

6.310: Oct. 7, 2026

FREQUENCY RESPONSE OF A CT LTI MODEL
(AND ITS MAGNITUDE/PHASE PARTS)

LAST LECTURE: SWITCH TO CT MODELING (FOR ANALYSIS/DESIGN)



|| SINCE "ω" WILL BE USED FOR "CT FREQUENCY"
USE "U" FOR "θ" TODAY

DT PICTURE

$$\begin{cases} d[n+1] = d[n] + \Delta_T V\theta[n] \\ \theta[n+1] = \theta[n] + \Delta_T U[n] \end{cases}$$

"ORIGINAL PLANT EQUATIONS"

$$\begin{cases} \frac{d[n+1] - d[n]}{\Delta_T} = V\theta[n] \\ \frac{\theta[n+1] - \theta[n]}{\Delta_T} = U[n] \end{cases}$$

CT PICTURE

$$\begin{cases} \dot{d}(t) = V\theta(t) \\ \dot{\theta}(t) = U(t) \end{cases}$$

"desired" $d[n]$

$$x_i[n+1] = x_i[n] + \Delta_T (r[n] - d[n]) \quad \text{OR} \quad \frac{x_i[n+1] - x_i[n]}{\Delta_T} = r[n] - d[n]$$

"ARTIFICIAL" INTEGRAL STATE

$$\dot{x}_i(t) = r(t) - d(t)$$

"PID CONTROL"

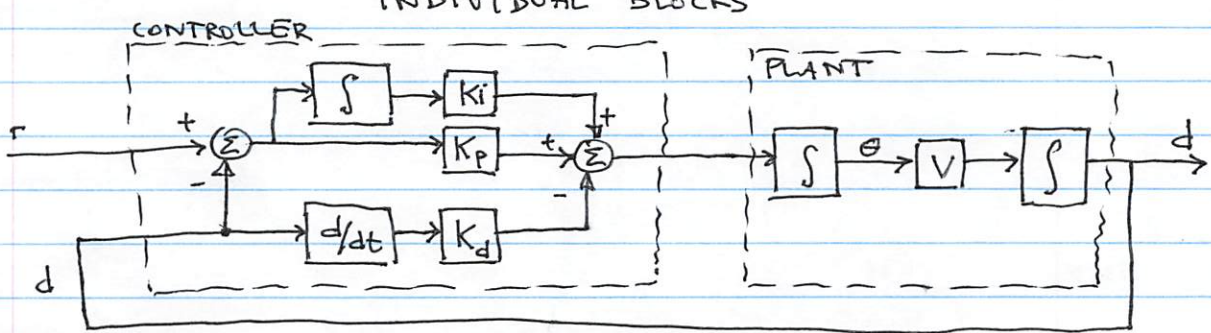
$$U[n] = K_p (r[n] - d[n]) - K_d \frac{d[n] - d[n-1]}{\Delta_T} + k_i x_i[n]$$

$$U(t) = K_p (r(t) - d(t)) - K_d \dot{d}(t) + k_i x_i(t)$$

CT AND DT VIEWS AGREE VERY WELL AS LONG AS $|\Delta_T \cdot \text{ANY COEFFICIENT}| \ll 1$

FREQUENCY RESPONSE

IS CONVENIENT FOR ANALYSING
COMBINATIONS OF LTI BLOCKS
BASED ON INPUT/OUTPUT INFO ABOUT
INDIVIDUAL BLOCKS



- | | | |
|----------------|--|---|
| <u>BLOCKS:</u> | CONSTANT GAIN
DIFFERENTIATION
INTEGRATION | } SINGLE INPUT / SINGLE OUTPUT |
| | SPLIT
SUM
FEEDBACK CONNECTION
SERIES CONNECTION | |
| | | } MIMO:
MULTI-INPUT / MULTI-OUTPUT
COMBINATIONS OF BLOCKS |

