

6.3100: Dynamic System Modeling and Control Design

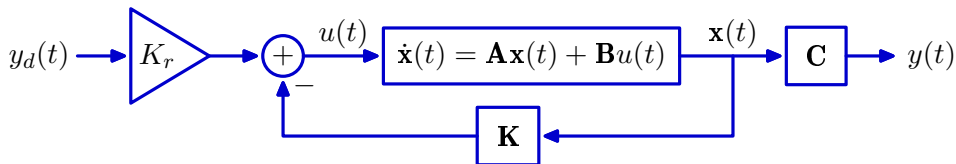
State-Space Characterization

- Transfer Functions
- Step Responses
- Matrix Exponentials

State-Space Approach

Last time, we introduced the **State-Space** approach to control:

- Describe a system by its **states**.
- Describe dynamics of a system by **first-order** relations among states.
- Collect the states and relations in a single first-order **matrix** equation.



Plant: state matrix \mathbf{A} , input vector \mathbf{B} , and output vector \mathbf{C} :

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t)$$

Feedback is characterized by a feedback vector \mathbf{K} and input scaler K_r :

$$u(t) = K_r y_d(t) - \mathbf{K}\mathbf{x}(t)$$

Combine to obtain **closed-loop** characterization:

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} - \mathbf{B}\mathbf{K} \right) \mathbf{x}(t) + \mathbf{B}\mathbf{K}_r y_d(t) \equiv \mathbf{A}_c \mathbf{x}(t) + \mathbf{B}_c y_d(t)$$

State-Space Analysis of Natural Frequencies

Find the natural frequencies of the **closed-loop** system.

Start with the homogeneous equation:

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) = \mathbf{A}_c\mathbf{x}(t)$$

Use the eigenvector/eigenvalue property:

$$\mathbf{A}_c\mathbf{v} = s\mathbf{v} = s\mathbf{I}\mathbf{v} \quad \mathbf{v} \rightarrow \boxed{\mathbf{A}_c} \rightarrow s\mathbf{v}$$

$$(s\mathbf{I} - \mathbf{A}_c)\mathbf{v} = 0$$

Either $\mathbf{v} = \mathbf{0}$ (trivial solution) or $(s\mathbf{I} - \mathbf{A}_c)$ is singular (determinant is zero):

$$\left| s\mathbf{I} - \mathbf{A}_c \right| = 0$$

Example: mass-spring-dashpot system:

$$\mathbf{A}_c = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} - \begin{bmatrix} 0 \\ k/m \end{bmatrix} [K_1 \quad K_2]$$

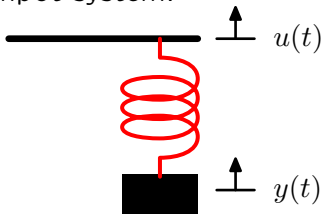
$$\left| s\mathbf{I} - \mathbf{A}_c \right| = \left| \begin{bmatrix} s & -1 \\ k(1+K_1)/m & s+(b+kK_2)/m \end{bmatrix} \right| = 0$$

Characteristic equation: $s^2 + (b+kK_2)s/m + k(1+K_1)/m = 0$ ✓

From State-Space to Transfer Function

Find the transfer function representation from the state-space description.

Example: mass-spring-dashpot system:



Given the state-space representation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ k/m \end{bmatrix} \quad \mathbf{C} = [1 \quad 0]$$

find transfer function:

$$H(s) = \frac{Y(s)}{U(s)}$$

From State-Space to Transfer Function

Find the transfer function representation from the state-space description.

Start with the state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

Convert to frequency domain:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

From State-Space to Transfer Function

Find the transfer function representation from the state-space description.

