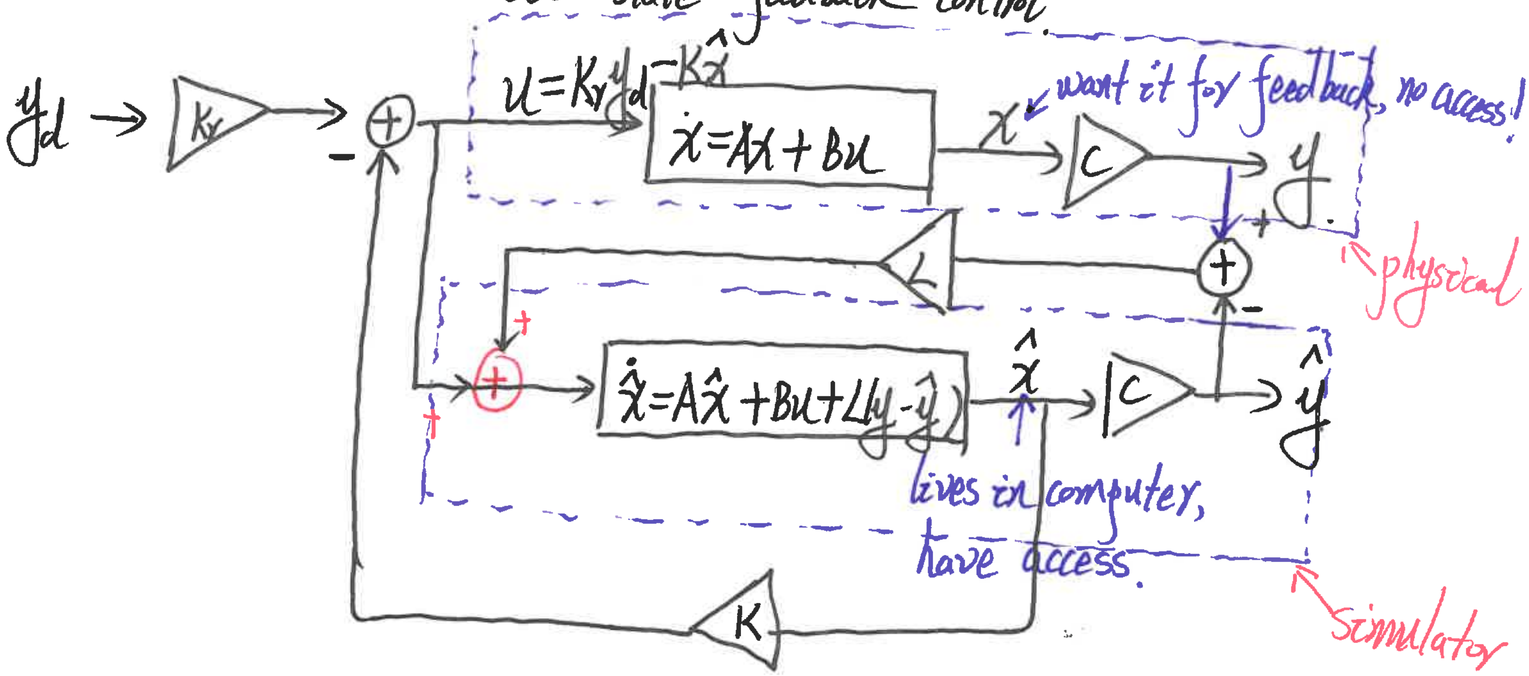
11/19/25

*. Relationship of K and L gains

*. D.T. implementation of CT system/observer.

1. Review: Observer based state feedback control



Physical system: $\frac{d}{dt}x = Ax + Bu$ ①

$y = Cx$

$u = K_r y_d - K\hat{x}$.

Estimator/observer/simulation: $\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(y-\hat{y})$ ②.

$\hat{y} = C\hat{x}$

If we let physical system and estimator evolve independently, very soon $\hat{x}$ does not represent x, due to noise, perturbation, initial state...

Solution: Introduce feedback on $\hat{x}$, add $L(y-\hat{y})$ onto u.

$y, \hat{y}$ — scalar $\quad \hat{x}$ — column vector, $\quad L = \begin{pmatrix} L_1 \\ \vdots \\ L_N \end{pmatrix}$.

How to choose L such that $\hat{x}$ tracks x?

Define $err(t) = x(t) - \hat{x}(t)$.

① − ②: $\frac{d}{dt} err(t) = A\, err(t) - L(\underbrace{y(t)}_{Cx} - \underbrace{\hat{y}(t)}_{C\hat{x}})$

$$\frac{d}{dt} err(t) = (A - LC)\, err(t)$$

Want $err(t)$ to converge to $0 \rightarrow$ pick $eig(A-LC)$ on left half plane.

