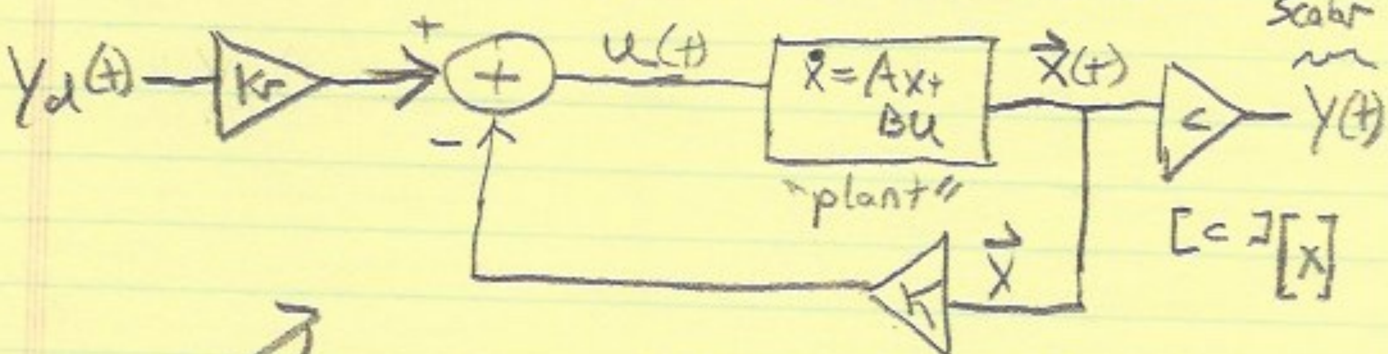
6.310 **11/17/25** ①

Today Observers in C.T. (D.T. after Holiday)

C.T. State Space Reminder



$Kx = [K][x]$
scalar

Diagram

Equation

$$\dot{\vec{x}} = A\vec{x} + Bu \quad u = k_r y_d - K\vec{x}$$

or

$$\dot{\vec{x}} = (A - BK)\vec{x} + BK_r y$$

state

output

AND

$$y(t) = C\vec{x}(t)$$

Ex. States

Arm-angle, umbrella position

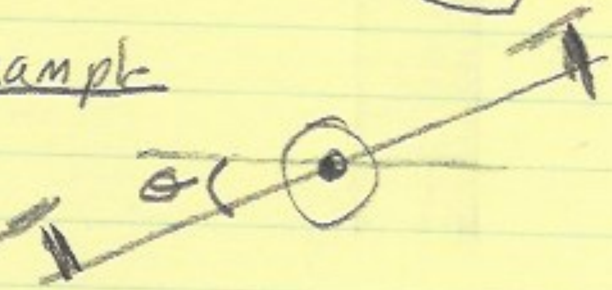
Arm Rot. Vel., Propellor Thrust

umbrella velocity, Coil Current

y(t)'s (that are also states)

What if x is $\boxed{\text{not}}$ measurable (or some of x isn't)

Example



propeller arm -driven

We can measure θ

We can approx ω_a $\quad \omega_a \approx \dfrac{\theta(T) - \theta(T-\Delta T)}{\Delta T}$

We can $\boxed{\text{not}}$ measure:

$\underbrace{\text{Differential}}$ Propeller speed (proportional to Torque)

Left prop speed − Right prop speed

Prop Speed Proportional to Thrust

Estimating States from output using transfer function inverses has noise issues (reminder on page 2A)

