Dynamic System Modeling and Control Design

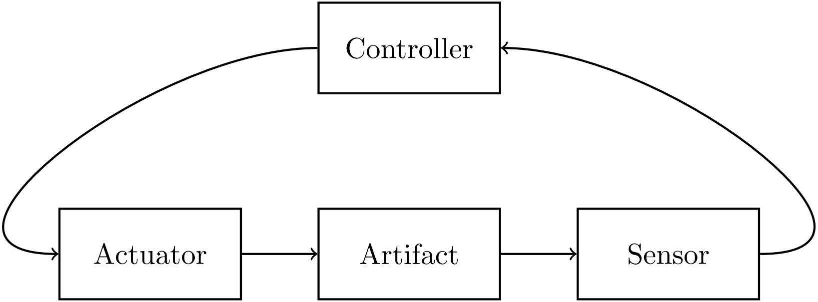
Linearity, Time Invariance, Parameter Identification

Sept. 10, 2025

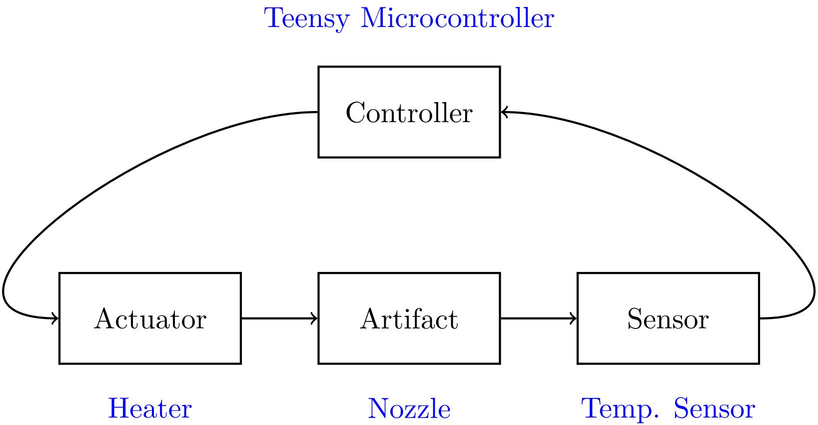
Outline

1. Recap of Last Lecture
2. Linearity and Time Invariance
3. Estimating System Parameters λ & γ

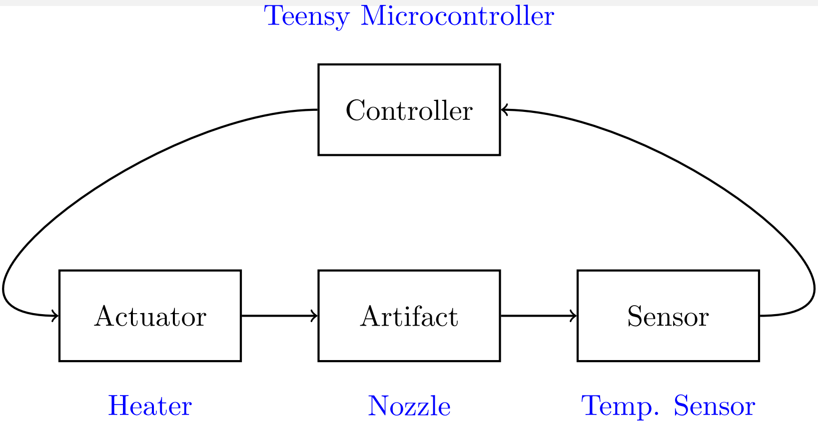
Recap: Generic Control System



Recap: Generic Control System



Recap: Generic Control System



$$T_m[n] = T_m[n-1] + \Delta T \gamma_{th} u[n-1] \underbrace{-\Delta T \beta T_m[n-1]}_{\text{heat loss term}}$$

Recap: General Form of First Order Difference Equation (FODE)

The general form of a first order difference equation:

$$y[n] = \lambda y[n-1] + \gamma u[n-1] \qquad (\#1)$$

Notes on the general form:

- Our goal is to solve for $y[n]$,
- $u[n]$ is the input or driving function we set,
- λ, γ are system parameters.

Recap: General Solutions of FODE

From the general form,

$$y[n] = \lambda y[n-1] + \gamma u[n-1] \qquad (\#1)$$

we found the general solution, for arbitrary n, to be:

$$y[n] = \lambda^n y[0] + \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$$

Additionally, two special cases:

- Zero Input Response: If $u[n] = 0 \; \forall n : \; y[n] = \lambda^n y[0]$
- Zero State Response: If $y[0] = 0 : \; y[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$

Recap: Steady State Response

If $u[n] = u_0 \; \forall n$ (i.e., constant input) AND $|\lambda| < 1$, then...