Recall: $\frac{d}{dt} x(t) = (A - BK)x(t) + \text{input}$

$K = place(A, B, poles)$.

$L = ?$ $\quad (A-LC)^T = A^T - C^T L^T$

$L = place(A^T, C^T, poles)^T$.

2. How to choose K and L?

Question: Are K and L strongly coupled together such that one determines them altogether, or there is some de-coupling?

$x - \hat{x} = err, \longrightarrow \hat{x} = x - err$

From ①: $\frac{d}{dt}x = Ax + B(K_r y_d - K\hat{x})$

$$\frac{d}{dt}x = Ax \mp BK(x - err) + BK_r y_d.$$

$$\frac{d}{dt}x = (A-BK)x(t) + BK \cdot err(t) + BK_r y_d \quad ④.$$

Copy ③: $\frac{d}{dt} err = (A - LC)\, err(t)$ ③.

Consider physical system + simulator in a big state space (2N states).

$$\frac{d}{dt}\begin{pmatrix} x \\ err \end{pmatrix}\begin{array}{l}\updownarrow N \\ \updownarrow N\end{array} = \begin{pmatrix} \overset{\overset{N}{\longleftrightarrow}}{A-BK} \updownarrow N & BK \\ 0 & \overset{\overset{N}{\longleftrightarrow}}{A-LC} \updownarrow N \end{pmatrix}\begin{pmatrix} x \\ err \end{pmatrix} + \begin{pmatrix} BK_r \\ 0 \end{pmatrix} y_d.$$

Results from linear algebra: $\det\begin{pmatrix} R & T \\ 0 & S \end{pmatrix} = \det(R)\det(S).$

poles controlled by K account for x stability

poles controlled by L, for err stability

K and L are largely decoupled when using e.g. place( ) to decide on their values.

3. D.T. modelling and D.T. simulation.

$\boxed{\text{Practical consideration includes making sure } err \text{ converges faster than } x!}$

Physical system is intrinsically C.T.

Controller, estimator realized with MCU — D.T.

← cross C.T. and D.T. boundary.

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Approach 1: model physical system in D.T., implement controller D.T. — Lab 1, Lab 2

Approach 2: — — C.T., ——→ D.T. — Lab 3, Lab 4
(with gains determined in C.T. model)

Now let's go back to Approach 1, particularly because MCU is not very powerful in solving differential Eq. as needed for estimator simulation.

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) \longrightarrow \frac{x[n]-x[n-1]}{\Delta T} \overset{\text{1st order Approx.}}{=} Ax[n-1] + Bu[n-1]$$

$$x[n] = \underbrace{(I + \Delta T A)}_{A_d} x[n-1] + \underbrace{\Delta T B}_{B_d} u[n-1] \quad ⑤ \text{ ------ D.T. state space model.}$$

------ $A_d, B_d \neq A, B$.

$$y = Cx(t) \longrightarrow y[n] = C_d x[n] \quad ⑥ \qquad C_d = C.$$

$$u(t) = K_r y_d - Kx(t) \longrightarrow u[n] = K_{rd} y_d[n] - K_d x[n] \quad ⑦$$

$K_d \overset{?}{\neq} K$, $K_{rd} \overset{?}{\neq} K_r$ Are they very different, or can be the same?

Substitute ⑦ into ⑤:

$$x[n] = (A_d - B_d K_d)x[n-1] + B_d K_{rd} y_d[n-1] \text{ —— closed loop D.T. S.S.M.}$$

Pick K_d: $K_d = \text{place}(A_d, B_d, \text{poles})$ ← within unit circle

Alternatively, one can start from C.T. model. $K = \text{place}(A, B, \text{poles})$ ← left half plane

A possible choice of K_d is:

a. $K_d = K$ b. $K_d = \Delta T\, K$ c. K does not provide info on K_d.

As long as ΔT is small, K is a good choice for K_d.

$$\frac{d}{dt}x(t) = (A - BK)x + BK_r y_d$$

$$\frac{d}{dt}x(t) \rightarrow \frac{x[n]-x[n-1]}{\Delta T} \qquad x \rightarrow x[n-1]$$

$$x[n] = (\underbrace{I + A\Delta T}_{A_d} - \underbrace{B\Delta T}_{B_d} K)\, x[n-1] + \underbrace{B\Delta T}_{B_d} K_r y_d[n-1]$$

The ΔT factor is absorbed in A_d and B_d definition.

K and K_d, K_r and K_{rd} can be similar.

Actually you use K and K_r from C.T. model for your controller (D.T.) in Lab 3, 4 and 5 unconsciously!

What about L and L_d? Hint: Compare relationship between B and B_d vs. C and C_d!

L_d^T = place (A_d^T, C_d^T, poles)

K_d = place (A_d, B_d, poles).