Dynamic System Modeling and Control Design

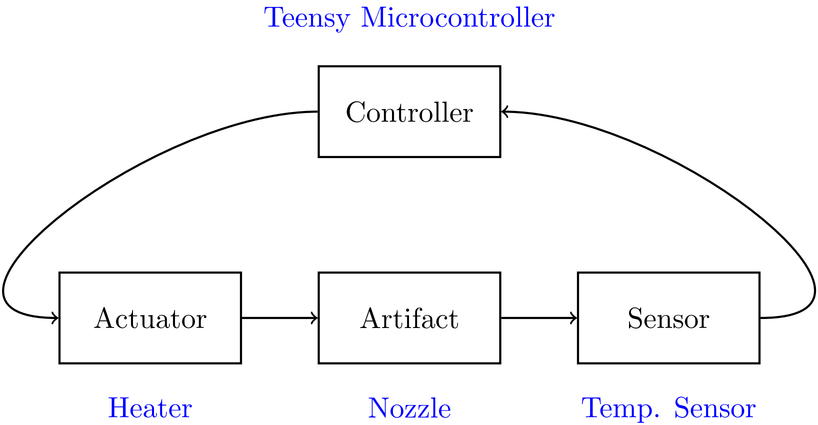
Heat Loss, Parameter Identification, Linearity and Time Invariance

Sept. 16, 2026

Outline

1. Recap of Last Lecture
2. Heat Loss and Nominal Operating Points
3. System Identification: Estimating λ and γ
4. Linearity and Time Invariance

Recap: Generic Control System



$$T_m[n] = T_m[n-1] + \Delta T \gamma_{th} \,\mathrm{Pow}[n-1] \qquad (\mathrm{Pow} = \text{heater power})$$

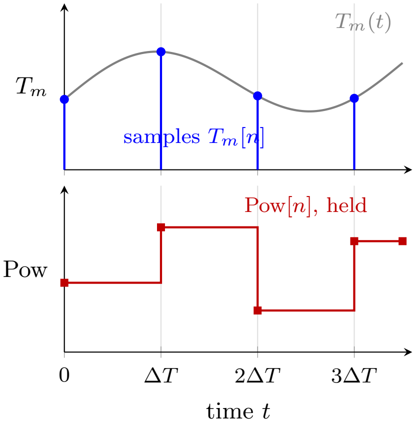
Recap: Clocked (Sampled) Systems

Every ΔT, the controller (Teensy 4.1):

1. reads the sensor,
2. computes a command,
3. updates the actuator.

Teensy: a few μs for this. Device: $\Delta T \approx 1\,\text{ms}$. Orders of magnitude apart, so we treat read–compute–update as **instantaneous**.

- Sampling: $T_m[n] = T_m(n\Delta T)$.
- Zero-order hold: Pow$[n]$ is held until the next update.



Recap: **Linear** First Order Difference Equation (FODE)

- General form, with input $u[n]$ and constant system parameters λ, γ:
$$y[n] = \lambda y[n-1] + \gamma u[n-1] \quad (\#1)$$
- **LINEAR**: y and u are only multiplied by constants and added. No products, no powers, no constant offsets (we will remove offsets by working around a nominal operating point).
- General solution, for arbitrary n:

$$y[n] = \underbrace{\lambda^n y[0]}_{\text{zero input response}} + \underbrace{\gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]}_{\text{zero state response}}$$

Recap: Steady State Response

If $u[n] = u_0$ for all n (i.e., constant input) AND $|\lambda| < 1$, then...

$$\lim_{n\to\infty} y[n] := y[\infty] = \frac{\gamma}{1-\lambda} u_0.$$

Why? At steady state $y[n] = y[n-1] = y[\infty]$, so (#1) becomes

$$y[\infty] = \lambda\, y[\infty] + \gamma u_0 \quad \Rightarrow \quad y[\infty] = \frac{\gamma}{1-\lambda} u_0.$$

Recap: Nozzle Heating with Proportional Control

Model (as of Monday), with $\text{Pow}[n]$ = heater power:

$$T_m[n] = T_m[n-1] + \Delta T \gamma_{th} \, \text{Pow}[n-1]$$

Proportional control ($T_d[n]$ = desired temperature; same index n = "instant update"):

$$\text{Pow}[n] = K_p \left(T_d[n] - T_m[n]\right)$$

Combined: a linear FODE with output T_m and input T_d:

$$T_m[n] = \underbrace{(1 - \Delta T \gamma_{th} K_p)}_{\lambda} T_m[n-1] + \underbrace{\Delta T \gamma_{th} K_p}_{\gamma} \, T_d[n-1]$$

