Dynamic System Modeling and Control Design

Matrix PID

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Outline

1. Recap of Last Lecture
2. Alignment Problem
3. Introducing Integral-Based Control

Recap: Disturbance Modeling

System Equation w/o Disturbance:

$$x[n] = Ax[n-1] + Bu[n-1]$$

A, B depend on the physical model and controller. **System Equation w/ Disturbance:**

$$x_{\text{dist}}[n] = Ax_{\text{dist}}[n-1] + Bu[n-1] + B_{\text{dist}}u_{\text{dist}}[n-1]$$

Disturb Response

$$\begin{aligned} e[n] &\triangleq x_{\text{dist}}[n] - x[n], \\ &= Ae[n-1] + B_{\text{dist}}u_{\text{dist}}[n-1]. \end{aligned}$$

Recap: Steady-State of Disturb Response

Suppose $u_{\text{dist}}[n] = u_{\text{dist}}[\infty],\ \forall n$. The steady-state disturb response is:

$$e[\infty] = (I - A)^{-1}\, B_{\text{dist}} u_{\text{dist}}[\infty]$$

In particular, we have,

$$\begin{bmatrix} e_1[\infty] \\ \vdots \\ e_N[\infty] \end{bmatrix} = \underbrace{(I - A)^{-1}}_{N \times N} \begin{bmatrix} B_{\text{dist,1}} \\ \vdots \\ B_{\text{dist,N}} \end{bmatrix} u_{\text{dist}}[\infty],$$

such that I can model a disturbance in any state, scaled by u_{dist}.

The Dream Controller

With our steady-state disturbance response:

$$e[\infty] = (I - A)^{-1} B_{\text{dist}} u_{\text{dist}}[\infty],$$

our controller in part defines the matrix A.

Goal: design a STABLE controller with $e[\infty] = 0$ for any disturbance B_{dist}.

- Equivalently, we want $(I - A)^{-1} B_{\text{dist}} = 0$ for all B_{dist}.
- Stability makes this impossible!
- What can't be done? And what can?

Impossibility: Zero Offset for *Every* Disturbance Direction

Claim: If A is stable (all $|\lambda_i(A)| < 1$), it is *impossible* to design a controller such that

$$e[\infty] = (I - A)^{-1} B_{\text{dist}}\, u_{\text{dist}}[\infty] = 0 \quad \text{for every nonzero } B_{\text{dist}}.$$

Reasoning (invertibility / nullspace): For stable A, $I - A$ is invertible, hence so is $(I - A)^{-1}$. An invertible linear map has a trivial nullspace:

$$(I - A)^{-1} v = 0 \;\Rightarrow\; v = 0.$$

Thus $(I - A)^{-1} B_{\text{dist}} u_{\text{dist}}[\infty] = 0$ *for all* B_{dist} would force $B_{\text{dist}} u_{\text{dist}}[\infty] = 0$.

Eigen-Decomposition View

Assume A is diagonalizable: $A = V\Lambda V^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $|\lambda_i| < 1$. Then

$$(I - A)^{-1} = V\,(I - \Lambda)^{-1} V^{-1}, \qquad (I - \Lambda)^{-1} = \mathrm{diag}\big((1 - \lambda_i)^{-1}\big).$$

Key point: For stability, $1 - \lambda_i \neq 0$, so each $(1 - \lambda_i)^{-1}$ is finite and nonzero. Therefore $(I - A)^{-1}$ is invertible with no nontrivial nullspace; it cannot annihilate every disturbance direction.

Impossibility: Zero Offset for a Single State for *Every* Disturbance

Recall, for constant disturbances:

$$e[\infty] \;=\; (I - A)^{-1} B_{\text{dist}}\, u_{\text{dist}}[\infty].$$

Let $r_i^\top$ be the ith row of $(I - A)^{-1}$. Then the ith component is

$$e_i[\infty] \;=\; r_i^\top B_{\text{dist}}\, u_{\text{dist}}[\infty].$$

Suppose we demand $e_i[\infty] = 0$ for *every* disturbance direction B_{dist} (and any nonzero $u_{\text{dist}}[\infty]$). Then

$$r_i^\top v = 0 \quad \text{for all } v \in \mathbb{R}^n \;\;\Longrightarrow\;\; r_i = 0.$$

But a zero row in $(I - A)^{-1}$ implies $\operatorname{rank}\big((I - A)^{-1}\big) \le n - 1$, i.e. $(I - A)^{-1}$ is *singular*.

What *Is* Possible?

You *can* achieve $e[\infty] = 0$ for a **chosen class** of disturbances by embedding their internal model in the controller.

- In other words, we can augment our system with a new state which can inform the design of a new controller.

But no stable controller can guarantee $e[\infty] = 0$ for *every* disturbance direction B_{dist}.

