad hoc algebra that we used to solve higher-order differential equations is replaced by the rules of linear algebra.

Check Yourself

Let $\mathbf{P}$ represent the following matrix:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

Which (if any) of the following matrices is equal to $e^{\mathbf{P}}$?

1. $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ 2. $\begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}$ 3. $\begin{bmatrix} 1/e & 0 \\ 0 & 2e^2 \end{bmatrix}$

4. $e^{\mathbf{P}}$ does not exist because the system is unstable

5. none of the above

Computing Matrix Exponentials

Finding the series expansion of a matrix exponential

$$e^{\mathbf{P}} = \mathbf{I} + \mathbf{P} + \mathbf{P^2}/2! + \mathbf{P^3}/3! + \mathbf{P^4}/4! + \cdots$$

is easy when $\mathbf{P}$ is **diagonal**:

$$\mathbf{P} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$$

$$e^{\mathbf{P}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}^2 + \cdots$$

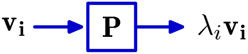
$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \ddots \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + \lambda_1 + \lambda_1^2/2! & & \\ & 1 + \lambda_2 + \lambda_2^2/2! & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots \end{bmatrix}$$

Computing Matrix Exponentials

Fortunately it's easy to **diagonalize** a matrix that is full-rank and has distinct eigenvalues. Start with the eigenvector/eigenvalue property:

$$\mathbf{P}\mathbf{v_i} = \lambda_i \mathbf{v_i}$$



where λ_i is the i^{th} eigenvalue and $\mathbf{v_i}$ is the i^{th} eigenvector (a column vector). If $\mathbf{P}$ is full rank and if none of the eigenvalues are repeated

$$\begin{aligned}
\mathbf{P}\left[\mathbf{v_1}|\mathbf{v_2}|\mathbf{v_3}|\cdots|\mathbf{v_n}\right] &= \left[\mathbf{P}\mathbf{v_1}|\mathbf{P}\mathbf{v_2}|\mathbf{P}\mathbf{v_3}|\cdots|\mathbf{P}\mathbf{v_n}\right] \\
&= \left[\lambda_1\mathbf{v_1}|\lambda_2\mathbf{v_2}|\lambda_3\mathbf{v_3}|\cdots|\lambda_n\mathbf{v_n}\right] \\
&= \left[\mathbf{v_1}\middle|\mathbf{v_2}\middle|\mathbf{v_3}\middle|\cdots\middle|\mathbf{v_n}\right] \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \\
\mathbf{P}\mathbf{V} &= \mathbf{V}\mathbf{\Lambda} \\
\mathbf{P} &= \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}
\end{aligned}$$

where $\mathbf{V} = \left[\mathbf{v_1}\middle|\mathbf{v_2}\middle|\mathbf{v_3}\middle|\cdots\middle|\mathbf{v_n}\right]$ and $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$.

Computing Matrix Exponentials

Substitute the diagonal expansion of $\mathbf{P}$:

$$\mathbf{P} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$$

into the series expansion of $e^{\mathbf{P}}$:

$$e^{\mathbf{P}} = \mathbf{I} + \mathbf{P} + \mathbf{P}^{2}/2! + \mathbf{P}^{3}/3! + \mathbf{P}^{4}/4! + \cdots$$

$$= \mathbf{I} + e^{\mathbf{P}t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow \mathbf{P}e^{\mathbf{P}t} + \frac{1}{2!} e^{\mathbf{P}t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow \mathbf{P}e^{\mathbf{P}t}e^{\mathbf{P}t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow$$

$$= \mathbf{I} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{1}{2!}\mathbf{V}\mathbf{\Lambda}^{\mathbf{v_i}} \rightarrow \boxed{\mathbf{P}} \rightarrow \lambda_i \mathbf{v_i}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{1}{3!}\mathbf{V}\mathbf{\Lambda}^{\mathbf{v_i}} \rightarrow \boxed{\mathbf{P}} \rightarrow \lambda_i \mathbf{v}$$