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

For the mass, spring, dashpot system:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ k/m \end{bmatrix} \quad \mathbf{C} = [1 \quad 0]$$

$$\begin{aligned} H(s) &= [1 \quad 0] \begin{bmatrix} s & -1 \\ k/m & s+b/m \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ k/m \end{bmatrix} \\ &= [1 \quad 0] \frac{1}{s^2 + sb/m + k/m} \begin{bmatrix} s+b/m & 1 \\ -k/m & s \end{bmatrix} \begin{bmatrix} 0 \\ k/m \end{bmatrix} \\ &= \frac{1}{s^2 + sb/m + k/m} [s+b/m \quad 1] \begin{bmatrix} 0 \\ k/m \end{bmatrix} \\ &= \frac{k/m}{s^2 + sb/m + k/m} \quad \checkmark \end{aligned}$$

Check Yourself

Consider a plant described by the following differential equation:

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t)$$

Which of the following show **A**, **B**, **C** matrices for this plant.

1. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\mathbf{C} = [1 \ 0]$
2. $\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{C} = [1 \ -1]$
3. $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{C} = [-1 \ 1]$
4. all of the above
5. none of the above

Check Yourself

Find the transfer function for the system described by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{C} = [1 \quad 0]$$

$$\begin{aligned} H(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\ &= [1 \quad 0] \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= [1 \quad 0] \frac{1}{s^2+5s+6} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2+5s+6} [s+5 \quad 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2+5s+6} \quad \checkmark \end{aligned}$$

Check Yourself

Find the transfer function for the system described by

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \mathbf{C} = [1 \quad -1]$$

$$H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$= [1 \quad -1] \begin{bmatrix} s+2 & 0 \\ 0 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [1 \quad -1] \frac{1}{s^2+5s+6} \begin{bmatrix} s+3 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+6} [s+3 \quad -s-2] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+6} \quad \checkmark$$

Check Yourself

Find the transfer function for the system described by

$$\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \mathbf{C} = [-1 \quad 1]$$

$$H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$= [-1 \quad 1] \begin{bmatrix} s+3 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [-1 \quad 1] \frac{1}{s^2+5s+6} \begin{bmatrix} s+2 & 0 \\ 0 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+6} [-s-2 \quad s+3] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+6} \quad \checkmark$$

Check Yourself

Consider a plant described by the following differential equation:

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t)$$

Which of the following show **A**, **B**, **C** matrices for this plant. 4.

1. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\mathbf{C} = [1 \ 0]$

2. $\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{C} = [1 \ -1]$

3. $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{C} = [-1 \ 1]$

4. all of the above

5. none of the above

Step Response

Find the step response $\mathbf{x}_s(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

where \mathbf{P} and \mathbf{Q} can represent open-loop matrices \mathbf{A} and \mathbf{B} or closed-loop matrices \mathbf{A}_c and \mathbf{B}_c .

Start by finding the step response of the scalar version of this system:

$$\dot{x}(t) = px(t) + qu(t)$$

Step Response

Find the step response $x_s(t)$ of the following scalar system equation:

$$\dot{x}(t) = px(t) + qu(t)$$

Assume the initial value of the step response $x_s(0) = 0$, and $u(t)=1$ for $t>0$.

Homogeneous equation: $\dot{x}_h(t) = px_h(t)$

$$x_h(t) = \alpha e^{\beta t}$$

$$\dot{x}_h(t) = \beta \alpha e^{\beta t} = px_h(t) = p \alpha e^{\beta t} \rightarrow \beta = p$$

$$x_h(t) = \alpha e^{pt}$$

Particular solution: $x_p(t) = \gamma$

$$\dot{x}_p(t) = 0 = p\gamma + q$$

$$\gamma = -q/p \quad (\text{provided that } p \neq 0)$$

Initial condition: $x_s(0) = \alpha - q/p = 0$

$$\alpha = q/p$$

Final solution:

$$x_s(t) = q(e^{pt} - 1)/p$$

Step Response

Find the step response $\mathbf{x}_s(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

The step response $x_s(t)$ of the scalar version of this system:

$$\dot{x}(t) = px(t) + qu(t)$$

is

$$x_s(t) = q(e^{pt} - 1)/p$$

provided $p \neq 0$.