Instead, Set up an Estimator

original system: $\dot{x} = Ax + Bu$

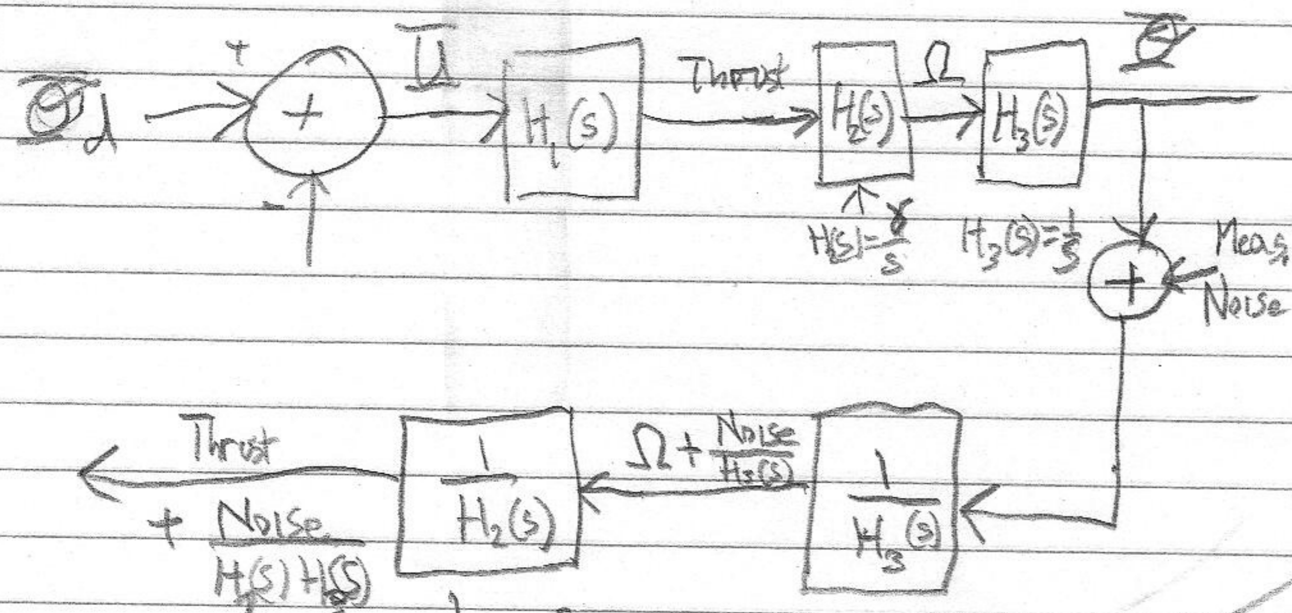
Estimation system (computed using simulation): $\dot{\hat{x}} = A\hat{x} + Bu$

estimate state

We generate u so we know it. Same for both systems

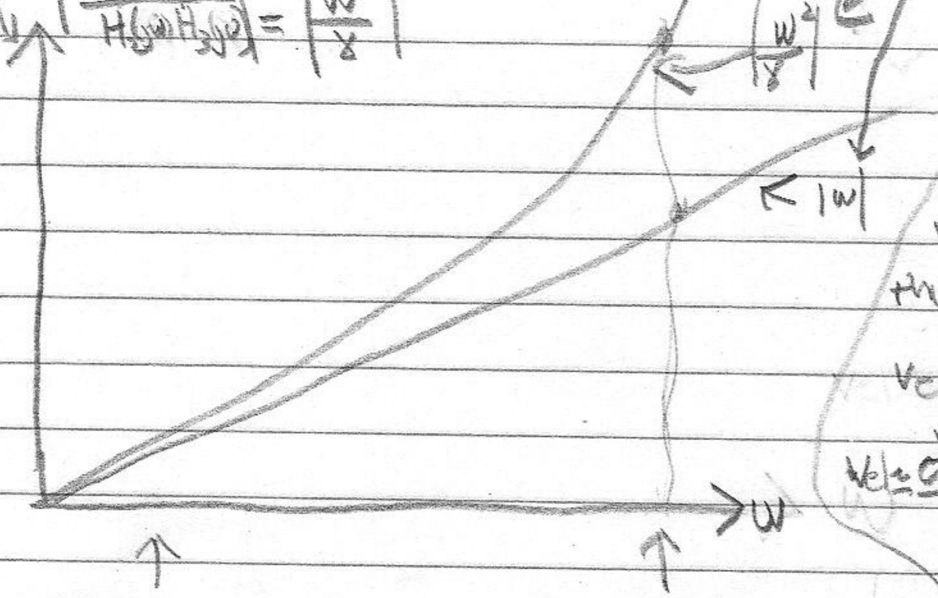
Getting states from output

Reminder of transfer function approach using prop-arm example



$\frac{1}{H_2(s)} = \frac{s}{\gamma}$ $\qquad$ $\frac{1}{H_3(s)} = s$

$|w| = \left|\frac{1}{H_3(jw)}\right|$, $\left|\frac{1}{H_2(jw)H_3(jw)}\right| = \left|\frac{w^2}{\gamma}\right|$



Thrust, Ω, Θ's are lower frequency

Noise is high frequency

Noise is Magnified

But we did this to estimate maglev velocity state

vel $= \frac{d(\text{cnt}) - d(\text{cnt}-1)}{\Delta T}$

and it is Noisy!