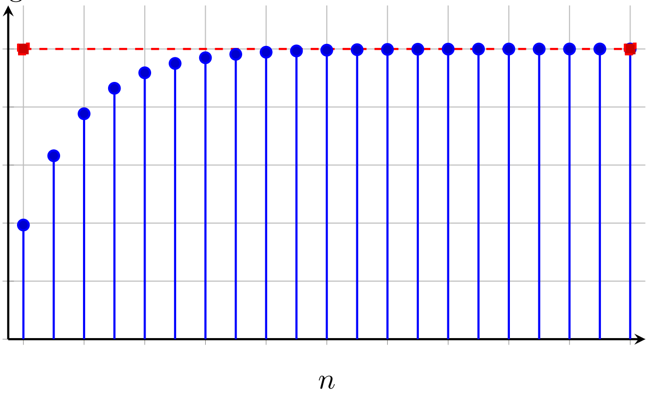
$$\lim_{n \to \infty} y[n] := y[\infty] = \frac{\gamma}{1 - \lambda} u_0.$$

For example, for the heating example with proportional control, with $u[n] = K_p(T_d[n] - T_m[n])$ and $|\lambda| = |1 - \Delta T \gamma_{th} K_p| < 1$,

$$T_m[\infty] = \frac{\gamma_{th} K_p}{\gamma_{th} K_p + \beta} T_{d_0}$$

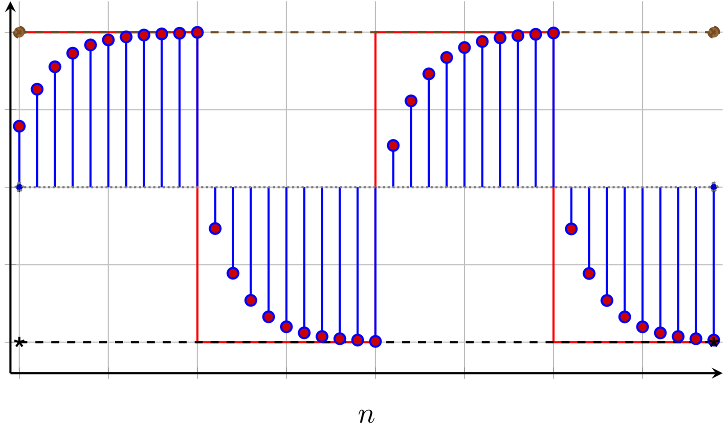
Today's First Objective

How do we go from this...



Today's First Objective

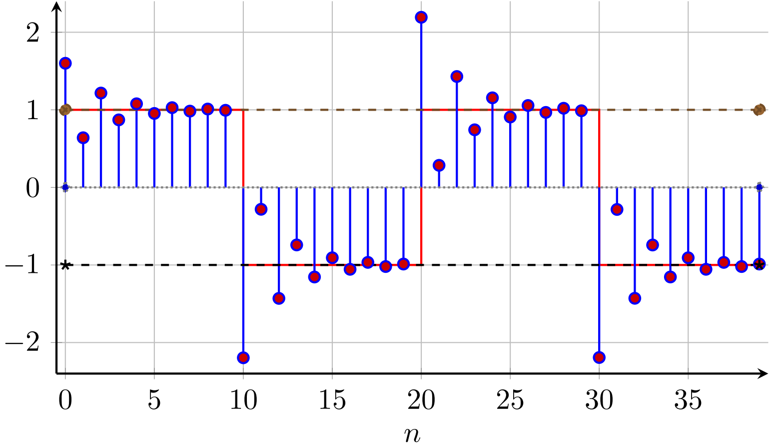
... to this??



Answer: Linearity and Time Invariance.

Today's First Objective

... or this??



Answer: Linearity and Time Invariance.

Today's Second Objective

We have this nice FODE:

$$y[n] = \lambda y[n-1] + \gamma u[n-1] \qquad (\#1)$$

What are λ and γ? How do we find them?

We will introduce how to estimate these system parameters today!

