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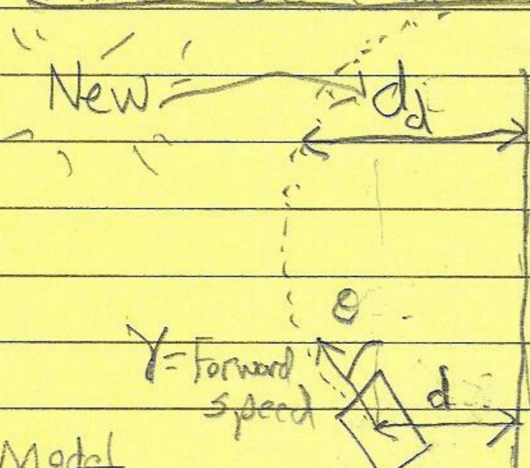
More General Path Following Robot



Model

$d[n] = d[n-1] + \Delta T \gamma \theta[n-1]$

$\theta[n] = \theta[n-1] + \Delta T \omega[n-1]$

Control

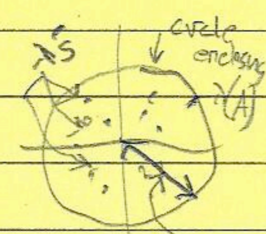
$\omega[n] = K_p(d_d[n] - d[n]) + K_\theta(-\theta[n])$

Matrix Form

$$\underbrace{\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix}}_{\vec{x}[n]} = \underbrace{\begin{bmatrix} 1 & \gamma \Delta T \\ -\Delta T K_p & 1-\Delta T K_\theta \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}}_{x[n-1]} + \underbrace{\begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix}}_{B} \underbrace{d_d[n-1]}_{u[n-1]}$$

Aside: Not general 2-D position control (would need $d_x[n]$, $d_y[n]$, and to control γ & ω)

Generically:



$$x[n] = A\,x[n-1] + B\,u[n-1]$$

Last Time

Stability (and oscillations) (Bounded input Bounded output) $\max_i |\underbrace{\lambda_i(A)}_{\text{eigenvalue}}| < 1$ ← spectral radius of A $sp(A)$

Why? $x[n] = \underbrace{A^n x[0]}_{} + \underbrace{\sum_{m=0}^{n} A^{n-m-1} B u[m]}_{}$

if $sp(A) < 1 \Rightarrow$ $\to 0$ as $n \to \infty$; is finite as $n \to \infty$ if $u[m]$ is bounded

Today - Disturbance Steady-State response

Steady-State

Assuming $sp(A) < 1$ & $\underbrace{u[n] = u[\infty]}_{\text{constant input}}$

what is $\lim_{n \to \infty} x[n]$?

$$\underbrace{\lim_{n \to \infty} x[n]}_{} = A \underbrace{\lim_{n \to \infty} [n-1]}_{} + B u[\infty]$$

$$x[\infty] \equiv A x[\infty] + B u[\infty]$$

or

$$(I - A)\, x[\infty] = B u[\infty]$$

or

$$x[\infty] = \underbrace{(I - A)^{-1}} B u[\infty]$$

Stability implies $(I-A)^{-1}$ exists   Assuming $I - A$ is invertible

~~Spectral Mapping theorem~~

for most functions f (e.g. those with taylor series)

$$\lambda(f(A)) = A(f(\lambda))$$

so $\lambda(I - A) = 1 - \lambda(A)$

$\Rightarrow$ Real(evals $(1 - \lambda(A))$) $> 0 \Rightarrow I-A$ is invertible

Back to Robot

$$A = \begin{bmatrix} 1 & \Delta T \gamma \\ -\Delta T \gamma K_p & 1 - \Delta T K_\theta \end{bmatrix}$$

$$\operatorname{eig}(A) \Rightarrow \lambda_{1,2} = 1 - \frac{\Delta T K_\theta}{2} \pm \Delta T \sqrt{\frac{K_\theta^2}{4} - K_p \gamma}$$