- Stable ($|\lambda| < 1$) if and only if $0 < \Delta T \gamma_{th} K_p < 2$, so the maximum gain is $K_p = \frac{2}{\Delta T \gamma_{th}}$.
- No oscillations ($0 < \lambda < 1$) if and only if $0 < \Delta T \gamma_{th} K_p < 1$.

Recap: Steady State of the Nozzle... Wait!

Hold the desired temperature constant: $T_d[n] = T_{d_0}$ for all n. If $|\lambda| < 1$ the system is stable and $T_m[\infty]$ exists:

$$T_m[\infty] = \frac{\gamma}{1-\lambda} T_{d_0} = \frac{\Delta T \gamma_{th} K_p}{1 - (1 - \Delta T \gamma_{th} K_p)} T_{d_0} = T_{d_0}.$$

The steady-state temperature matches the desired temperature for *any* $0 < K_p < \frac{2}{\Delta T \gamma_{th}}$!

Wait! At steady state, $\mathrm{Pow}[\infty] = K_p \left(T_{d_0} - T_m[\infty]\right) = 0$. Can we really turn the heater *off* and have the nozzle stay hot?

And: whenever $T_m[n] > T_d[n]$, the command $\mathrm{Pow}[n]$ is negative. What would negative heater power mean?

The math is correct. The **model** is not!

Include Heat Loss

A hot nozzle loses heat to its surroundings, and loses it faster the hotter it is:

$$T_m[n] = T_m[n-1] + \Delta T \gamma_{th} \, \mathrm{Pow}[n-1] \underbrace{- \Delta T \beta \, T_m[n-1]}_{\text{heat loss}}, \qquad \beta > 0.$$

(Temperatures are measured relative to room temperature, so there is no heat loss at $T_m = 0$.)

A heater can only *add* heat, and only so much: $0 \leq \mathrm{Pow}[n] \leq P_{\max}$. But the proportional command $K_p \, (T_d[n] - T_m[n])$ is negative whenever the nozzle is hotter than desired. The real heater just sits at 0 there, and the linear model breaks.

Fix: run the heater around a *nominal* power $P_0 > 0$, and let the controller push it up or down from there.

Nominal Operating Point

Pick a **nominal operating point** (T_0, P_0) where heat in = heat out, so the nozzle would stay at T_0 with the heater at P_0:

$$\Delta T \gamma_{th} P_0 = \Delta T \beta \, T_0 \quad \Rightarrow \quad P_0 = \frac{\beta}{\gamma_{th}} T_0.$$

Then measure everything *relative* to it:

$$\delta T[n] = T_m[n] - T_0, \qquad \delta T_d[n] = T_d[n] - T_0, \qquad \delta P[n] = \mathrm{Pow}[n] - P_0.$$

Substitute into the model. The constants cancel, because $\Delta T \gamma_{th} P_0 - \Delta T \beta T_0 = 0$:

$$\delta T[n] = \delta T[n-1] + \Delta T \gamma_{th} \, \delta P[n-1] - \Delta T \beta \, \delta T[n-1].$$

Same equation, now with *no constant offset.* $\delta P[n] < 0$ just means "less than nominal"; the model holds as long as $0 \leq P_0 + \delta P[n] \leq P_{\max}$.

Heat Loss: Closed Loop

Controller = nominal power + proportional correction:

$$\mathrm{Pow}[n] = P_0 + K_p\left(T_d[n] - T_m[n]\right) \quad \Longleftrightarrow \quad \delta P[n] = K_p\left(\delta T_d[n] - \delta T[n]\right).$$

(Same idea in the OPTO lab code: `motorCmd = nomCmd +` *correction*.)
Closed loop, in deviation variables:

$$\delta T[n] = \underbrace{\big(1 - \Delta T(\gamma_{th} K_p + \beta)\big)}_{\lambda}\,\delta T[n-1] + \underbrace{\Delta T \gamma_{th} K_p}_{\gamma}\,\delta T_d[n-1].$$

Still a linear FODE; compared to Monday, only λ changed.