- Instead, design a controller so that $e_i[\infty] = 0$ for some states and some B_{dist}'s.

Countering Steady-State Error

Disturbances cause a non-zero steady-state error. What can we do?
Introduce new state (accumulation of distance error):

$$q[n] = q[n-1] - \Delta T\,(d_d[n] - d[n]).$$

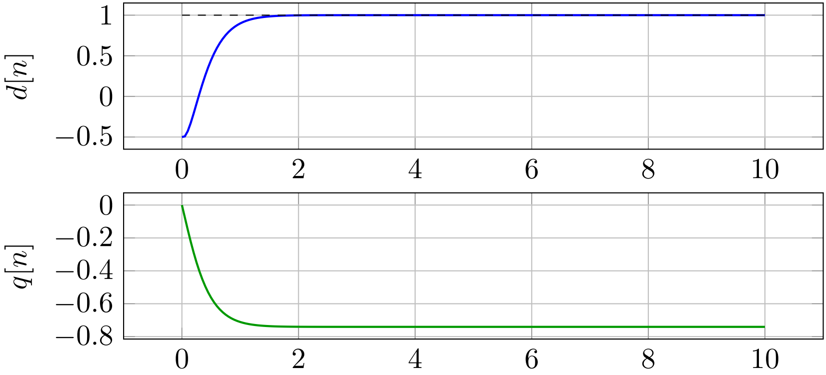
Full system now has three states:

$$\begin{aligned} d[n] &= d[n-1] + \Delta T\,\nu\,\theta[n-1], \\ \theta[n] &= \theta[n-1] + \Delta T\,\omega[n-1], \\ q[n] &= q[n-1] - \Delta T\,(d_d[n] - d[n]). \end{aligned}$$

What is $q[n]$?

Suppose we use $\omega[n] = K_p(d_d[n] - d[n]) - K_\theta \theta[n]$, and track,

$$q[n] = q[n-1] - \Delta T \left(d_d[n] - d[n]\right) :$$



$q[n]$ accumulates the distance error over time.

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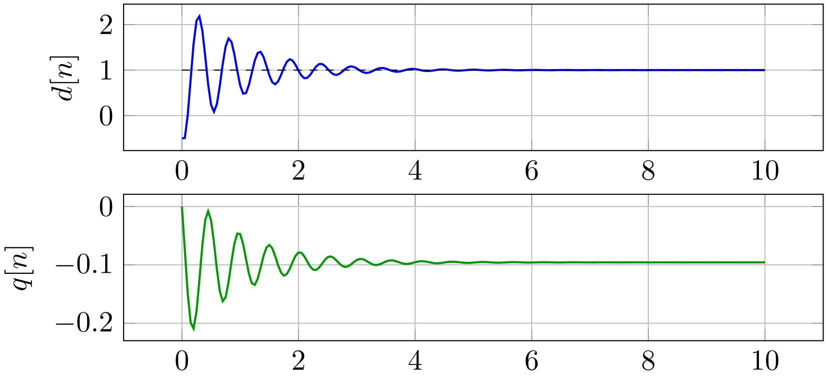
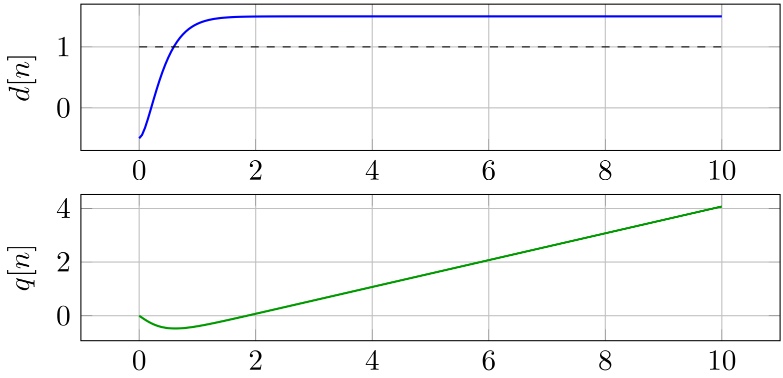


$q[n]$ accumulates the distance error over time.

What is $q[n]$? *With Disturbance.*

Suppose we use $\omega[n] = K_p(d_d[n] - d[n]) - K_\theta \theta[n]$, and track,

$$q[n] = q[n-1] - \Delta T \left(d_d[n] - d[n]\right):$$



$q[n]$ accumulates the distance error over time.

Adding Integral State

Full system now has three states:

$$\begin{aligned} d[n] &= d[n-1] + \Delta T\, \nu\, \theta[n-1], \\ \theta[n] &= \theta[n-1] + \Delta T\, \omega[n-1], \\ q[n] &= q[n-1] - \Delta T\, (d_d[n] - d[n]). \end{aligned}$$

New controller:

$$\omega[n] = K_p\, (d_d[n] - d[n]) - K_\theta\, \theta[n] - K_i\, q[n].$$

Interpretation: $q[n]$ accumulates (sums) the distance error, so including $-K_i q[n]$ ensures the controller reacts to long-term offsets and drives steady-state error to zero.

