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More on LQR

LQR: Given $\begin{cases} A, B \text{ in } \dot{x}(t) = Ax(t) + Bu(t) \\ Q, R, N \text{ in } \sigma(x,u) = x^T Q x + u^T R u + 2x^T N u \end{cases}$

Find $K = K_{opt}$ such that

(a) control feedback $u(t) = -Kx(t)$ stabilizes $\dot{x}(t) = Ax(t) + Bu(t)$
(i.e., $\max(\text{real}(\text{eig}(A-BK))) < 0$)

(b) the integral $J(K, x_0) = \int_0^\infty \sigma(x(t), u(t))\, dt$, subject to $\dot{x}(t) = Ax(t) + Bu(t)$, $u(t) = -Kx(t)$, $x(0) = x_0$ is minimal at $K = K_{opt}$
(i.e., $J(K_{opt}, x_0) \le J(K, x_0)$ for all x_0, and all K: $\max(\text{real}(\text{eig}(A-BK))) < 0$)

In "real life": * $\dot{x}(t) = Ax(t) + Bu(t)$ models hardware/laws of nature

* K determines a linear feedback
  - ☐ formally: $u(t) = -Kx(t)$
  - ☐ practically: $u(t) = k_r Y_d(t) - Kx(t)$ (k_r designed separately)
  - ☐ ultimately: $u(t) = k_r Y_d(t) - K\hat{x}(t)$ ← estimated $x(t)$, using whatever info is available

Do you see how $\dot{x}(t) = Ax(t) + Bu(t)$, $u(t) = k_r Y_d(t) - Kx(t)$ leads to "$A-BK$"?

!!! $\dot{x}(t) = Ax(t) + B(k_r Y_d(t) - Kx(t)) = (A-BK)x(t) + Bk_r Y_d(t)$

Calculating k_r (to assure $Y_\infty = Y_{d,\infty}$)

* Assume $Y(t)$ [controled output] is defined by $Y(t) = Cx(t)$
* When $Y_d(t) = Y_{d,\infty} = \text{const}$, the limit $x_\infty = \lim_{t\to\infty} x(t)$ is given by ← known matrix

  $0 = (A-BK)x_\infty + Bk_r Y_{d,\infty}$, i.e., $x_\infty = -(A-BK)^{-1} B k_r Y_{d,\infty}$
* Hence $Y_\infty = Cx_\infty = -C(A-BK)^{-1} B k_r Y_{d,\infty}$, i.e., need $k_r = -\dfrac{1}{C(A-BK)^{-1}B}$
* This works whenever $P_{u2y}(0) \neq 0$, where $P_{u2y}(s) = C(sI-A)^{-1}B$ is the transfer function (open loop) $u \to y$

The main issue/challenge when using LQR: how to select "σ"

"Silly" costs not allowed, i.e.,

* $\sigma(x,u) = -u^2$
* $\sigma(x,u) = |x|^2$
* $\sigma(x,u) = u^2$ when $\dot{x} = u$

} No optimal K, for obvious reasons (common sense)

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TYPICAL WAY OF DEFINING LQR COST "σ":

$$\sigma(x,u) = |e|^2 = \left\| \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix} \right\|^2 = e_1^2 + e_2^2 + \ldots + e_p^2 \quad , \quad e = C_e x + D_e u$$

$$|e|^2 = (C_e x + D_e u)^T (C_e x + D_e u) = x^T C_e^T C_e x + u^T D_e^T D_e u + 2 x^T C_e^T D_e u$$

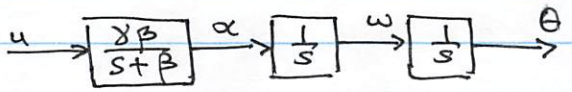
i.e, e_i's ARE SOME LINEAR COMBINATIONS OF STATE AND CONTROL COMPONENTS

$Q = C_e^T C_e$, $R = D_e^T D_e$, $N = C_e^T D_e$

MOST COMMON (EVEN IF SIMPLISTIC):

$$e = \begin{bmatrix} y \\ \rho u \end{bmatrix} = \begin{bmatrix} Cx \\ \rho u \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ 0 \end{bmatrix}}_{C_e} x + \underbrace{\begin{bmatrix} 0 \\ \rho \end{bmatrix}}_{D_e} u \qquad (\text{MOST HAVE } \rho \neq 0)$$

Ex. (PROP MODEL):



USE $\gamma = 50$, $\beta = 25$

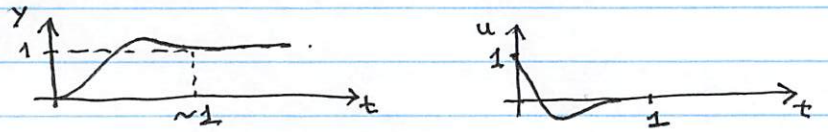
$$x = \begin{bmatrix} \theta \\ \omega \\ \alpha \end{bmatrix}, \quad \dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\beta \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \gamma\beta \end{bmatrix}}_{B} u(t), \quad y(t) = \underbrace{[1, 0, 0]}_{C} x(t)$$