G.310: Oct. 7, 2026

EXPECTED FROM AN LTI SYSTEM "S":

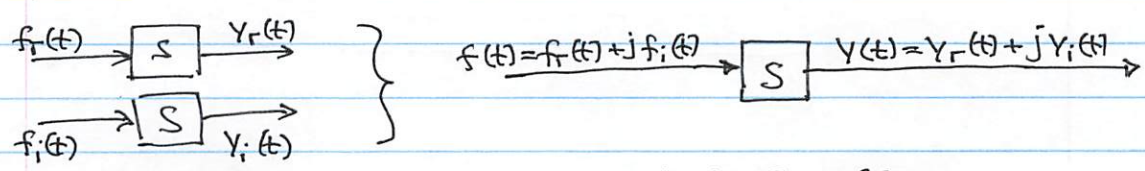


$\frac{A_{out}}{A}$ AND $\phi_{out} - \phi$
DEPEND ON ω ONLY!

USUALLY: FOR EVERY GIVEN SINUSOIDAL INPUT THERE EXISTS A UNIQUE INITIAL STATE THAT PRODUCES A SINUSOIDAL OUTPUT, AS ABOVE

SOMETIMES: ZERO INPUT WITH CERTAIN INITIAL CONDITIONS PRODUCES A SINUSOIDAL OUTPUT

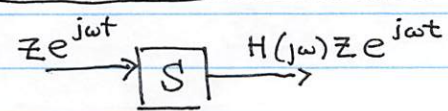
FOR CONVENIENCE: ALLOW LTI ^{MODELS} SYSTEMS TO ACCEPT COMPLEX INPUTS



? WHAT'S THE RESPONSE OF S TO $\overbrace{(a+jb)(f_r(t)+j f_i(t))}^{\text{CONSTANT } f(t)} = \tilde{y}(t)$?

! SINCE $(a+jb)(f_r+jf_i) = (af_r-bf_i) + j(af_i+bf_r)$, BY LINEARITY, THE RESPONSE IS
 $\tilde{y}(t) = (ay_r-by_i) + j(ay_i+bf_r) = (a+jb)f(t)$

LTI RESPONSE TO SINUSOIDS (IN COMPLEX EXPONENT NOTATION):

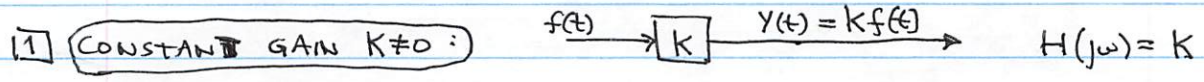


$$|H(j\omega)| = \frac{A_{out}}{A}$$

$$\angle H(j\omega) = \phi_{out} - \phi$$

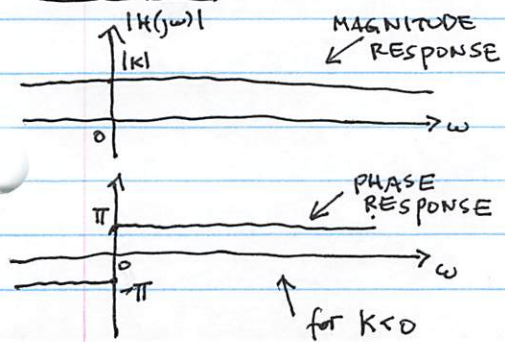
$H(j\omega)$ IS CALLED THE FREQUENCY RESPONSE OF S

A ZOO OF FREQUENCY RESPONSES

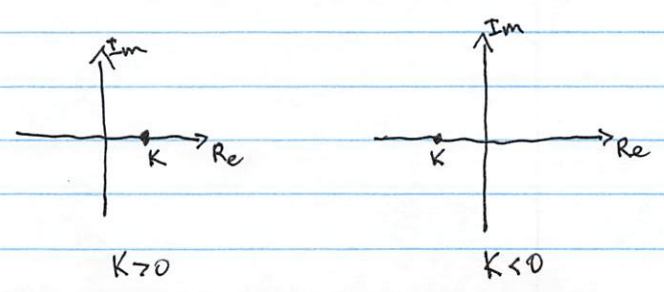


$|H(j\omega)| = |K|$, $\angle H(j\omega) = \angle K \in \{0, \pi, -\pi\}$

Bode plot:



Nyquist plot:



THE TRAJECTORY OF $H(j\omega)$
AS ω GROWS FROM $-\infty$ TO $+\infty$

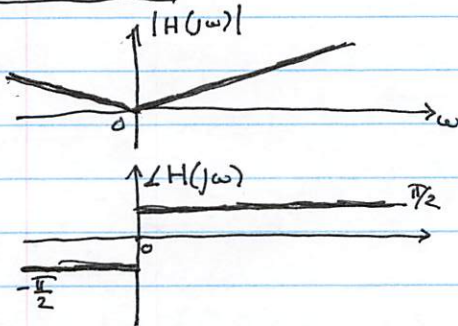
A ZOO OF FREQUENCY RESPONSES (CONTINUED)

2 DIFFERENTIATION: $f(t) \rightarrow \left[\frac{d}{dt} \right] \rightarrow y(t) = \dot{f}(t)$

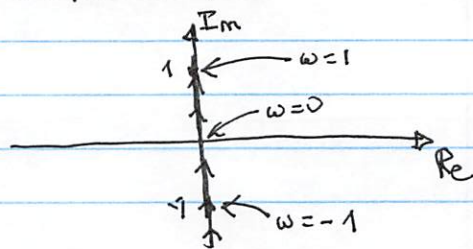
$$f(t) = Ze^{j\omega t} \rightarrow y(t) = j\omega Ze^{j\omega t}, \text{ i.e., } H(j\omega) = j\omega$$

$$|H(j\omega)| = |\omega|, \quad \angle H(j\omega) = \begin{cases} \pi/2, & \omega > 0 \\ -\pi/2, & \omega < 0 \\ \text{ANYTH,} & \omega = 0 \end{cases}$$

Bode plot:



Nyquist plot:



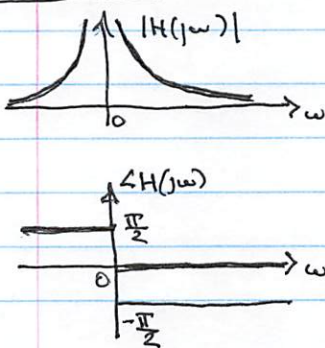
3 INTEGRATION:

$$f(t) \rightarrow \left[\int \right] \rightarrow y(t) = y_0 + \int_0^t f(\tau) d\tau$$

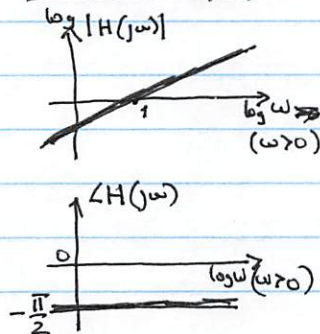
↑
INITIAL CONDITION

$$f(t) = Ze^{j\omega t} \rightarrow y(t) = \begin{cases} \frac{1}{j\omega} Ze^{j\omega t}, & \text{WHEN } \omega \neq 0 \\ y_0 e^{j\omega t}, & \text{WHEN } \omega = 0 \text{ AND } Z = 0 \end{cases}$$

Bode plot:



Log-log Bode plot



BECAUSE $\angle H$ IS ALREADY A LOGARITHM!
 $\log(re^{j\phi}) = \log(r) + j\phi$
 ANGLE

4 SERIES CONNECTION:

$$f \rightarrow \left[H_1(j\omega) \right] \rightarrow \left[H_2(j\omega) \right] \rightarrow y$$