Step Response

Find the step response $\mathbf{x}_s(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

The step response $x_s(t)$ of the scalar version of this system:

$$\dot{x}(t) = px(t) + qu(t)$$

is

$$x_s(t) = q(e^{pt} - 1)/p$$

provided $p \neq 0$.

What's the matrix equivalent of the exponential function e^{pt} ?

Scalar Exponential Function

Exponential functions are eigenfunctions of the derivative operator:

$$e^{pt} \rightarrow \boxed{\frac{d}{dt}} \rightarrow pe^{pt}$$

Express the exponential function as a power series:

$$e^{pt} = 1 + \frac{pt}{1!} + \frac{p^2t^2}{2!} + \frac{p^3t^3}{3!} + \dots$$

Differentiate term-by-term:

$$\begin{aligned} \frac{d}{dt} e^{pt} &= 0 + \frac{p}{1!} + \frac{2p^2t}{2!} + \frac{3p^3t^2}{3!} + \dots \\ &= p + p \frac{pt}{1!} + p \frac{p^2t^2}{2!} + p \frac{p^3t^3}{3!} + \dots \\ &= p \left(1 + \frac{pt}{1!} + \frac{p^2t^2}{2!} + \frac{p^3t^3}{3!} + \dots \right) \\ &= pe^{pt} \quad \checkmark \end{aligned}$$

Matrix Exponential Function

Matrix exponentials are eigenfunctions of the matrix derivative operator:

$$e^{\mathbf{P}t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow \mathbf{P}e^{\mathbf{P}t}$$

The matrix exponential function can also be expanded as a power series:

$$e^{\mathbf{P}t} = \mathbf{I} + \frac{\mathbf{P}t}{1!} + \frac{\mathbf{P}^2t^2}{2!} + \frac{\mathbf{P}^3t^3}{3!} + \dots$$

Differentiate term-by-term:

$$\begin{aligned} \frac{d}{dt}e^{\mathbf{P}t} &= 0 + \frac{\mathbf{P}}{1!} + \frac{2\mathbf{P}^2t}{2!} + \frac{3\mathbf{P}^3t^2}{3!} + \dots \\ &= \mathbf{P} + \mathbf{P}\frac{\mathbf{P}t}{1!} + \mathbf{P}\frac{\mathbf{P}^2t^2}{2!} + \mathbf{P}\frac{\mathbf{P}^3t^3}{3!} + \dots \\ &= \mathbf{P}\left(\mathbf{I} + \frac{\mathbf{P}t}{1!} + \frac{\mathbf{P}^2t^2}{2!} + \frac{\mathbf{P}^3t^3}{3!} + \dots\right) \\ &= \mathbf{P}e^{\mathbf{P}t} = e^{\mathbf{P}t}\mathbf{P} \quad \checkmark \end{aligned}$$

Step Response

Find the step response $\mathbf{x}_s(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

Assume the initial value of the step response $\mathbf{x}_s(0) = \mathbf{0}$, and $u(t)=1$ for $t>0$.

Homogeneous equation: $\dot{\mathbf{x}}_h(t) = \mathbf{P}\mathbf{x}_h(t)$

$$\mathbf{x}_h(t) = e^{\mathbf{P}t}\mathbf{\Psi}$$

Particular solution: $\mathbf{x}_p(t) = \mathbf{\Phi}$

$$\dot{\mathbf{x}}_p(t) = \mathbf{0} = \mathbf{P}\mathbf{\Phi} + \mathbf{Q}$$

$$\mathbf{\Phi} = -\mathbf{P}^{-1}\mathbf{Q} \quad (\text{provided that } \mathbf{P} \text{ is not singular})$$

Initial condition: $\mathbf{x}(0) = \mathbf{\Psi} - \mathbf{P}^{-1}\mathbf{Q} = \mathbf{0}$

$$\mathbf{\Psi} = \mathbf{P}^{-1}\mathbf{Q}$$

Step response:

$$\begin{aligned}\mathbf{x}_s(t) &= (e^{\mathbf{P}t} - \mathbf{I})\mathbf{P}^{-1}\mathbf{Q} \\ &= \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}\end{aligned}$$

Step Response

Find the step response $\mathbf{x}_s(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

Assume the initial value of the step response $\mathbf{x}_s(0) = \mathbf{0}$, and $u(t)=1$ for $t>0$.