System Equations with Accumulator State

State vector (with accumulator state included):

$$x[n] = \begin{bmatrix} d[n] \\ \theta[n] \\ q[n] \end{bmatrix}.$$

System equations:

$$x[n] = \underbrace{\begin{bmatrix} 1 & \Delta T\,\nu & 0 \\ -\Delta T K_p & 1 - \Delta T K_\theta & -\Delta T K_i \\ \Delta T & 0 & 1 \end{bmatrix}}_{A} x[n-1] + \underbrace{\begin{bmatrix} 0 \\ \Delta T K_p \\ -\Delta T \end{bmatrix}}_{B} d_d[n-1].$$

Stability of A?

What are the eigenvalues of A?

$$\underbrace{\begin{bmatrix} 1 & \Delta T\,\nu & 0 \\ -\Delta T K_p & 1-\Delta T K_\theta & -\Delta T K_i \\ \Delta T & 0 & 1 \end{bmatrix}}_{A} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \Delta T \underbrace{\begin{bmatrix} 0 & \nu & 0 \\ -K_p & -K_\theta & -K_i \\ 1 & 0 & 0 \end{bmatrix}}_{M}$$

- We can use Spectral Mapping Theorem:

$$\text{evals}(A) = 1 - \Delta T \text{evals}(M).$$

- How do we pick K_p, K_θ, K_i? Let's use numerical tools.

Computing $(I - A)^{-1}$

Integral-augmented system matrix:

$$A = \begin{bmatrix} 1 & \Delta T\,\nu & 0 \\ -\Delta T K_p & 1 - \Delta T K_\theta & -\Delta T K_i \\ \Delta T & 0 & 1 \end{bmatrix}$$

$$\Rightarrow I - A = \Delta T \begin{bmatrix} 0 & -\nu & 0 \\ K_p & K_\theta & K_i \\ -1 & 0 & 0 \end{bmatrix}.$$

We can find a very simple inverse,

$$\boxed{(I - A)^{-1} = \frac{1}{\Delta T} \begin{bmatrix} 0 & 0 & -1 \\ -\frac{1}{\nu} & 0 & 0 \\ \frac{K_\theta}{K_i\,\nu} & \frac{1}{K_i} & \frac{K_p}{K_i} \end{bmatrix}.}$$

Steady-State Checks (Without Disturbance)

Input matrix for constant d_d:

$$(I - A)^{-1} = \frac{1}{\Delta T} \begin{bmatrix} 0 & 0 & -1 \\ -\frac{1}{\nu} & 0 & 0 \\ \frac{K_\theta}{K_i \, \nu} & \frac{1}{K_i} & \frac{K_p}{K_i} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \Delta T K_p \\ -\Delta T \end{bmatrix}, \; (I - A)^{-1} B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

For any constant desired distance $d_d[\infty]$,

$$\begin{bmatrix} d[\infty] \\ \theta[\infty] \\ q[\infty] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d_d[\infty]$$

Disturbance: Constant Lateral Drift ("Wind")

Additive offset in d: $d[n] = d[n-1] + \Delta T \nu \, \theta[n-1] + \underbrace{\Delta T(???)}_{\text{wind?}}$.

As a constant input: $B_{\text{dist}} = \begin{bmatrix} \Delta T \\ 0 \\ 0 \end{bmatrix}$.

Steady-state:

$$(I - A)^{-1} B_{\text{dist}} = \begin{bmatrix} 0 \\ -\dfrac{1}{\nu} \\ \dfrac{K_\theta}{K_i \, \nu} \end{bmatrix}.$$

Interpretation: Integral action drives the *distance* error to zero, but a constant drift induces a steady *heading* bias $\theta[\infty] = -1/\nu$, with $q[\infty] = (K_\theta/(K_i \, \nu))$.

Disturbance: Constant Bias in θ ("Cyclone")

Additive offset in θ: $\theta[n] = \theta[n-1] + \Delta T\,(\omega[n-1]) + \underbrace{\Delta T(???)}_{\text{cyclone?}}$.

As a constant input: $B_{\text{dist}} = \begin{bmatrix} 0 \\ \Delta T \\ 0 \end{bmatrix}$.

Steady-state:

$$(I - A)^{-1} B_{\text{dist}} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{K_i} \end{bmatrix}.$$

Interpretation: The integrator ramps to $q[\infty] = 1/K_i$ and cancels the bias; thus $d[\infty] = \theta[\infty] = 0$ despite the constant angle offset.