WITH $K = [K_\theta, K_\omega, K_\alpha]$, TO HAVE $x_\infty = \begin{bmatrix} y_{d,\infty} \\ 0 \\ 0 \end{bmatrix}$, NEED $\overbrace{K_y - K x_\infty}^{u_\infty} = 0$, i.e., $y_{d,\infty}$

$$\boxed{K_r = K_\theta}$$

① TRYING LQR WITH $\rho = 1$, $e = \begin{bmatrix} y \\ \rho u \end{bmatrix}$, i.e., $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $R = 1$, $N = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

WILL GET CLOSED LOOP STEP RESPONSES ($Y_d \to Y$, $Y_d \to u$) LIKE

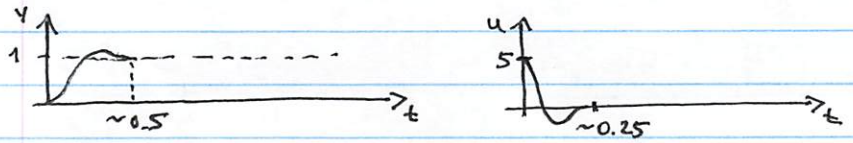


IS THIS "FAST ENOUGH"? IS THIS "SMOOTH ENOUGH"? IT DEPENDS!

ASSUME: *CONTROL EFFORT MUST BE BELOW $u_{max} = 5$
*STEPS BY MORE THEN 1 UNIT WILL NOT BE REQUIRED

THEN WE ARE UNDERUTILIZING CONTROL AUTHORITY!
HOW TO FIX? TO "ALLOW" 5x CONTROL EFFORT, REDUCE ρ BY 5x!

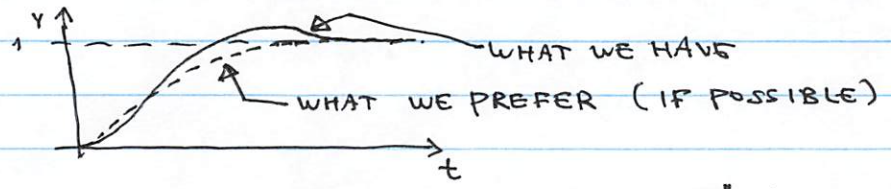
② TRYING LQR WITH $\rho = 0.2$, $e = \begin{bmatrix} y \\ \rho u \end{bmatrix}$, i.e., $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $R = 0.04$, $N = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$



THE OVERSHOOT IS ABOUT 6%. WHAT IF THIS IS TOO MUCH?

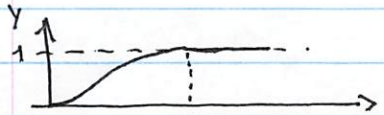
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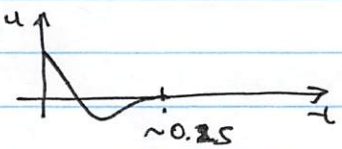
ONE USUALLY PREFERS THE "EXPONENTIAL DECAY" RESPONE :



HOW ABOUT REPLACING "y" BY "$y+T\dot{y}$" IS THE DFN. OF "e"?

(3) TRYING LQR WITH $\rho=0.2$, $T=0.06$, $e=\begin{bmatrix} y+0.06\dot{y} \\ \rho u \end{bmatrix}$, i.e., $Q=\begin{bmatrix} 1 & T & 0 \\ T & T^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $R=0.04$, $N=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$





Note: since $\dot{x}_1 = x_2$ IN THIS MODEL $e=\begin{bmatrix} y+T\dot{y} \\ \rho u \end{bmatrix}$ is SAME AS $e=\begin{bmatrix} y \\ T\dot{y} \\ \rho u \end{bmatrix}$

AS $\int_0^\infty 2x_1x_2\,dt = \int_0^\infty \frac{d}{dt}[x_1^2]\,dt = x_1^2\Big|_0^\infty = -x_1(0)^2$ DOES NOT DEPEND ON "K"

DOES LQR HAVE INTERPRETATION IN TERMS OF FREQUENCY RESPONSES?

SURE IT DOES!

