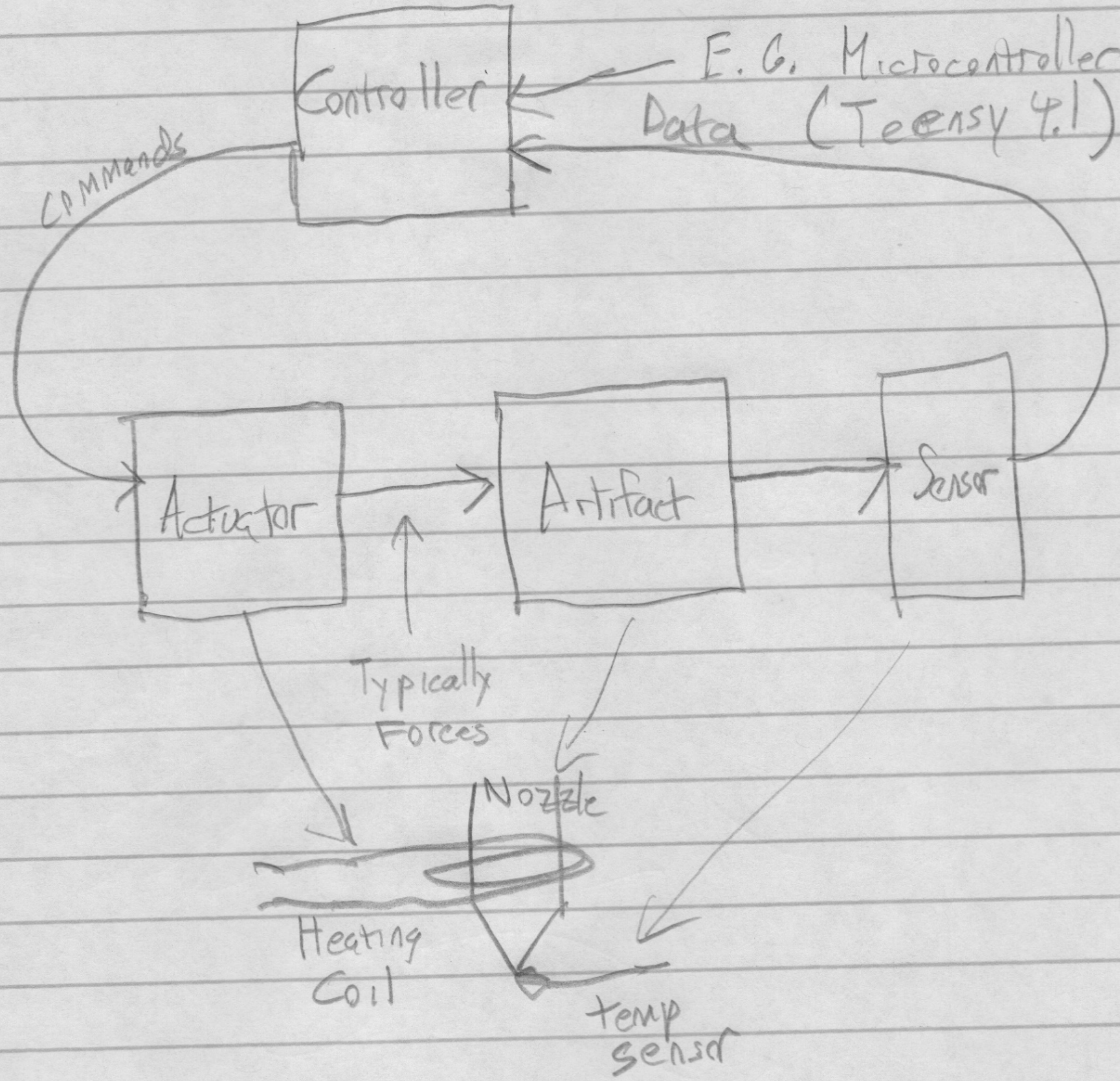
6.3100/2 9/14/26

Generic Control System



Diff by orders of magnitude

Clocked Systems

Every ΔT:

Read Sensor
Compute Command
Update the actuator

We Assume
Read
Calculation
Update
in an instant!