- Stable iff $0 < \Delta T(\gamma_{th} K_p + \beta) < 2$, i.e., $K_p < \frac{2-\Delta T\beta}{\Delta T \gamma_{th}}$.
- No oscillations iff $\Delta T(\gamma_{th} K_p + \beta) < 1$, i.e., $K_p < \frac{1-\Delta T\beta}{\Delta T \gamma_{th}}$.

Steady State with Heat Loss

If $T_d[n] = T_{d_0}$ for all n, then $\delta T_d[n] = T_{d_0} - T_0 =: \delta T_{d_0}$. If also $|\lambda| < 1$, then (using $1 - \lambda = \Delta T(\gamma_{th} K_p + \beta)$):

$$\delta T[\infty] = \frac{\gamma}{1-\lambda}\,\delta T_{d_0} = \frac{\Delta T \gamma_{th} K_p}{\Delta T(\gamma_{th} K_p + \beta)}\,\delta T_{d_0} = \frac{\gamma_{th} K_p}{\gamma_{th} K_p + \beta}\,\delta T_{d_0},$$

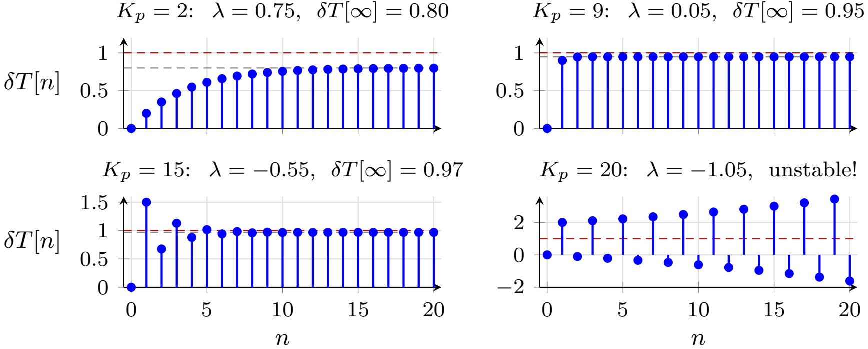
so the steady-state error is

$$T_{d_0} - T_m[\infty] = \delta T_{d_0} - \delta T[\infty] = \frac{\beta}{\gamma_{th} K_p + \beta}\,(T_{d_0} - T_0)\,.$$

- The steady-state error is proportional to how far the target is from the operating point.
- Target *at* the operating point ($T_{d_0} = T_0$): zero error. The nominal power P_0 alone holds the temperature.
- Otherwise, the error shrinks as $\gamma_{th} K_p$ grows relative to β.
- The heater stays on: $\mathrm{Pow}[\infty] = \frac{\beta}{\gamma_{th}}\,T_m[\infty]$ (heat in = heat out).
- Room temperature as the operating point ($T_0 = 0$, $P_0 = 0$): $T_m[\infty] = \frac{1}{1+\beta/(\gamma_{th} K_p)}\,T_{d_0} < T_{d_0}$.

Bigger K_p: Better Match... Until It Isn't

Example: $\Delta T \gamma_{th} = 0.1$, $\Delta T \beta = 0.05$; start at the operating point, $\delta T[0] = 0$, and raise the target by 1: $\delta T_d[n] = 1$ (red: δT_d; gray: $\delta T[\infty]$).



Stable only for $K_p < \frac{2-\Delta T \beta}{\Delta T \gamma_{th}} = 19.5$; no oscillations for $K_p < 9.5$. Large K_p also means large (and negative) δP: the heater must stay within $0 \leq P_0 + \delta P \leq P_{\max}$, or it saturates.

Today's Main Objective

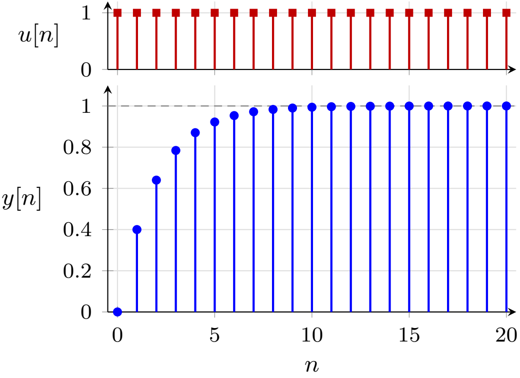
We have this nice FODE:

$$y[n] = \lambda y[n-1] + \gamma u[n-1] \qquad (\#1)$$

What are λ and γ for *your* lab system? How do we find them from measured data?