Then the solution is

$$\mathbf{x}_s(t) = \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

provided \mathbf{P} is not singular.

Step Response

Find the step response $\mathbf{x}_s(t)$ of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t)$$

Assume the initial value of the step response $\mathbf{x}_s(0) = \mathbf{0}$, and $u(t)=1$ for $t>0$.

Then the solution is

$$\mathbf{x}_s(t) = \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

provided \mathbf{P} is not singular.

Notice that this solution to the matrix problem matches the scalar solution if the matrix problem is first order.

$$x_s(t) = q(e^{pt} - 1)/p$$

But unlike the scalar approach (homogeneous and particular solutions, initial conditions, etc.), **the matrix solution works for any order.**

Step Response: Second-Order Example

Find the step response of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x}_s(t) = \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

Step Response: Second-Order Example

Find the step response of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x}_s(t) = \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

Find \mathbf{P}^{-1} : (easy because \mathbf{P} is diagonal)

$$\mathbf{P}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

Step Response: Second-Order Example

Find the step response of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x}_s(t) = \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

Find the matrix exponential:

$$\begin{aligned} e^{\mathbf{P}t} &= \mathbf{I} + \mathbf{P}t + \mathbf{P}^2 t^2 / 2! + \mathbf{P}^3 t^3 / 3! + \mathbf{P}^4 t^4 / 4! + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} t + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^2 \frac{t^2}{2!} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^3 \frac{t^3}{3!} + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} t + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} \frac{t^3}{3!} + \dots \\ &= \begin{bmatrix} 1 - t + t^2/2! - t^3/3! + \dots & 0 \\ 0 & 1 - 2t + (2t)^2/2! - (2t)^3/3! + \dots \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

Step Response: Second-Order Example

Find the step response of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x}_s(t) = \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

Substitute into the general expression for the step response of the state:

$$\begin{aligned} \mathbf{x}_s(t) &= \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix} \left(\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} e^{-t}-1 & 0 \\ 0 & e^{-2t}-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-e^{-t} & 0 \\ 0 & (1-e^{-2t})/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-e^{-t} \\ (1-e^{-2t})/2 \end{bmatrix} \end{aligned}$$

Step Response: Second-Order Example

Find the step response of the following matrix system equation:

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{Q}u(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x}_s(t) = \mathbf{P}^{-1}(e^{\mathbf{P}t} - \mathbf{I})\mathbf{Q}$$

Much of the ad hoc algebra that we used to solve higher-order differential equations is replaced by the rules of linear algebra.

Check Yourself

Let \mathbf{P} represent the following matrix:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

Which (if any) of the following matrices is equal to $e^{\mathbf{P}}$?

1. $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ 2. $\begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}$ 3. $\begin{bmatrix} 1/e & 0 \\ 0 & 2e^2 \end{bmatrix}$

4. $e^{\mathbf{P}}$ does not exist because the system is unstable

5. none of the above

Check Yourself

Find $e^{\mathbf{P}}$ for $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

$$e^{\mathbf{P}} = \mathbf{I} + \mathbf{P} + \mathbf{P}^2/2! + \mathbf{P}^3/3! + \mathbf{P}^4/4! + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}^2 \frac{1}{2!} + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}^3 \frac{1}{3!} + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \frac{1}{2!} + \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix} \frac{1}{3!} + \dots$$

$$= \begin{bmatrix} e^1 & 0 \\ 0 & e^{-2} \end{bmatrix}$$

Check Yourself

Let \mathbf{P} represent the following matrix:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

