Dynamic System Modeling and Control Design

Eigenvalues and Natural Frequencies

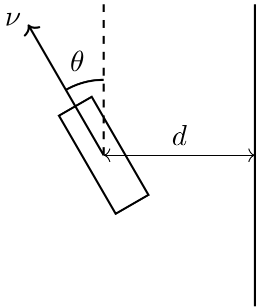
Sept. 17, 2025

Outline

1. Recap of Last Lecture
2. Eigenvalues of 2×2 System Matrix
3. Natural Frequencies
4. New Controller!

Recap: Path Following Robot

Consider a line following example illustrated below:



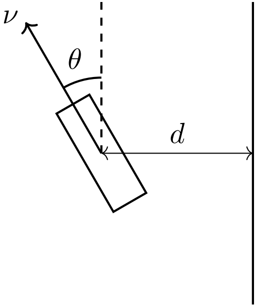
$$d[n] = d[n-1] + \Delta T \nu \theta[n-1],$$
$$\theta[n] = \theta[n-1] + \Delta T \underbrace{\omega[n-1]}_{\text{we control}}.$$

- Goal: control the angular velocity ($\omega[n]$) to follow the line.
- Assume we have an optical sensor to measure the distance, $d[n]$.

Recap: Matrix Form

We can model our system in matrix form.

- Let's try our proportional controller, $\omega[n] = K_p(\underbrace{d_d[n]}_{=0} - d[n])$



$$\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \nu\Delta T \\ -K_p\Delta T & 1 \end{bmatrix}}_{A} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}$$

- Is this a stable system? How can we tell?

Recap: Eigenvalues of A Determine Stability

$$\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \nu\Delta T \\ -K_p\Delta T & 1 \end{bmatrix}}_{A} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}$$

$$\text{evals}(A) = \lambda_1, \lambda_2 = 1 \pm j\Delta T\sqrt{K_p\nu}$$

- We can use an analogous result from our first order system:
  - If $|\lambda_i| < 1,\ i = 1, 2$, then our system is stable.
- ... which does not hold using proportional control. ☹

Today's Objectives

Can we find an equation for the response, $d[n]$?

$$\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \nu\Delta T \\ -K_p\Delta T & 1 \end{bmatrix}}_{A} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}$$

$$\text{evals}(A) = \lambda_1, \lambda_2 = 1 \pm j\Delta T\sqrt{K_p\nu}$$

Why are there complex eigenvalues in my "real" system??

- We can use an analogous result from our first order system:
  - If $|\lambda_i| < 1, \ i = 1, 2$, then our system is stable.
- ... which does not hold using proportional control. ☹
- Can we find a better controller?

Recall the Definition of Eigenvalues and Eigenvectors

For a matrix A, λ_i, v_i are an eigenvalue and eigenvector, resp., of A if

$$Av_i = \lambda_i v_i$$

Suppose that we initialized with $x[0] = c_1 v_1$. Then,

$$\begin{aligned} x[1] &= Ax[0] \\ &= Ac_1 v_1 \\ &= c_1 \lambda_1 v_1 \\ x[2] &= Ax[1] \\ &= Ac_1 \lambda_1 v_1 \\ &= c_1 \lambda_1^2 v_1 \\ &\vdots \\ x[n] &= c_1 \lambda_1^n v_1. \end{aligned}$$

Eigenvalues and Eigenvectors

For **any** vector, we can write it in terms of a linear combination of v_1, v_2:

$$x[0] = c_1 v_1 + c_2 v_2$$

So, the general solution for $x[n]$ is

$$x[n] = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2.$$

What if I only care about $d[n]$, the first element of $x[n]$?

$$d[n] = \tilde{c}_1 \lambda_1^n + \tilde{c}_2 \lambda_2^n,\ \tilde{c}_1 = c_1 v_1[1],\ \tilde{c}_2 = v_2[1].$$

Key things to note:

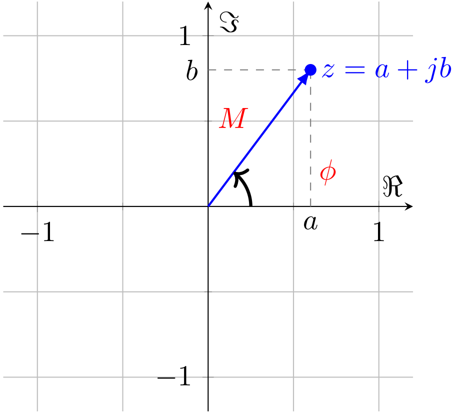
- Can solve for $\tilde{c}_1, \tilde{c}_2$ from initial conditions. Need $d[0], d[1]$.
- λ_1, λ_2 can either be (1) both real or (2) both complex.