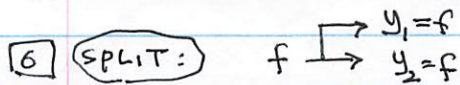
$$H(j\omega) = H_2(j\omega) H_1(j\omega)$$

5 PARALLEL CONNECTION:

$$f \rightarrow \left[H_1(j\omega) \right] \rightarrow \oplus \rightarrow y$$

$$\left[H_2(j\omega) \right] \rightarrow \oplus$$

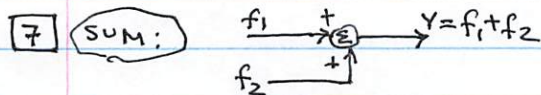
$$H(j\omega) = H_1(j\omega) + H_2(j\omega)$$

A ZOO OF FREQUENCY RESPONSES (CONTINUED)

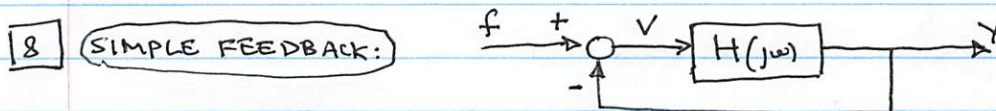
2x1 MATRIX (1 INPUT, 2 OUTPUTS)

$$f(t) = z e^{j\omega t} \rightarrow Y(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} z e^{j\omega t}$$

$$H(j\omega) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$\begin{cases} f_1(t) = z_1 e^{j\omega t} \\ f_2(t) = z_2 e^{j\omega t} \end{cases}, \text{ i.e., } f(t) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} e^{j\omega t} \rightarrow Y(t) = (z_1 + z_2) e^{j\omega t} = [1, 1] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} e^{j\omega t}$$



if $f = F e^{j\omega t}$, $V = V e^{j\omega t}$, $Y = Y e^{j\omega t}$ THEN

$$\begin{cases} Y = H(j\omega) V \\ V = F - Y \end{cases} \left\{ \begin{array}{l} \text{solve} \\ \text{for } Y, \\ \text{given } F \end{array} \right.$$

$$V + H(j\omega) V = F$$

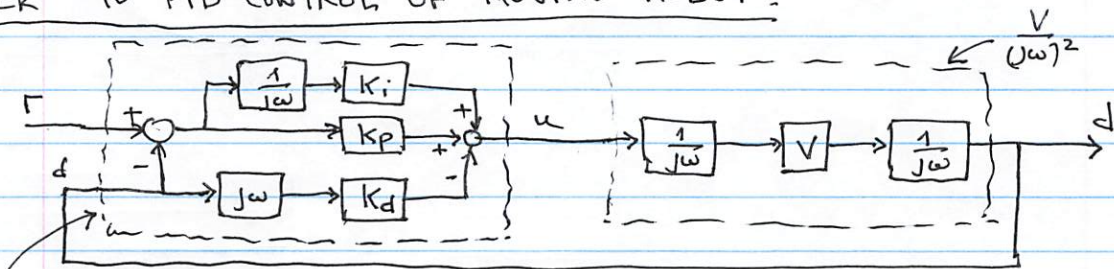
$$\downarrow (1 + H(j\omega)) V = F$$

$$V = [1 + H(j\omega)]^{-1} F \Rightarrow Y = \frac{H(j\omega)}{1 + H(j\omega)} F$$

"BLACK'S FORMULA":

$$\text{CLOSED-LOOP-GAIN} = \frac{\text{FORWARD-GAIN}}{1 - \text{TOTAL-LOOP-GAIN}} = \frac{H(j\omega)}{1 - (-1) H(j\omega)}$$

BACK TO PID CONTROL OF MOVING ROBOT:



$$\left[\frac{K_i}{j\omega} + K_p, -\frac{K_i}{j\omega} - K_p - j\omega K_d \right]$$

$$H_{r \rightarrow d}(j\omega) = \frac{(K_p + \frac{K_i}{j\omega}) \frac{V}{(j\omega)^2}}{1 + (K_p + \frac{K_i}{j\omega} + j\omega K_d) \frac{V}{(j\omega)^2}} = \frac{(j\omega K_p + K_i) V}{(j\omega)^3 + K_p V (j\omega)^2 + K_d V (j\omega) + K_i V}$$

E.g.,

ASSUMING STABILITY, WHAT IS THE "DC GAIN"?

$$! H_{r \rightarrow d}(0) = 1$$

How to ASSESS STABILITY IN CT?