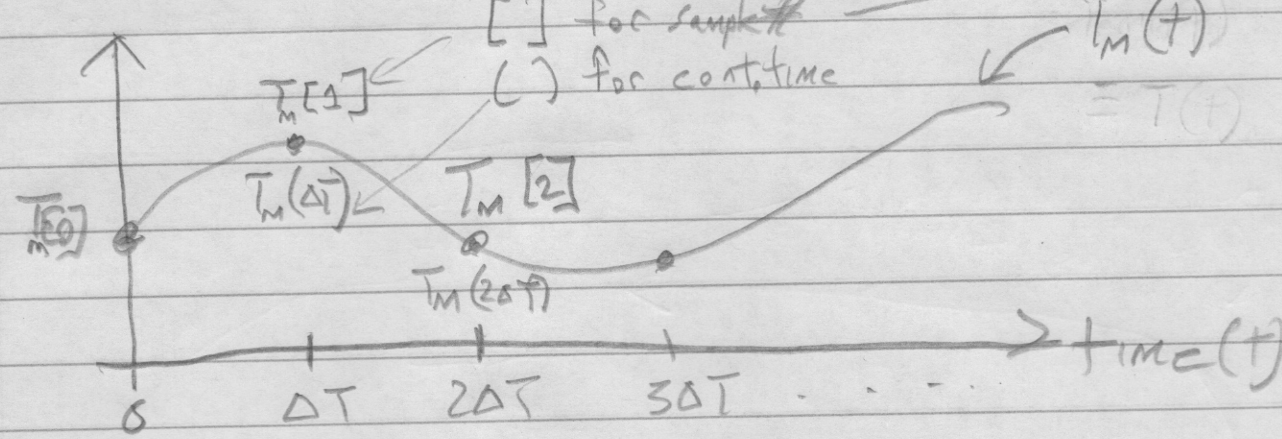
On a Teensy 4.1
Read → Calc → Update
few μsec's

ΔT for physical Device
1000x a second (every millisec)

Nozzle Temp Sensing
(Sampling)

$T_M \equiv$ Measured Temp

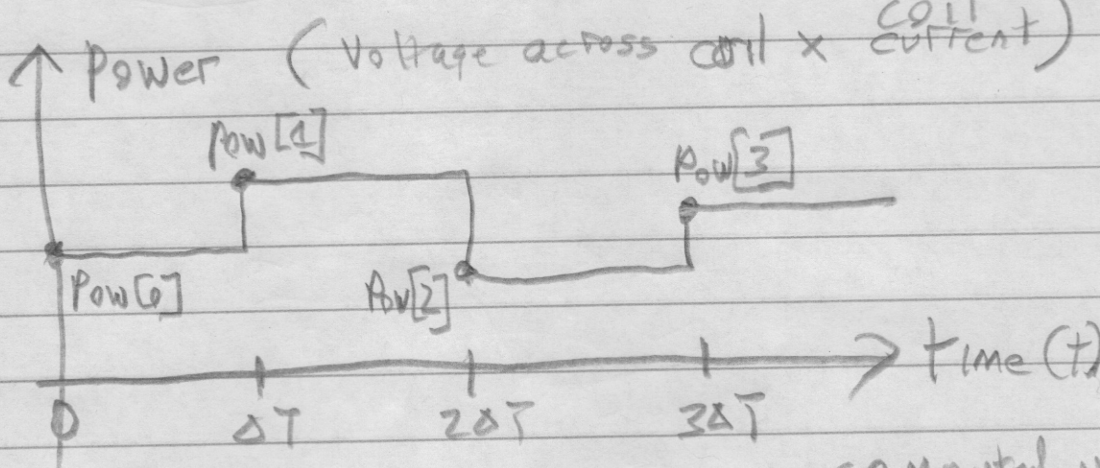
Sampling a continuous function of time



Controller gets $T_m[1], T_M[2], \ldots T_M[n], \ldots$

Nozzle Heater (actuator)

Piecewise Constant Updating (Zero-Order Hold)



computed update

Control : Proportional Control

Same index "Instant Update"

$$\underset{\text{Updated power}}{Pow[n]} = \underset{\text{proportional Gain}}{K_p} \left(\underbrace{T_d[n]}_{\text{Desired Temp}} - \underset{\text{read sensor}}{T_m[n]} \right)$$

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Model of Nozzle Heating

$$T_m[n] = T_m[n-1] + \Delta T \overset{\text{summarizes the physics}}{\gamma_{th}} \underbrace{Pow[n-1]}_{\text{Coil Heat}}$$

$\uparrow$ nozzle temp at $n\Delta T$

$\uparrow$ nozzle temp at $(n-1)\Delta T$

or

$$\frac{dT_m}{dt} \simeq \frac{T_m[n] - T_m[n-1]}{\Delta T} = \gamma_{th} Pow[n-1]$$