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Use estimated (e.g. simulated) state for control

$$u(t) = \underbrace{K_r\, y_d(t)}_{\text{Known}} - K\hat{x}$$

estimated state

"Obvious Problems"

1) is $x(t)\big|_{t=0} = \hat{x}(t)\big|_{t=0}$?
(Do initial conds match)

2) Must simulate estimation system as part of control

Examine Initial Condition Issue

Define $err(t) = x(t) - \hat{x}(t)$

(1) $\frac{d}{dt}x(t) = Ax(t) + Bu(t)$
(2) $\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t)$

Subtracting (2) from (1) (Assuming A,B model match exactly)

$$\frac{d}{dt}\underbrace{(x(t) - \hat{x}(t))}_{err(t)} = A\underbrace{(x(t) - \hat{x}(t))}_{err(t)} + \cancel{Bu(t) - Bu(t)}$$

$$\frac{d}{dt}err(t) = A\, err(t)$$

Recall $e^{At} \to 0$ if Re(eigenvals(A)) < 0
                                  ↑ nat freqs

So

if $err(t)\big|_{t=0} = x(0) - \hat{x}(0) \neq 0$

and $\text{real}(\text{eig}(A)) < 0$ ← strictly stable

then $err(t) \to 0$

So initial condition error decays for a strictly stable system!

"Less Obvious Problems"

1) Modeling Errors will cause mismatches
2) Physical system disturbances will not be in estimation (simulation), so No disturbance rejection

Fix with Feedback

We always measure $y(t) = Cx(t)$
if single output system
then $y(t)$ is a scalar
but $x(t)$, $\hat{x}(t)$ are vectors!
How do we use a scalar measurement to correct vectors

Correct $\hat{x}(t)$ using $y(t) = C\vec{x}(t)$ 5

(1) physical system $\dot{x} = Ax + Bu$

(2) Estimation system $\dot{\hat{x}} = A\hat{x} + \underbrace{Bu}_{\text{We KNOW}} + [L](\underset{y=Cx}{y} - \underset{\hat{y}=C\hat{x}}{\hat{y}})$

Recall estimation error

$$err(t) \equiv x(t) - \hat{x}(t)$$

(1)-(2)

$$\dot{\vec{e}}rr(t) = A\vec{e}rr(t) - L(y(t) - \hat{y}(t))$$

$$= A\vec{e}rr(t) - L(Cx(t) - C\hat{x}(t))$$

$$= A\vec{e}rr(t) - LC(\vec{e}rr(t))$$

$$= (A - LC)\vec{e}rr(t)$$

Pick L so that $\vec{e}rr(t) \to 0$ fast

Note

$$\underbrace{(A - \underset{n\times 1}{[B]}\underset{1\times n}{[K]})}$$

Transpose

$$(A - \underset{n}{[L]}\underset{n}{[C]})$$

K = place(A, B, poles)
K = lqr(A, B, weights)

$L = \text{place}(A^T, C^T, \text{poles})^T$
$= \text{lqr}(A^T, C^T, \text{weights})^T$

LQR $L = \text{lqr}(A^T, C^T, Q, R)^T$?

what's being minimized?