$|\lambda_i| < 1$ for "good" K_p, K_θ pairs

Assuming $d_d[n] = d_d[\infty]\ \forall n$

$$\begin{bmatrix} d[\infty] \\ \theta[\infty] \end{bmatrix} = \underbrace{\left(I - A\right)}^{-1} \underbrace{B} \, d_d[\infty]$$

$$= \begin{bmatrix} 0 & -\Delta T \gamma \\ \Delta T K_p & \Delta T K_\theta \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix} d_d[\infty]$$

$$= \underbrace{\left(\frac{1}{\Delta T} \begin{bmatrix} \frac{K_\theta}{\gamma K_p} & \frac{1}{K_p} \\ -\frac{1}{\gamma} & 0 \end{bmatrix} \right)}_{(I - A)^{-1}} \underbrace{\begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix}}_{B} d_d[\infty]$$

$$\overbrace{\begin{bmatrix} d[\infty] \\ \\ \theta[\infty] \end{bmatrix}}^{X[\infty]} = \overbrace{\begin{bmatrix} d_d[\infty] \\ \\ 0 \end{bmatrix}}^{(I-A)^{-1}Bu[\infty]}$$

$d[\infty] = d_d[\infty]$ ✓ $\theta[\infty] = 0$ ✓

<u>Perfect</u> for any K_p, K_θ

(as long as K_p, K_θ stable)

<u>But</u> Our Model does <u>not</u> include disturbances (Like the fan blowing on the encoder in Lab 1).

<u>No disturbance system</u>

(I) $$X[n] = A\,X[n-1] + B\,u[n-1]$$

<u>Add another input to model disturbance</u>

(II) $$x_{dist}[n] = A\,X_{dist}[n-1] + B\,u[n-1] + B_d\,u_d[n-1]$$

How disturbance effects equations

Disturber

<u>Subtract II - I</u>

change in x due to disturbance

$$x_{dist}[n] - X[n] \equiv e[n] = A\,e[n-1] + B_d u_d[n-1]$$

Steady-State Disturbance Response

Assume $\underbrace{u[n] = u[\infty], \quad u_d[n] = u_d[\infty]}_{\forall n}$

$$x_{dist}[\infty] - x[\infty] \equiv e[\infty]$$ ← steady-state disturbance response

$$e[\infty] = A\, e[\infty] + B_d\, u_d[\infty]$$

or

$$e[\infty] = (I - A)^{-1} B_d\, u_d[\infty]$$

For the path-following robot

$$\begin{bmatrix} d_{dist}[\infty] - d[\infty] \\ \theta_{dist}[\infty] - \theta[\infty] \end{bmatrix} = \underbrace{\frac{1}{\Delta T}\begin{pmatrix} \frac{K_\theta}{V K_p} & \frac{1}{K_p} \\ -1/V & 0 \end{pmatrix}}_{(I-A)^{-1}} \underbrace{\begin{bmatrix} B_{d_1} \\ B_{d_2} \end{bmatrix}}_{\text{impact on distance \& angle equation}} \overbrace{u_d[\infty]}^{\text{disturbance scale factor}}$$

Suppose wind blows robot

$$d_{dist}[n] = d_{dist}[n-1] + \Delta T\, V \theta[n-1] + \Delta T \underbrace{Wind_{vel}[n-1]}_{\text{wind velocity}}$$

$$\theta_{dist}[n] = \underbrace{\theta_{dist}[n-1] + \Delta T\, \omega[n-1]}_{\text{wind does not change angle eqn}}$$

↑ robot distance including disturbances

Wind Blowing Disturbance B_d

$$B_d = \begin{bmatrix} \Delta T \\ 0 \end{bmatrix}$$

only affects distance equation

$$\begin{bmatrix} d_{dist}[\infty] - d[\infty] \\ \theta_{dist}[\infty] - \theta[\infty] \end{bmatrix} = \frac{1}{\Delta T} \begin{pmatrix} \frac{K_\theta}{\gamma K_p} & \frac{1}{K_p} \\ -\frac{1}{\gamma} & 0 \end{pmatrix} \begin{bmatrix} \Delta T \\ 0 \end{bmatrix} u_d[\infty]$$

scales the Wind

$$= \begin{bmatrix} \frac{K_\theta}{\gamma K_p} u_d[\infty] \\ -\frac{u_d[\infty]}{\gamma} \end{bmatrix}$$

As $K_p \to \infty$ $d_{dist}[\infty] - d[\infty] \to 0$

Bigger K_p, Smaller Distance Disturbance Response!

But

Robot angle Disturbance response is <u>unaffected</u> by K_p