Combining Model & Control

$$T_m[n] = T_m[n-1] + \Delta T \gamma_{th} \underbrace{K_p\left(T_d[n-1] - T_m[n-1]\right)}_{Pow[n-1]}$$

$$T_m[n] = \left(1 - \Delta T \gamma_{th} K_p\right) T_m[n-1] + \Delta T \gamma_{th} K_p T_d[n-1]$$

Linear 1st Order Difference Eqn

General 1^{st}-Order Linear Diff Eqn

"Input" (Usually known): $u[0], u[1], \ldots$

"Output" (to be computed): $y[0], y[1]$

initial condition

LFODE

$$\forall n \qquad y[n] = \lambda y[n-1] + \gamma u[n-1]$$

$n=1 \qquad y[1] = \lambda y[0] + \gamma u[0]$

$n=2 \qquad y[2] = \lambda y[1] + \gamma u[1] = \lambda(\lambda y[0] + \gamma u[0]) + \gamma u[1]$

$$= \lambda^2 y[0] + \lambda \gamma u[0] + \gamma u[1]$$

$$\vdots$$

For arb n $\qquad y[n] = \lambda^n y[0] + \gamma \sum_{m=0}^{n} \lambda^{(n-m-1)} u[m]$

If $u[n]=0 \; \forall n$ (ZIR): $\qquad y[n] = \lambda^n y[0]$

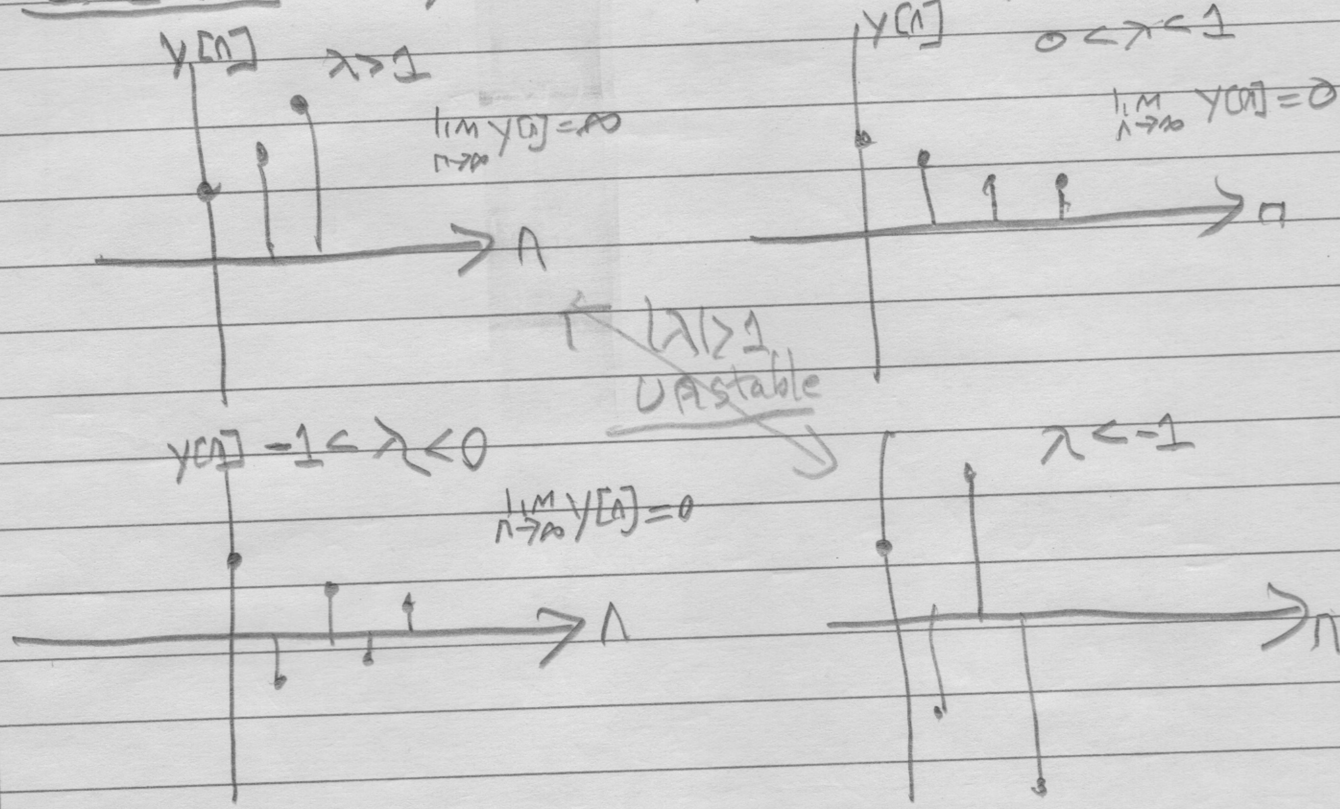
zero input response

If $y[0]=0$ (ZSR): $\qquad y[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m-1)} u[m]$