Property I: Decomposition of General Solution

Recall our two special cases:

- Zero Input Response: If $u[n] = 0 \; \forall n :\; y[n] = \lambda^n y[0]$
- Zero State Response: If $y[0] = 0 :\; y[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$

Then we can decompose $y[n]$ into its ZIR and ZSR:

$$y[n] = \lambda^n y[0] + \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$$

$$\text{(I): } y[n] = y_{ZIR}[n] + y_{ZSR}[n]$$

Property II: Linearity & Time Invariance of ZSR

Suppose that I have two different input functions $u_A[n], u_B[n]$. Then,

$$y_{A,ZSR}[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u_A[m],$$
$$y_{B,ZSR}[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u_B[m].$$

Assume $u_A[n] = u_B[n] = 0 \; \forall n < 0$ and $N_A, N_B > 0$.

If $u[n] = \alpha u_A[n - N_A] + \beta u_B[n - N_B]$, then

$$\text{(II): } y_{ZSR}[n] = \alpha y_{A,ZSR}[n - N_A] + \beta y_{B,ZSR}[n - N_B].$$

Property III: An Aspect of Time Invariance of ZIR

If $u[n] = 0 \ \forall n \geq N$, then

$$\text{(III)}: \ y[n] = \lambda^{n-N} y[N], \ \forall n > N.$$

Proof sketch:

$$\begin{aligned} y[N+1] &= \lambda y[N] + \gamma u[N] \\ y[N+2] &= \lambda y[N+1] + \gamma u[N+1] \\ &= \lambda^2 y[N] \\ y[N+3] &= \lambda^3 y[N] \\ &\ \vdots \end{aligned}$$

Steady State for General System (Using (I) - (III))

Suppose that I have $|\lambda| < 1$ and an input function $u_1[n]$ defined by,

$$u_1[n] = 0,\ n < N$$
$$u_1[n] = 1,\ n \geq N.$$

with an initial state of $y_1[0] = 0$. What is $y_1[n]$?

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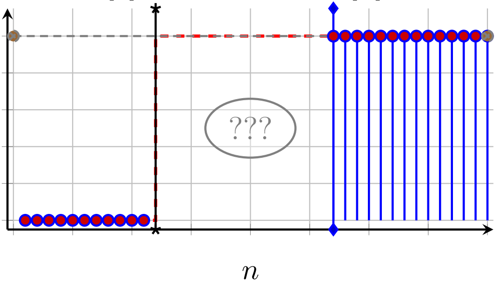

Let's find out using Properties (I)-(III)!

Steady State (Cont.)

Consider a system with

$$y_2[0] = \frac{\gamma}{1-\lambda},$$
$$u_2[n] = 1 \ n < N,$$
$$u_2[n] = 0 \ n \geq N.$$

What is $y_2[n]$ for $n > N$?

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$$u_2[n] = 1 \; n < N,$$
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What is $y_2[n]$ for $n > N$?
We can use Property (III): Time Invariance of ZIR:

$$y_2[n] = \frac{\gamma}{1-\lambda} \; n \leq N$$
$$y_2[N+1] = \lambda y_2[N] = \lambda \frac{\gamma}{1-\lambda}$$
$$y_2[N+n] = \lambda^{(n-N)} \frac{\gamma}{1-\lambda}$$

Steady State (Cont.)

Consider a system with

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What is $y_3[n]$?

Steady State (Cont.)

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$$u_3[n] = 1 \ \forall n,$$

What is $y_3[n]$?
Since we initialized at steady state, and the input function $u_3[n]$ does not change, we will remain in steady state.

$$y_3[n] = \frac{\gamma}{1-\lambda}.$$

Steady State for General System (Using (I) - (III))

Recall that $|\lambda| < 1, u[n] = u_0 \ \forall n > N$, then $y[\infty] = \frac{\gamma}{1-\lambda} u_0$.
Suppose that I have $|\lambda| < 1$ and an input function $u_1[n]$ defined by,

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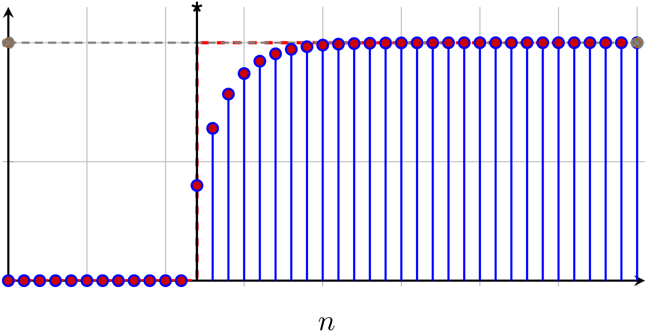
with an initial state of $y_1[0] = 0$. What is $y_1[n]$?
We can use Property (II): Linearity of ZSR