$$= \mathbf{I} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{1}{2!}\mathbf{V}\mathbf{\Lambda}^{2}\mathbf{V}^{-1} + \frac{1}{3!}\mathbf{V}\mathbf{\Lambda}^{3}\mathbf{V}^{-1} + \cdots$$

$$= \mathbf{V}\mathbf{V}^{-1} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{1}{2!}\mathbf{V}\mathbf{\Lambda}^{2}\mathbf{V}^{-1} + \frac{1}{3!}\mathbf{V}\mathbf{\Lambda}^{3}\mathbf{V}^{-1} + \cdots$$

$$= \mathbf{V}\left(\mathbf{I} + \mathbf{\Lambda} + \frac{1}{2!}\mathbf{\Lambda}^{2} + \frac{1}{3!}\mathbf{\Lambda}^{3} + \cdots\right)\mathbf{V}^{-1}$$

$$= \mathbf{V}e^{\mathbf{\Lambda}}\mathbf{V}^{-1}$$

The matrix exponential of $\mathbf{P}$ can be directly computed from the eigenvalues and eigenvectors of $\mathbf{P}$.

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Step 1: Find the eigenvalues of $\mathbf{P}$.

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Step 1: Find the eigenvalues of $\mathbf{P}$: -1 and -2.

$$|s\mathbf{I} - \mathbf{P}| = \left| \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \right| = s^2 + 3s + 2 = (s+1)(s+2) = 0$$

$$s_{1,2} = -1, -2$$

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Step 1: Find the eigenvalues of $\mathbf{P}$: -1 and -2.

Step 2: Find the eigenvectors of $\mathbf{P}$

$$\mathbf{P}\mathbf{v} = \mathbf{P}\begin{bmatrix} a \\ b \end{bmatrix} = s\mathbf{v} = s\begin{bmatrix} a \\ b \end{bmatrix}$$

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Step 1: Find the eigenvalues of $\mathbf{P}$: -1 and -2.

Step 2: Find the eigenvectors of $\mathbf{P}$

$$\mathbf{Pv} = \mathbf{P} \begin{bmatrix} a \\ b \end{bmatrix} = s\mathbf{v} = s \begin{bmatrix} a \\ b \end{bmatrix}$$

$s = -1$:

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -1 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{aligned} b &= -a \\ -2a - 3b &= -b \end{aligned}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \ \forall \alpha$$

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Step 1: Find the eigenvalues of $\mathbf{P}$: -1 and -2.

Step 2: Find the eigenvectors of $\mathbf{P}$

$$\mathbf{P}\mathbf{v} = \mathbf{P} \begin{bmatrix} a \\ b \end{bmatrix} = s\mathbf{v} = s \begin{bmatrix} a \\ b \end{bmatrix}$$

$s = -2$:

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$b = -2a$$

$$-2a - 3b = -2b$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -2 \end{bmatrix}; \ \forall \beta$$

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Step 1: Find the eigenvalues of $\mathbf{P}$: -1 and -2.

Step 2: Find the eigenvectors of $\mathbf{P}$: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Step 3: Find $e^{\mathbf{P}t} = \mathbf{V} e^{\mathbf{\Lambda} t} \mathbf{V}^{-1}$.

$$\begin{aligned}
e^{\mathbf{P}t} &= \left[\begin{array}{c|c} 1 & 1 \\ -1 & -2 \end{array}\right] \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \left[\begin{array}{c|c} 1 & 1 \\ -1 & -2 \end{array}\right]^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}
\end{aligned}$$

Check Yourself

Let

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Determine $e^{\mathbf{P}t}$

Summary

Characterizing systems described by state-space representations.

Relation between state-space and **transfer function** representations

- specifying $\mathbf{A}, \mathbf{B}, \mathbf{C}$ uniquely determines $H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$.
- specific transfer functions can be represented with different $\mathbf{A}, \mathbf{B}, \mathbf{C}$: depends on how we choose the state.

Determined **step response** of a state-space model

- found explicit representation using **matrix exponential**
- diagonalized system matrix to simplify computing matrix exponential

Next Time: Finding "optimal" gains for state-space control.