state

ZIR

$$y[n] = \lambda y[n-1] = \lambda^n y[0]$$



Steady-state response

if $u[n] = u_0 \;\forall n$ (Constant input)

&

$|\lambda| < 1$

then

$$\underbrace{\lim_{n\to\infty} y[n]}_{y[\infty]} = \lim_{n\to\infty} \lambda^n y[0] + \lim_{n\to\infty} \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u_0$$

$\to 0$; sum

$$y[\infty] = \frac{\gamma}{1-\lambda} u_0$$

Easier if $y[\infty]$ exists then

$$y[n] = \lambda y[n-1] + \gamma u_0$$

$$y[\infty] = \lambda y[\infty] + \gamma u_0$$

$$y[\infty] = \frac{\gamma}{1-\lambda} u_0$$

For Nozzle Example

$$\lambda = (1 - \Delta T \gamma_{th} K_p) \qquad \gamma = \Delta T K_p \gamma_{th}$$

So $\underbrace{|\lambda| < 1}_{\text{stable}}$ iff $0 < \Delta T \gamma_{th} K_p < 2$

$\Delta T > 0$
$\gamma_{th} > 0$
$K_p > 0$

$$\Rightarrow \text{Max } K_p = \frac{2}{\Delta T \gamma_{th}}$$

And if $\quad 0 < \Delta T \gamma_{th} K_p < 1$

$$0 < \lambda < 1 \longleftarrow \text{No oscillations}$$

Max K_p for No oscillations $\leftarrow$ $\boxed{\text{Quiz Question Example}}$

If $\begin{cases} Td[n] = Td_0 \ \forall n \\ |\lambda| < 1 \Rightarrow \text{System Stable} \Rightarrow \lim_{n\to\infty} T_m[n] \text{ exists} \end{cases}$

$$T_m[\infty] = \frac{\gamma}{1-\lambda} Td_0$$

$$= \frac{\Delta T \gamma_{th} K_p}{1 - (1 - \Delta T \gamma_{th} K_p)} Td_0$$

$$= 1 \cdot Td_0 = Td_0$$

Wait!

$K_p(T[\infty] - Td_0) = 0$
$= pow[\infty]$

Huh?
can we really turn the heater off when we get to steady-state?

Steady State Temp Matches Desired Temp for any $0 < K_p < \frac{2}{\Delta T \gamma_{th}}$

The Math is correct, the Model is Not!

Include Heat Loss

$$T_M[n] = T_M[n-1] + \Delta T\, \gamma_{th}\, \overbrace{Pow[n-1]}^{K_p(T_d[n-1]-T_M[n-1])}$$

$$- \underbrace{\Delta T\, \beta\, T_M[n-1]}_{\text{Heat Loss}}$$

$$T_M[n] = \overbrace{\left(1-\Delta T(\gamma_{th}K_p+\beta)\right)}^{\lambda}\, T_M[n-1]$$

$$+ \underbrace{\Delta T\, \gamma_{th} K_p}_{\gamma}\, T_d[n-1]$$

If $\lambda < 1$ & $T_d[n] = T_{d0}$

$$T_M[\infty] = \frac{\overbrace{\Delta T\, \gamma_{th} K_p}^{\gamma}}{1-(1-\Delta T\,\gamma_{th}K_p+\beta)}\, T_{d0}$$

$$= \frac{\cancel{\Delta T}\,\gamma_{th}K_p}{\cancel{\Delta T}\,\gamma_{th}K_p+\beta}\, T_{d0}$$

$$= \frac{1}{1+\frac{\beta}{\gamma_{th}K_p}}\, T_{d0}$$

$$\simeq 1 \quad \text{if} \quad K_p\gamma_{th} \gg \beta$$

Bigger K_p : Better Match

Until $(\gamma_{th}K_p+\beta) > 2/\Delta T$

As $\Delta T \to 0$, $K_p \to \infty$ and $T_M[\infty] - T_{d0} \to 0$!