Which (if any) of the following matrices is equal to $e^{\mathbf{P}}$? **5.**

1. $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ 2. $\begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}$ 3. $\begin{bmatrix} 1/e & 0 \\ 0 & 2e^2 \end{bmatrix}$

4. $e^{\mathbf{P}}$ does not exist because the system is unstable

5. none of the above

Computing Matrix Exponentials

Finding the series expansion of a matrix exponential

$$e^{\mathbf{P}} = \mathbf{I} + \mathbf{P} + \mathbf{P}^2/2! + \mathbf{P}^3/3! + \mathbf{P}^4/4! + \dots$$

is easy when \mathbf{P} is **diagonal**:

$$\begin{aligned}\mathbf{P} &= \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \\ e^{\mathbf{P}} &= \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}^2 + \dots \\ &= \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \ddots \end{bmatrix} + \dots \\ &= \begin{bmatrix} 1 + \lambda_1 + \lambda_1^2/2! & & \\ & 1 + \lambda_2 + \lambda_2^2/2! & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots \end{bmatrix}\end{aligned}$$

Computing Matrix Exponentials

Fortunately it's easy to **diagonalize** a matrix that is full-rank and has distinct eigenvalues. Start with the eigenvector/eigenvalue property:

$$\mathbf{P}\mathbf{v}_i = \lambda_i\mathbf{v}_i \qquad \mathbf{v}_i \longrightarrow \boxed{\mathbf{P}} \longrightarrow \lambda_i\mathbf{v}_i$$

where λ_i is the i^{th} eigenvalue and \mathbf{v}_i is the i^{th} eigenvector (a column vector). If \mathbf{P} is full rank and if none of the eigenvalues are repeated

$$\begin{aligned}\mathbf{P}[\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\cdots|\mathbf{v}_n] &= [\mathbf{P}\mathbf{v}_1|\mathbf{P}\mathbf{v}_2|\mathbf{P}\mathbf{v}_3|\cdots|\mathbf{P}\mathbf{v}_n] \\ &= [\lambda_1\mathbf{v}_1|\lambda_2\mathbf{v}_2|\lambda_3\mathbf{v}_3|\cdots|\lambda_n\mathbf{v}_n] \\ &= [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\cdots|\mathbf{v}_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}\end{aligned}$$

$$\mathbf{P}\mathbf{V} = \mathbf{V}\mathbf{\Lambda}$$

$$\mathbf{P} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$$

where $\mathbf{V} = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\cdots|\mathbf{v}_n]$ and $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$.

Computing Matrix Exponentials

Substitute the diagonal expansion of \mathbf{P} :

$$\mathbf{P} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$$

into the series expansion of $e^{\mathbf{P}}$:

$$e^{\mathbf{P}} = \mathbf{I} + \mathbf{P} + \mathbf{P}^2/2! + \mathbf{P}^3/3! + \mathbf{P}^4/4! + \dots$$

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$$= \mathbf{I} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{1}{2!} \mathbf{V}\mathbf{\Lambda} \boxed{\mathbf{V}^{-1}\mathbf{V}} \mathbf{\Lambda}\mathbf{V}^{-1} + \frac{1}{3!} \mathbf{V}\mathbf{\Lambda} \boxed{\mathbf{V}^{-1}\mathbf{V}} \mathbf{\Lambda} \boxed{\mathbf{V}^{-1}\mathbf{V}} \mathbf{\Lambda}\mathbf{V}^{-1} + \dots$$

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$$= \mathbf{I} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{1}{2!} \mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^{-1} + \frac{1}{3!} \mathbf{V}\mathbf{\Lambda}^3\mathbf{V}^{-1} + \dots$$

$$= \mathbf{V}\mathbf{V}^{-1} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{1}{2!} \mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^{-1} + \frac{1}{3!} \mathbf{V}\mathbf{\Lambda}^3\mathbf{V}^{-1} + \dots$$

$$= \mathbf{V} \left(\mathbf{I} + \mathbf{\Lambda} + \frac{1}{2!} \mathbf{\Lambda}^2 + \frac{1}{3!} \mathbf{\Lambda}^3 + \dots \right) \mathbf{V}^{-1}$$