LET $g=g(t)$ BE THE ZIR OF $\dot{x}=Ax+Bu$, $x(0)=B_0$, $u=-Kx$, $g=C_gx+D_gu$

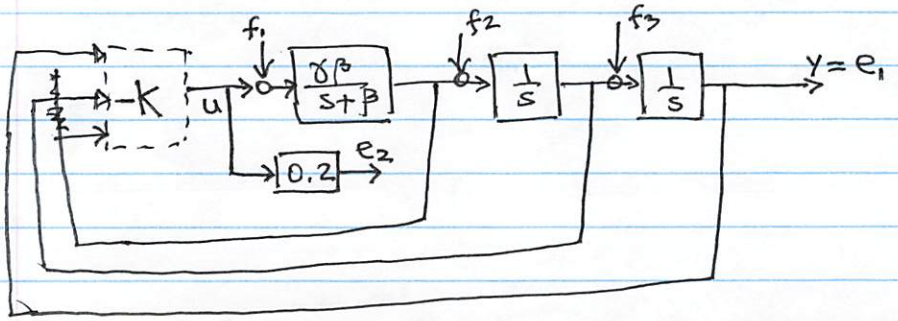
i.e., $g(t)=(C_g-D_gK)x(t)$, WHERE $\dot{x}(t)=(A-BK)x(t)$, $x(0)=B_0$

THEN $$\int_0^\infty g(t)^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(j\omega)|^2 d\omega$$

WHERE $G=G(s)$ IS THE TRANSFER FUNCTION $f \longrightarrow C_gx+D_gu=g$ IN

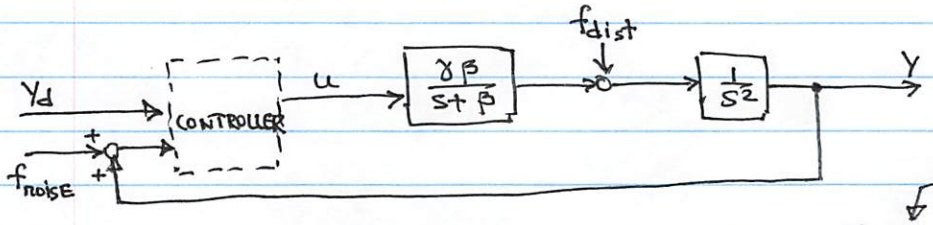
$$\dot{x}(t) = Ax(t)+Bu(t)+B_0f(t),\quad g(t) = C_gx(t)+D_gu(t)$$

Ex. WHAT DID LQR MINIMIZE, WHEN $e=\begin{bmatrix} y \\ 0.2u \end{bmatrix}$ WAS USED?



LQR PUSHED DOWN THE WEIGHTED SUM OF INTEGRALS OF SQUARED ABSOLUTE VALUES OF CLOSED LOOP FREQUENCY RESPONSES FROM f TO y AND FROM f TO u

IN THE "PROP" EXAMPLE, WE DO NOT MEASURE ω, α DIRECTLY, HENCE LQR DOES NOT MATCH THE SETUP EXACTLY:



"LOW PASS FILTER" (KILLS HIGH FREQS, LIKE $L(s)=1/(s+1)$)

WHAT WE ACTUALLY WANT:

$Y_d \longrightarrow y-(LPF)Y_d$

$f_{noise} \longrightarrow u$

$f_{noise} \longrightarrow y$

$f_{dist} \longrightarrow y$

$f_{dist} \longrightarrow u$

} DESIRED TO HAVE SMALL FREQUENCY RESPONCES

QUIZ: IN THE LQR SETUP WITH $A=0, B=1, Q=R=1, N=0, x_0=1$

i.e., $\dot{x}(t)=u(t)$, $x(0)=1$, $\int_0^\infty [x(t)^2+u(t)^2]\,dt \longrightarrow \min$

WHAT IS THE PERFORMANCE CORRESPONDING TO FEEDBACK (CONTROLLER) $u(t)=-Kx(t)$, WHERE $K>0$ IS A PARAMETER?

HINTS: * $\dot{x}(t)=-Kx(t)$, $x(0)=1$ IMPLIES $x(t)=e^{-Kt}$

* FOR $h>0$, $\int_0^\infty e^{-ht}\,dt = \frac{1}{h}$

SOLUTION: WITH $u(t)=-Kx(t)$, EQUATION $\dot{x}(t)=u(t)$, $x(0)=1$ YIELDS $\dot{x}(t)=-Kx(t)$, $x(0)=1$, HENCE $x(t)=e^{-Kt}$, $u(t)=-Ke^{-Kt}$

THEREFORE

$$\int_0^\infty [x(t)^2+u(t)^2]\,dt = \int_0^\infty [e^{-2Kt}+K^2e^{-2Kt}]\,dt$$

$$= \frac{1}{2K}+\frac{K^2}{2K} = \frac{1}{2}\left(\frac{1}{K}+K\right)$$

(TO MINIMIZE W.R.T. K: $\frac{d}{dK}\left(\frac{1}{K}+K\right) = 1-\frac{1}{K^2}$,

HENCE THE OPTIMAL $K>0$ IS $K_{opt}=1$)