Dynamic System Modeling and Control Design

Frequency Response

Oct. 8, 2025

Review: s-Domain Description

- We describe an LTI system by

$$Y = H(s)\,U, \qquad H(s) = \frac{n(s)}{d(s)},$$

where $H(s)$ is a rational function of polynomials,

$$n(s) = b_{\hat{\ell}} s^{\hat{\ell}} + \ldots + b_1 s + b_0, \qquad d(s) = a_\ell s^\ell + \ldots + a_1 s + a_0,$$

- using the heuristic rule

$$\frac{d}{dt} \;\longleftrightarrow\; s.$$

Turns a differential equation into algebra in the variable s.

Natural Frequencies of Our System

In general,

$$(a_\ell s^\ell + \ldots + a_1 s + a_0)Y = (b_{\hat{\ell}} s^{\hat{\ell}} + \ldots + b_1 s + b_0)U.$$

Suppose $u(t) = 0 \ \forall t \Rightarrow U = 0$ and $y(t) = \sum_i \alpha_i e^{\lambda_i t}$.

Then for each term,

$$a_\ell s^\ell Y \longleftrightarrow a_\ell \frac{d^\ell}{dt^\ell} y(t), \qquad \frac{d^\ell}{dt^\ell} y(t) = \sum_i \lambda_i^\ell \alpha_i e^{\lambda_i t}.$$

$\alpha_i \neq 0$ iff λ_i is a root of $d(s)\big|_{s=\lambda_i} = 0$

- The roots of $d(s)$ are called the “natural frequencies.”

Motivation: Why (Complex) Exponential Inputs?

We will choose an input function of the form: $u(t) = \sum_i \beta_i e^{s_i t}$.

- Many signals in engineering are oscillatory: vibrations, AC power, rotating machinery, etc.
- Because the system is linear and time-invariant; knowing the response to one sinusoid tells us how the system behaves for any sum of sinusoids.

Idea: We can probe our system with exponentials of our choosing, $e^{s_0 t}$, and analyze how the system output responds.

Decomposing the System Response

- For any input,

$$y(t) = \underbrace{\sum_i \alpha_i e^{\lambda_i t}}_{\text{natural response}} + (\text{response due to an input}).$$

- If the natural frequencies λ_i have negative real parts, the natural response **decays**.
- Today's objective: understand the response due to a **sinusoidal input**.

Complex Exponential In, Complex Exponential Out

Recall our setup:

$$(a_\ell s^\ell + a_{\ell-1} s^{\ell-1} + \cdots + a_1 s + a_0)Y = (b_{\hat{\ell}} s^{\hat{\ell}} + \cdots + b_1 s + b_0)U,$$

or equivalently,

$$d(s)Y = n(s)U.$$

We want to see how the system responds to an **exponential input:**

$$u(t) = e^{s_0 t}.$$

Because $s \leftrightarrow \frac{d}{dt}$,

$$s^k U \;\longleftrightarrow\; \frac{d^k}{dt^k} e^{s_0 t} = s_0^k e^{s_0 t}.$$

Try a particular solution: assume $y(t) = Ce^{s_0 t}$, where C is a complex constant to be found.

Complex Exponential In, Complex Exponential Out

Substitute $y(t) = Ce^{s_0 t} \longleftrightarrow Y$ and $u(t) = e^{s_0 t} \longleftrightarrow U$ into

$$d(s)\big|_{s=s_0} Y = n(s)\big|_{s=s_0} U.$$

Then:

$$d(s_0)\, Ce^{s_0 t} = n(s_0)\, e^{s_0 t}.$$

If $d(s_0) \neq 0$, we can cancel $e^{s_0 t}$ to obtain

$$C = \frac{n(s_0)}{d(s_0)}, \Rightarrow \boxed{y(t) = \frac{n(s_0)}{d(s_0)}\, e^{s_0 t}}.$$

Special case: Complex exponential input.

If $d(s_0) \neq 0$, we can cancel $e^{s_0 t}$ to obtain

$$C = \frac{n(s_0)}{d(s_0)}, \Rightarrow \boxed{y(t) = \frac{n(s_0)}{d(s_0)}\, e^{s_0 t}}.$$

Let $s_0 = j\omega$:

$$H(j\omega) \equiv \frac{n(j\omega)}{d(j\omega)}, \qquad y(t) = H(j\omega)e^{j\omega t}.$$

- Recall Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta$
- The complex exponential $e^{j\omega t}$ NEVER decays to zero.

Transient and Steady-State Responses

- For a persisting sinusoidal input,

$$y(t) = \underbrace{\sum_i \alpha_i e^{\lambda_i t}}_{\text{natural response}} + (\text{sinusoidal steady state response}).$$

- If the natural frequencies λ_i have negative real parts, the natural response **decays**.
  - Dies out with time; referred to as the **transient** response.
- We have the tools necessary to determine the **sinusoidal steady state** response.

Sinusoidal Steady State

Suppose our input takes the form $u(t) = \cos \omega t$. What is $y(t)$?

Euler's identity: $u(t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$

- Through superposition, we know that,

$$\begin{aligned}
y(t) &= \frac{1}{2}\left(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right), \\
&= \Re\left\{H(j\omega)e^{j\omega t}\right\}, \\
&= \Re\left\{|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t}\right\}, \\
&= \Re\left\{|H(j\omega)|e^{j(\omega t + \angle H(j\omega))}\right\}, \\
&= |H(j\omega)| \cos(\omega t + \angle H(j\omega)),
\end{aligned}$$

- Compared with $u(t)$, output $y(t)$'s magnitude scales by $|H(j\omega)|$ and its phase changes by $\angle H(j\omega)$.