Computing Matrix Exponentials

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$$= \mathbf{V} \left(\mathbf{I} + \mathbf{\Lambda} + \frac{1}{2!} \mathbf{\Lambda}^2 + \frac{1}{3!} \mathbf{\Lambda}^3 + \dots \right) \mathbf{V}^{-1}$$

$$= \mathbf{V} e^{\mathbf{\Lambda}} \mathbf{V}^{-1}$$

The matrix exponential of \mathbf{P} can be directly computed from the eigenvalues and eigenvectors of \mathbf{P} .

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Step 1: Find the eigenvalues of \mathbf{P} .

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Step 1: Find the eigenvalues of \mathbf{P} .

$$|s\mathbf{I} - \mathbf{P}| = \left| \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \right| = s^2 + 3s + 2 = (s+1)(s+2) = 0$$

$$s_{1,2} = -1, -2$$

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Step 1: Find the eigenvalues of \mathbf{P} : -1 and -2 .

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Step 2: Find the eigenvectors of \mathbf{P}

$$\mathbf{P}\mathbf{v} = \mathbf{P} \begin{bmatrix} a \\ b \end{bmatrix} = s\mathbf{v} = s \begin{bmatrix} a \\ b \end{bmatrix}$$

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$$\mathbf{P}\mathbf{v} = \mathbf{P} \begin{bmatrix} a \\ b \end{bmatrix} = s\mathbf{v} = s \begin{bmatrix} a \\ b \end{bmatrix}$$

$s = -1$:

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -1 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$b = -a$$

$$-2a - 3b = -b$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \forall \alpha$$

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

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Step 2: Find the eigenvectors of \mathbf{P}

$$\mathbf{P}\mathbf{v} = \mathbf{P} \begin{bmatrix} a \\ b \end{bmatrix} = s\mathbf{v} = s \begin{bmatrix} a \\ b \end{bmatrix}$$

$s = -2$:

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$b = -2a$$

$$-2a - 3b = -2b$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -2 \end{bmatrix}; \quad \forall \beta$$

Computing Matrix Exponentials: Example

Determine $e^{\mathbf{P}t}$ for $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

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Step 3: Find $e^{\mathbf{P}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}$.

$$\begin{aligned} e^{\mathbf{P}t} &= \left[\begin{array}{c|c} 1 & 1 \\ -1 & -2 \end{array} \right] \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \left[\begin{array}{c|c} 1 & 1 \\ -1 & -2 \end{array} \right]^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \end{aligned}$$

Check Yourself

Let

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Determine $e^{\mathbf{P}t}$

Check Yourself

Let

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Determine $e^{\mathbf{P}t}$

Step 1: Find the eigenvalues of \mathbf{P} .

$$|s\mathbf{I} - \mathbf{P}| = 0$$

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right| = \left| \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \right| = s^2 + 1 = 0$$

$$s_{1,2} = \pm j$$

Check Yourself

Let

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Determine $e^{\mathbf{P}t}$

Step 1: Find the eigenvalues of \mathbf{P} : $s_{1,2} = \pm j$

Step 2: Find the eigenvectors of \mathbf{P} .

Check Yourself

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Determine $e^{\mathbf{P}t}$

Step 1: Find the eigenvalues of \mathbf{P} : $s_{1,2} = \pm j$

Step 2: Find the eigenvectors of \mathbf{P}

$$\mathbf{P}\mathbf{v} = s\mathbf{v}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = j \begin{bmatrix} a \\ b \end{bmatrix}$$

$$b = ja$$

$$-a = jb$$

The second equation is j times the first. The eigenvector can be taken as any scalar multiple of $\begin{bmatrix} 1 \\ j \end{bmatrix}$

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Check Yourself

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$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Determine $e^{\mathbf{P}t}$

Step 1: Find the eigenvalues of \mathbf{P} : $s_{1,2} = \pm j$

Step 2: Find the eigenvectors of \mathbf{P} : $\begin{bmatrix} 1 \\ j \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -j \end{bmatrix}$.