① STATE SPACE MODELING STILL AVAILABLE:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

e.g. FOR $\begin{cases} \dot{d} = v(t) \\ \dot{\theta} = u(t) \end{cases}$ USING $x(t) = \begin{bmatrix} d(t) \\ \theta(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

WITH EXTRA STATE x_i : $\tilde{x}_i(t) = \theta(t) - d(t)$, $x(t) = \begin{bmatrix} d(t) \\ \theta(t) \\ x_i(t) \end{bmatrix}$

$$\dot{x}(t) = \begin{bmatrix} 0 & v & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

WITH PID FEEDBACK $u(t) = K_p(r(t) - d(t)) - K_d \dot{d}(t) + K_i x_i(t) =$

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & v & 0 \\ -K_p & -K_d v & K_i \\ -1 & 0 & 0 \end{bmatrix}}_{A_{cl}} x(t) + \underbrace{\begin{bmatrix} 0 \\ K_p \\ 1 \end{bmatrix}}_{B_{cl}} r(t)$$

NEED ALL EIGS OF A_{cl} TO HAVE NEGATIVE REAL PART ($\text{Re}(\lambda_k) < 0$)

BECAUSE SOLUTIONS OF $\dot{x}(t) = A_{cl} x(t)$ ARE SUMS OF EXPONENTS $e^{\lambda_k t}$ WITH SOME COEFFICIENTS (ADD $t^m e^{\lambda_k t}$ WITH $m < \text{mult}(\lambda_k)$ WHEN MULTIPLICITY $\text{MULT}(\lambda_k)$ OF λ_k IS LARGER THAN 1)

② INTRODUCE $s = j\omega$, AND LOOK AT $H(s)$ FOR s WITH $\text{Re}(s) \geq 0$

IN THE EXAMPLE: $H_{r \rightarrow d}(s) = \frac{(K_i + sK_p)V}{s^3 + K_d V s^2 + K_p V s + K_i V}$

REQUIRE FOR STABILITY: ALL ROOTS OF DENOMINATOR

$$q(s) = s^3 + K_d V s^2 + K_p V s + K_i V \text{ TO HAVE } \text{Re}(\lambda_k) < 0$$

!! NOT ACCIDENTALLY: $\det(sI - \begin{bmatrix} 0 & v & 0 \\ -K_p & -K_d v & K_i \\ -1 & 0 & 0 \end{bmatrix}) = s^3 + K_d V s^2 + K_p V s + K_i V$

USEFUL: ROUTH-HURWITZ CRITERION (FOR 3RD ORDER POLY):

$$q(s) = s^3 + a_2 s^2 + a_1 s + a_0 \text{ HAS ALL ROOTS } \lambda_k \text{ WITH } \text{Re}(\lambda_k) < 0$$

IF AND ONLY IF $a_i > 0$ AND $a_2 a_1 > a_0$

IN OUR CASE: $K_d V K_p V > K_i V$, i.e., $\boxed{K_d K_p V > K_i}$ CRITERION OF STABILITY

QUESTION LTI SYS. RESPONSE TO $f(t) \equiv \cos^2(t)$ IS $y(t) \equiv 1$ WHAT IS $H(0)$? WHAT IS $H(2j)$?ANSWER SINCE $\cos^2(t) = \frac{1}{2} + \frac{1}{2} \cos(2t)$, WE HAVE $H(0) = 2$, $H(2j) = 0$