Two unknowns, so let's find two equations and solve. You will need this for Friday's lab!

Warm-Up: Estimate λ from a Step Response



Zero state response ($y[0] = 0$) of FODE (#1) to a unit step: $u[n] = 1$ for $n \geq 0$.

Here $\gamma = 1 - \lambda$, so $y[\infty] = 1$.

Question: can you estimate λ from the figure?

$y[1] = \gamma = 0.4$, so
$\lambda = 1 - \gamma = 0.6$.

Or: the gap to 1 shrinks by a factor λ each step:
$1,\ 0.6,\ 0.36,\ \ldots$

Key Tool: The Gap to Steady State

Suppose $|\lambda| < 1$ and the input is constant from time N on: $u[n] = u_0$ for $n \geq N$. Its steady state $y_{ss} = \frac{\gamma}{1-\lambda}\, u_0$ satisfies $y_{ss} = \lambda y_{ss} + \gamma u_0$.

Subtract this from (#1): for $n > N$,

$$\underbrace{y[n] - y_{ss}}_{e[n]} = \lambda\big(\underbrace{y[n-1] - y_{ss}}_{e[n-1]}\big) \quad \Rightarrow \quad e[n] = \lambda^{n-N}\, e[N].$$

The gap to the new steady state shrinks by a factor λ every step, wherever we start.

Special case, input off ($u_0 = 0$, so $y_{ss} = 0$): $y[n] = \lambda^{n-N}\, y[N]$.

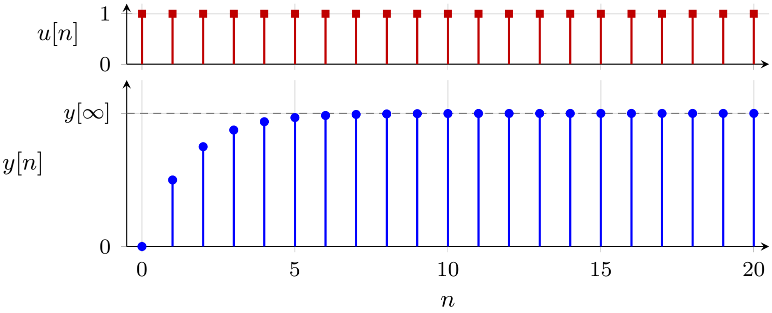
Evaluate Steady State Response

Assuming (ideal case) FODE (#1) with

$$y[0] = 0, \qquad 0 < \lambda < 1, \qquad u[n] = 1 \text{ for } n \geq 0,$$

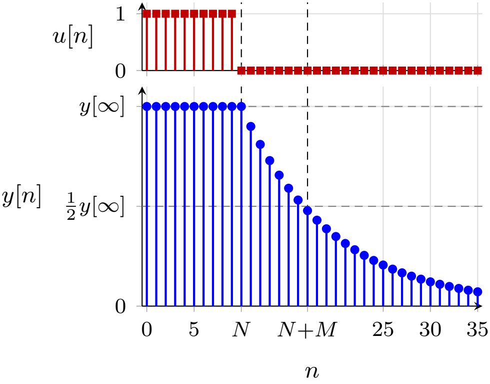
we already have one relationship!

$$y[\infty] = \frac{\gamma}{1-\lambda} \Rightarrow \gamma = y[\infty](1-\lambda).$$



Evaluate Decay

Start at steady state ($u[n] = 1$ for $n < N$), then turn the input off: $u[n] = 0$ for $n \geq N$. How many steps M until y has decayed halfway?