Complex numbers and polar form

For a complex number z, we have that

$$z = a + jb = |z|e^{j\angle z} = Me^{j\phi}$$

$$M = \sqrt{a^2 + b^2}$$
$$\arcsin(\phi) = \frac{a}{M}$$
$$\arccos(\phi) = \frac{b}{M}$$



Complex eigenvalues in conjugate pairs

Claim. If $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{C}$ is an eigenvalue, then $\bar{\lambda}$ is also an eigenvalue.

Proof. If $Av = \lambda v$ with $v \neq 0$ and A real, then $\overline{Av} = \overline{\lambda v} \Rightarrow A\bar{v} = \bar{\lambda}\bar{v}$. So $\bar{\lambda}$ is also an eigenvalue (with eigenvector $\bar{v}$). $\square$

Consequence: Nonreal eigenvalues of real matrices always occur in conjugate pairs.

Coefficients of complex modes

- General solution with a conjugate eigenvalue pair:

$$x[n] = c_1\,\lambda_1^n v_1 \;+\; c_2\,\lambda_2^n v_2.$$

- If $\lambda_2 = \bar{\lambda}_1$ and $v_2 = \bar{v}_1$, then

$$x[n] = c_1\,\lambda^n v \;+\; c_2\,\bar{\lambda}^n \bar{v}.$$

- For $x[n]$ to remain real, the coefficients must also be conjugates:

$$c_2 = \bar{c}_1.$$

- So the contribution of a conjugate pair always has the form

$$c\,\lambda^n v \;+\; \bar{c}\,\bar{\lambda}^n \bar{v},$$

which is guaranteed to be real.

Conjugate pair contribution

Suppose a real system has a conjugate pair of eigenvalues:

$$\lambda_{1,2} = Me^{\pm j\phi}, \qquad 0 < M < 1.$$

With coefficients c and $\bar{c}$:

$$x[n] = c\,M^n e^{jn\phi} \;+\; \bar{c}\,M^n e^{-jn\phi}.$$

Expanding with Euler's identity

Euler's identity: $e^{j\phi} = \cos\phi + j\sin\phi$.

$$x[n] = c\,M^n(\cos n\phi + j\sin n\phi) + \bar{c}\,M^n(\cos n\phi - j\sin n\phi).$$

Factor M^n and collect cosine and sine terms:

$$x[n] = M^n\Big[\underbrace{(c+\bar{c})}_{\text{real}}\cos(n\phi) \;+\; \underbrace{j(c-\bar{c})}_{\text{real...?}}\sin(n\phi)\Big].$$

Where does the j go?

Let $c = \alpha + j\beta$ with $\alpha, \beta \in \mathbb{R}$.

$$c + \bar{c} = 2\alpha \quad \text{(real)}, \qquad c - \bar{c} = 2j\beta.$$

So

$$j(c - \bar{c}) = j(2j\beta) = -2\beta \quad \text{(real)}.$$

Final real sinusoidal form

Both coefficients are real, so the result is real:

$$x[n] = M^n\big((2\alpha)\cos(n\phi)\ +\ (-2\beta)\sin(n\phi)\big).$$

$$\boxed{x[n] = M^n\big(\alpha'\cos(n\phi) + \beta'\sin(n\phi)\big),}$$

with real α', β' determined by the initial condition.

Another perspective: a phase–shifted cosine

Start from the decaying sinusoidal form

$$x[n] \;=\; M^n\big(\alpha\cos(n\phi) + \beta\sin(n\phi)\big), \qquad 0 < M < 1.$$

Define

$$R \;=\; \sqrt{\alpha^2+\beta^2}, \qquad \delta \;=\; \mathrm{atan2}(\beta,\alpha).$$

Then, using $\cos(A-B) = \cos A\cos B + \sin A\sin B$,

$$\alpha\cos(n\phi) + \beta\sin(n\phi) \;=\; R\cos\big(n\phi - \delta\big),$$

so $x[n]$ can be written as the *phase–shifted cosine*

$$\boxed{x[n] \;=\; M^n\,R\,\cos\big(n\phi - \delta\big)}$$

- M sets the decay per step,
- ϕ sets the radians per sample (oscillation rate),
- the phase parameter (δ) absorbs the initial mix of cosine/sine.