Defining the Frequency Response

- The **frequency response** of a continuous-time LTI system is

$$H(j\omega) \equiv H(s)\Big|_{s=j\omega}.$$

- $H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$ is complex.
- $|H(j\omega)|$: amplitude ratio (gain) $\angle H(j\omega)$: phase shift.
- Together, they describe the steady-state output for any sinusoidal input.

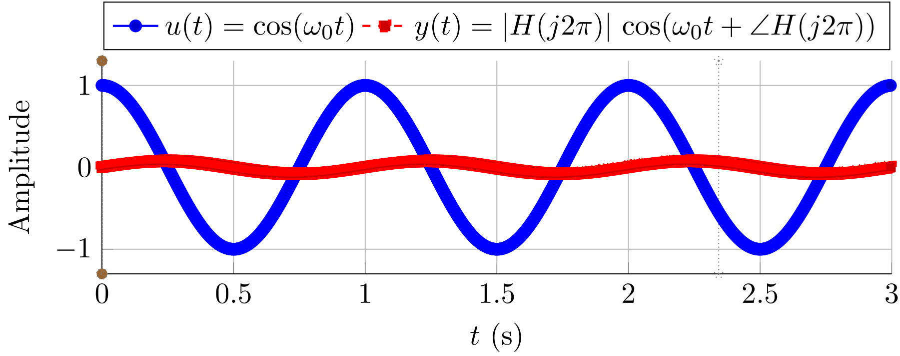
Interpreting the Frequency Response

- $|H(j\omega)|$: how much the system amplifies or attenuates each input frequency.
- The shape of $|H(j\omega)|$ reveals which input frequencies excite the system strongly (those near its natural frequencies).
- $\angle H(j\omega)$: how much the system shifts each frequency in time.
- The frequency response is therefore the bridge between the system's natural dynamics and its steady-state behavior under sinusoidal forcing.

Example: $H(s) = \dfrac{1}{1+2s}$ for single input

Suppose $H(s) = \dfrac{1}{1+2s}$. We input $u(t) = \cos(2\pi t)$.

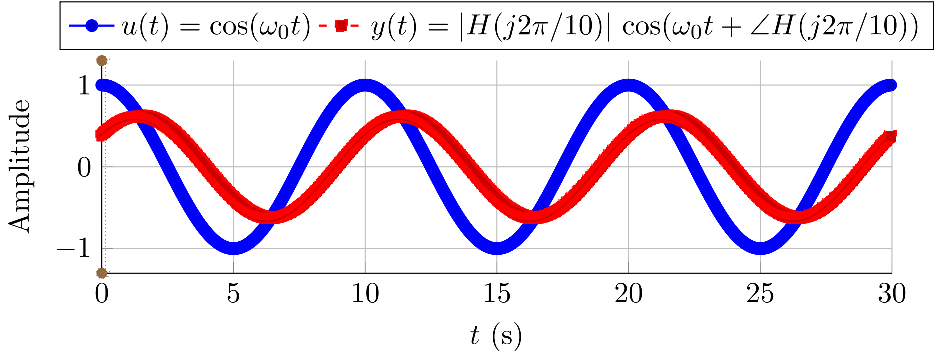
- $|H(j2\pi)| \approx 0.079$
- $\angle H(j2\pi) \approx -1.49$ rad



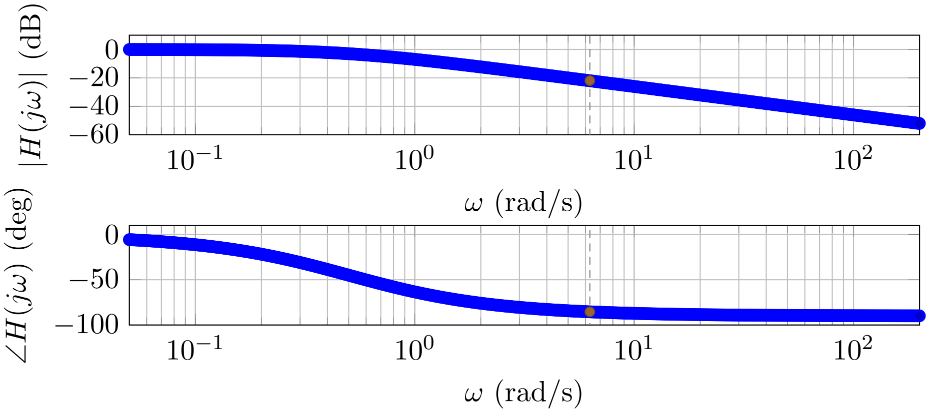
Example: $H(s) = \dfrac{1}{1+2s}$ for single input

Suppose $H(s) = \dfrac{1}{1+2s}$. We input $u(t) = \cos(\frac{2\pi}{10}t)$.

- $|H(j2\pi/10)| \approx 0.625$
- $\angle H(j2\pi/10) \approx -0.90$ rad



Example: $H(s) = \dfrac{1}{1+2s}$ for different inputs



At $\omega_0 = 2\pi$: $|H(j\omega)| \approx -22.0$ dB, $\angle H(j\omega) \approx -85.4°$.

We will learn more about generating these plots next week.

Summary

- Natural frequencies: describe the system's own free oscillations (transients).
- Frequency response: $H(j\omega)$ — the system's gain and phase shift for forced sinusoidal inputs.
- The total response:

$$y(t) = \underbrace{\sum_i \alpha_i e^{\lambda_i t}}_{\text{transient (natural)}} + \underbrace{|H(j\omega)| \sin(\omega t + \angle H(j\omega))}_{\text{steady-state (forced)}}.$$

- As transients decay, the steady-state sinusoid remains.
- $H(j\omega)$ fully characterizes this steady-state relationship.