Pole Placement $L = \text{place}(A^T, C^T, \text{poles})^T$

$$\text{eig}(A-LC) = \text{eig}((A-LC)^T)$$

$$= \text{eig}(A^T - C^T L^T)$$

<u>so place works the usual way</u>

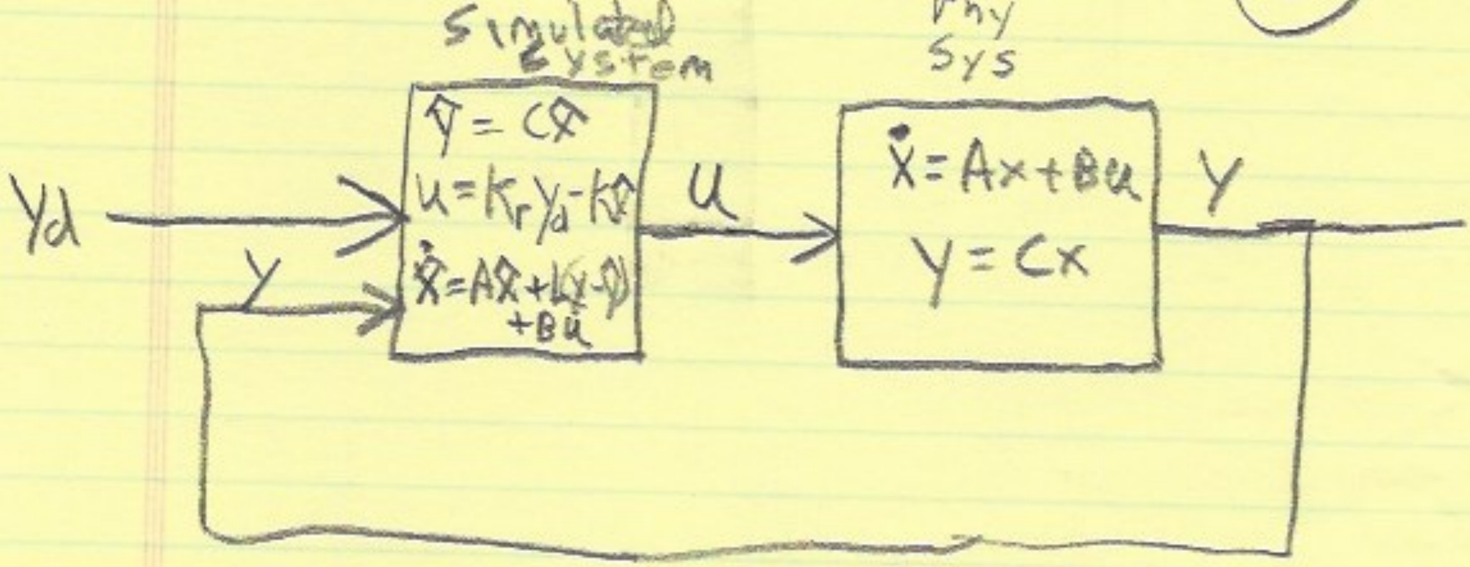
<u>Estimator given L & K</u>

$$u = K_r y_d - K\hat{x}$$

Est $\dot{\hat{x}} = (A - BK)\hat{x} + K_r y_d + L(y - \overset{C\hat{x}}{\hat{y}})$

Phys $\dot{\hat{x}} = (A - BK - LC)\hat{x} + K_r y_d + Ly$

Hmmm..



<u>Analyzing</u>

(1) Phys $\dot{x} = Ax + Bu = Ax + B(K_r y_d - K\hat{x})$

(2) Est $\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$

$err(t) \equiv x(t) - \hat{x}(t)$ $\quad\quad \hat{x}(t) = x(t) - err(t)$

(1)-(2) $\dot{err}(t) = (A - LC)\, err(t)$

$$\dot{x} = Ax + BK_r y_d - BK\,(\overbrace{x(t) - err(t)}^{\hat{x}(t)})$$

$$= (A - BK)x + BK\, err(t) + BK_r y_d$$

$$\begin{bmatrix} \dot{x} \\ \dot{err} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ err \end{bmatrix} + \begin{bmatrix} BK_r \\ 0 \end{bmatrix} y_d$$

Decouples control from estimator!

- **Recall Pole Placement Tradeoff**

Trade off

Larger K → For a faster controller pick K to make $Re(eig(A-BK))$ as negative as possible

Smaller K → To keep $|u|$ below hardware limits pick K so that $|u| = |K_r y_d - K\vec{x}| <$ hardware limit

- **Trade off for the observer**

Trade off

Larger L → Pick L so that $Re(eig(A-LC))$ is negative as possible

Because $\dot{err}(t) = (A-LC)\,err(t)$

Ignoring measurement Noise

$\Rightarrow err(t) = e^{(A-LC)t}\,err(0)$

$err(t) \to 0$ fast if $Re(eig(A-LC)) \ll 0$

Smaller L → What limits L?

Recall

$$\dot{x}(t) = Ax(t) + Bu$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu + L(y - \hat{y})$$

$$\underbrace{\dot{x}(t) - \dot{\hat{x}}(t)}_{err(t)} = A\underbrace{(x(t) - \hat{x}(t))}_{err(t)} - LC(x(t) - \hat{x}(t)) - L \cdot Noise(t)$$

y Measured $+ Noise(t)$

Larger L magnifies Noise!