Step 3: Calculate the complex exponential.

Check Yourself

Let

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Determine $e^{\mathbf{P}t}$

Step 1: Find the eigenvalues of \mathbf{P} : $s_{1,2} = \pm j$

Step 2: Find the eigenvectors of \mathbf{P} : $\begin{bmatrix} 1 \\ j \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -j \end{bmatrix}$.

Step 3: Calculate the complex exponential.

$$\begin{aligned} e^{\mathbf{P}t} &= \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{jt} & 0 \\ 0 & e^{-jt} \end{bmatrix} \begin{bmatrix} 1/2 & -j/2 \\ 1/2 & j/2 \end{bmatrix} \\ &= \begin{bmatrix} e^{jt} & e^{-jt} \\ je^{jt} & -je^{-jt} \end{bmatrix} \begin{bmatrix} 1/2 & -j/2 \\ 1/2 & j/2 \end{bmatrix} = \begin{bmatrix} (e^{jt}+e^{-jt})/2 & -j(e^{jt}-e^{-jt})/2 \\ j(e^{jt}-e^{-jt})/2 & (e^{jt}+e^{-jt})/2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Check Yourself

Let

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Determine $e^{\mathbf{P}t}$

Alternative method: use series expansion

$$\begin{aligned} e^{\mathbf{P}t} &= \mathbf{I} + \frac{\mathbf{P}t}{1!} + \frac{\mathbf{P}^2 t^2}{2!} + \frac{\mathbf{P}^3 t^3}{3!} + \frac{\mathbf{P}^4 t^4}{4!} + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{t^3}{3!} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^4}{4!} + \dots \\ &= \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - + \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - + \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} - \frac{t^9}{9!} \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - + \dots \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Compare Methods

Let

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Determine $e^{\mathbf{P}t}$

Series method requires closing infinite sums. ✗

$$\begin{aligned} e^{\mathbf{P}t} &= \mathbf{I} + \frac{\mathbf{P}t}{1!} + \frac{\mathbf{P}^2 t^2}{2!} + \frac{\mathbf{P}^3 t^3}{3!} + \frac{\mathbf{P}^4 t^4}{4!} + \dots \\ &= \mathbf{I} + \mathbf{P}t + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{t^3}{3!} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^4}{4!} + \dots \\ &= \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - + \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - + \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} - \frac{t^9}{9!} \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - + \dots \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Check Yourself

Let

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Determine $e^{\mathbf{P}t}$

The diagonalizing method uses conventional linear algebra. ✓

Step 1: Find the eigenvalues \mathbf{P} : $s_{1,2} = \pm j$

Step 2: Find the eigenvectors of \mathbf{P} : $\begin{bmatrix} 1 \\ j \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -j \end{bmatrix}$.

Step 3: Calculate the complex exponential $e^{\mathbf{P}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}$

- assemble \mathbf{V} = matrix of eigenvectors
- assemble $e^{\mathbf{\Lambda}t}$ = matrix of eigenfunctions
- determine the inverse of \mathbf{V}
- matrix multiply

Easily done with numerical methods in Python, MATLAB, ...

Summary

Characterizing systems described by state-space representations.

Relation between state-space and **transfer function** representations

- specifying **A, B, C** uniquely determines $H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$.
- specific transfer functions can be represented with different **A, B, C**: depends on how we choose the state.

Determined **step response** of a state-space model

- found explicit representation using **matrix exponential**
- diagonalized system matrix to simplify computing matrix exponential

Next Time: Finding “optimal” gains for state-space control.