Welcome Back Pathbot

Let's revisit our path following robot:

$$d[n] = d[n-1] + \Delta T\, \nu\, \theta[n-1],$$
$$\theta[n] = \theta[n-1] + \Delta T\, \omega[n-1],$$

Goal: make $d[n] \to 0$, $\theta[n] \to 0$ by choosing angular velocity $\omega[n]$.

Suppose that we have access to $\theta[n]$. We can penalize large $\theta[n]$:

$$\omega[n] \ = \ -K_p\, d[n] \ - \ K_\theta \theta[n].$$

Proportional-Angle Control?

Then our system equation becomes:

$$x[n] = A\,x[n-1], \qquad A = \begin{bmatrix} 1 & \Delta T \nu \\ -\Delta T K_\theta & 1 - \Delta T K_d \end{bmatrix}.$$

We can find the roots of $\det(\lambda I - A)$ to determine the eigenvalues:

$$(\lambda - 1)(\lambda - (1 - \Delta T K_\theta)) + \Delta T^2 \nu K_p = 0$$

$$\Rightarrow \lambda_1, \lambda_2 = 1 - \frac{\Delta T K_\theta}{2} \pm \frac{\Delta T}{2} \sqrt{K_\theta^2 - 4 K_p \nu}$$

When will the system be stable?

With the following eigenvalues, when is the system stable?

$$\Rightarrow \lambda_1, \lambda_2 = 1 - \frac{\Delta T K_\theta}{2} \pm \frac{\Delta T}{2}\sqrt{K_\theta^2 - 4K_p\nu}$$

- If $K_\theta^2 = 4K_p\nu$, both eigenvalues are $1 - \frac{\Delta T K_\theta}{2}$.
- If $K_\theta^2 < 4K_p\nu$, we have complex-valued roots.
- If $K_\theta^2 > 4K_p\nu$, we have two distinct real eigenvalues.

Takeaway: larger $K_\theta \Rightarrow$ larger safe K_p

For decaying oscillations (complex eigenvalues) with $|\lambda| < 1$ it suffices to pick

$$\boxed{\frac{K_\theta^2}{4\nu} < K_p < \frac{K_\theta}{\nu\,\Delta T}, \qquad K_\theta < \frac{4}{\Delta T}}.$$

Interpretation.

- The upper bound on K_p grows *linearly* with K_θ: $K_p^{\max} = \dfrac{K_\theta}{\nu\,\Delta T}$.
- The lower bound grows *quadratically* with K_θ: $K_p^{\min} = \dfrac{K_\theta^2}{4\nu}$.
- As long as $K_\theta < 4/\Delta T$, there is a nonempty band of K_p values yielding $|\lambda| < 1$ (stable, decaying oscillations).

Example: Eigenvalues in the complex plane

Let $\Delta T = 0.05$ s, $\nu = 0.5$ m/s. Then $4/\Delta T = 80$.

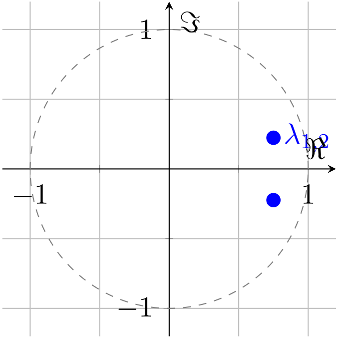
Choose $K_\theta = 10$ (< 80).

$$K_p \in \left(\frac{10^2}{4{\cdot}0.5},\ \frac{10}{0.5{\cdot}0.05}\right) = (50,\ 400).$$

Pick $K_p = 100$. Then

$$\lambda_{1,2} = 0.75 \pm j\,0.2236,$$

with magnitude $|\lambda| \approx 0.783 < 1$.



Larger K_θ enlarges the admissible K_p range

Same parameters as before: $\Delta T = 0.05$ s, $\nu = 0.5$ m/s, $4/\Delta T = 80$.