Since $u_1[n] = u_3[n] - u_2[n]$, we know that $y_1[0] = y_3[0] - y_2[0]$.
Therefore,

$$\begin{aligned} y_1[n] &= y_3[n] - y_2[n] \\ &= \frac{\gamma}{1-\lambda} - \lambda^{n-N} \frac{\gamma}{1-\lambda}, \; n > N \\ &= \frac{\gamma}{1-\lambda}(1 - \lambda^{n-N}), \; n > N. \end{aligned}$$

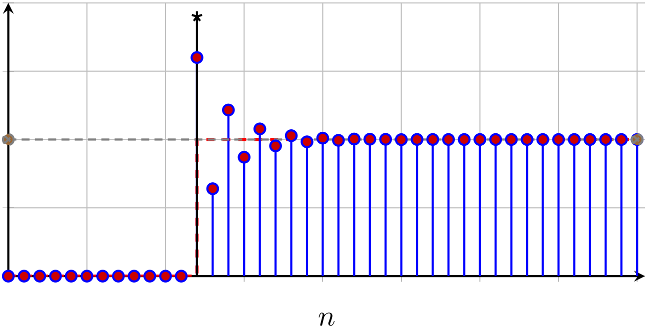
Now, we can fill in the gap...

$$y_1[n] = \frac{\gamma}{1-\lambda}(1-\lambda^{n-N}),\ n > N, \underline{0 < \lambda < 1}.$$



Now, we can fill in the gap...

$$y_1[n] = \frac{\gamma}{1-\lambda}(1-\lambda^{n-N}),\ n > N, \underline{-1 < \lambda < 0}.$$



Recall Today's Second Objective

We have this nice FODE:

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We can (experimentally) estimate λ, γ in many different ways.

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In particular, we have two unknowns. Let's find two equations and solve.

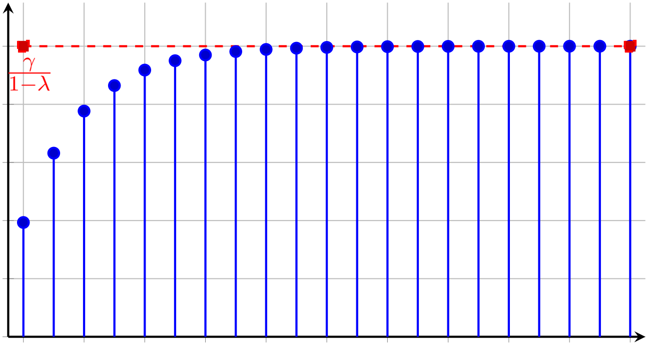
Evaluate Steady State Response

Assuming...

$$y[0] = 0, 0 < \lambda < 1, y[n] = \lambda y[n-1] + \gamma u[n-1], u[n] = 1,$$

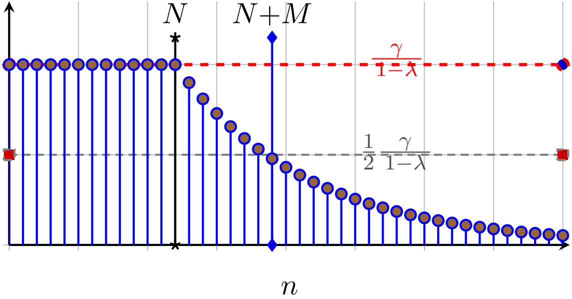
we already have one relationship!

$$y[\infty] = \frac{\gamma}{1-\lambda} \Rightarrow \gamma = y[\infty](1-\lambda).$$



Evaluate Decay

Let $u[n] = 1\ \forall n \leq N$. How many steps does it take to decay halfway?



From Property (III) Time Invariance of ZIR:

$$\frac{\gamma}{1-\lambda}\lambda^M = 0.5\frac{\gamma}{1-\lambda}$$
$$\Rightarrow \lambda = 0.5^{1/M}.$$
$$\Rightarrow \gamma = y[\infty](1 - 0.5^{1/M})$$

Closing Thoughts

How can we generate the previous two plots?

- We get to pick which controller we use (and set the gains)!
- We can find an expression for λ which will (probably) be a function of the gain(s) of our controller.
- We can pick gains such that we truly reach zero steady state error.