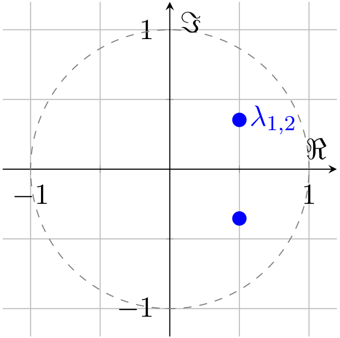
Now choose a **larger** angle gain: $K_\theta = 20$ (< 80).

$$K_p \in \left(\frac{20^2}{4 \cdot 0.5},\ \frac{20}{0.5 \cdot 0.05}\right) = (200,\ 800).$$

Pick $K_p = 300$ (inside the band). Then the eigenvalues are

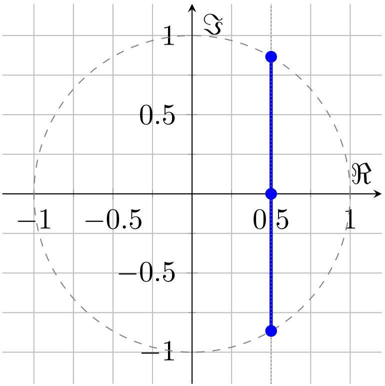
$$\lambda_{1,2} = 1 - \frac{\Delta T K_\theta}{2} \pm j\,\frac{\Delta T}{2}\sqrt{4\nu K_p - K_\theta^2}$$
$$= 0.5 \pm j\,0.3536,$$

with magnitude $|\lambda| = \sqrt{0.5^2 + 0.3536^2} \approx 0.612 < 1$.



Eigenvalue locus as K_p varies (fixed K_θ)

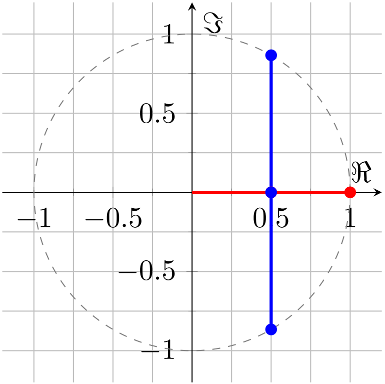
As K_p increases from $K_{p,\min}$ to $K_{p,\max}$, the pair moves straight up/down along $\Re\{\lambda\} = 0.50012$, from $\Im = 0$ to $\Im = \pm 0.8658$.



Eigenvalue locus as K_p varies (fixed K_θ)

For $0 < K_p < K_{p,\min}$**:** eigenvalues are real, moving on the real axis.

For $K_{p,\min} < K_p < K_{p,\max}$**:** eigenvalues are complex, forming a vertical line at $\Re = 0.50012$.



Proportional Derivative (PD) control

What if we cannot measure $\theta[n]$? Penalize the rate of change of $d[n]$.

$$\boxed{\omega[n] = -K_p\, d[n] \;-\; K_d\, \frac{d[n] - d[n-1]}{\Delta T}}.$$

Note that this is a way to approximate the angle:

$$\frac{d[n] - d[n-1]}{\Delta T} \;\approx\; \nu\, \theta[n].$$

Thus, *for analysis/intuition*, PD on $d[n]$ is approximately

$$\omega[n] \;\approx\; -K_p\, d[n] \;-\; \underbrace{(K_d\, \nu)}_{:= K_\theta}\, \theta[n],$$

PD on distance $d[n]$ (exact discrete-time law)

$$\boxed{\omega[n] \;=\; -K_p\, d[n] \;-\; K_d\, \frac{d[n]-d[n-1]}{\Delta T}}$$

Augment $x[n]$ to carry $d[n-1]$: let $x[n] = \begin{bmatrix} d[n] \\ \theta[n] \\ d[n-1] \end{bmatrix}$. Then our system is:

$$x[n] \;=\; \underbrace{\begin{bmatrix} 1 & \Delta T\,\nu & 0 \\ -(\Delta T K_p + K_d) & 1 & K_d \\ 1 & 0 & 0 \end{bmatrix}}_{A} x[n-1].$$

Eigenvalues of A?

So, what are the eigenvalues of A for PD control? When are they less than 1 in magnitude?

$$\underbrace{\begin{bmatrix} 1 & \Delta T\,\nu & 0 \\ -(\Delta T K_p + K_d) & 1 & K_d \\ 1 & 0 & 0 \end{bmatrix}}_{A}$$

Let's use computational tools to analyze the eigenvalues and stability.