Key tool with the input off: $y[N+M] = \lambda^M\, y[\infty]$, so

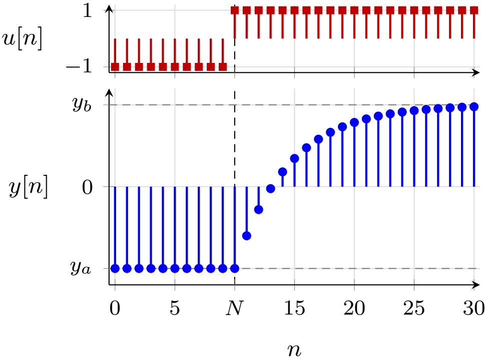
$$\frac{\gamma}{1-\lambda}\lambda^M \approx \frac{1}{2}\,\frac{\gamma}{1-\lambda}$$
$$\Rightarrow \lambda \approx 0.5^{1/M},$$
$$\Rightarrow \gamma \approx y[\infty]\left(1 - 0.5^{1/M}\right).$$

(M is an integer, so these are estimates.)

Here: $M \approx 7 \Rightarrow \lambda \approx 0.91$ (true value: $\lambda = 0.9$).

In Practice (a): From One Steady State to Another

In the lab, transients rarely start from zero: the input steps $u_a \to u_b$ (e.g., $-1 \to +1$) and the output moves between two *nonzero* steady states $y_a \to y_b$.



Steady states $y_{a,b} = \frac{\gamma}{1-\lambda}\, u_{a,b}$, so

$$\frac{\gamma}{1-\lambda} = \frac{y_b - y_a}{u_b - u_a}.$$

Key tool, $e[n] = y[n] - y_b$: fraction of the step still to go is

$$\frac{y_b - y[n]}{y_b - y_a} = \lambda^{n-N}, \quad n \geq N.$$

Halfway after M steps: $\lambda \approx 0.5^{1/M}$.

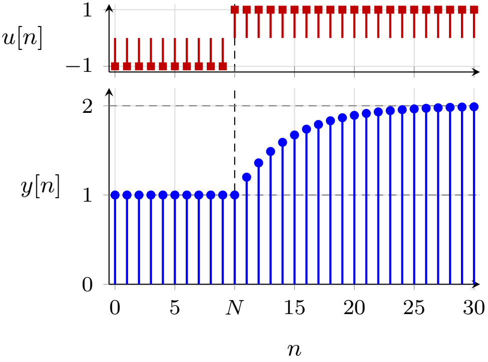
Here: ratio $= \frac{1}{2}$, $\;M \approx 3 \Rightarrow \lambda \approx 0.79$.

In Practice (b): Unknown Offset

Even if the lab system is linear, what we measure is better modeled as

$$y[n] = \lambda y[n-1] + \gamma u[n-1] + c,$$

where c is an unknown constant: e.g., sensor bias, or a nominal pair that is not exactly in balance ($c = \Delta T(\gamma_{th} P_0 - \beta T_0)$ for the nozzle).



Same system as (a), with $c = 0.3$. Steady state is now

$$y_{ss} = \frac{\gamma u_0 + c}{1-\lambda}.$$

Naive: $y[\infty] = 2$ with $u_0 = 1$, so $\frac{\gamma}{1-\lambda} = 2$? Wrong!

Differences:
$y_b - y_a = \frac{\gamma}{1-\lambda}(u_b - u_a)$, and c cancels: $\frac{2-1}{1-(-1)} = \frac{1}{2}$ ✓

In Practice: Trust Only Differences

The key tool still works with the offset: $y_b = \lambda y_b + \gamma u_b + c$, so for $n > N$

$$e[n] = y[n] - y_b = \lambda\, e[n-1] \qquad \text{(both } u \text{ and } c \text{ drop out).}$$

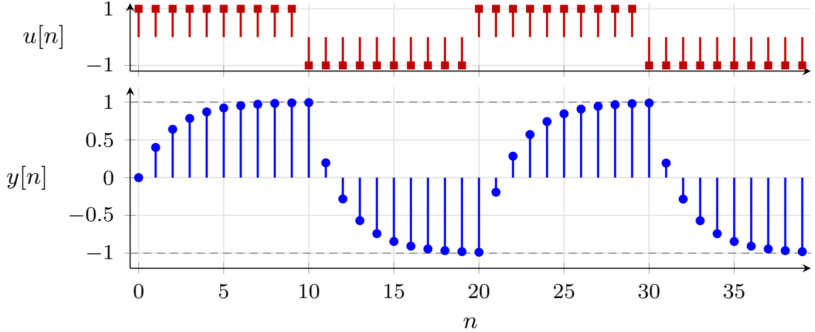
The offset shifts every level by $\frac{c}{1-\lambda}$ but does not change the transient.
Recipe for estimating λ, γ from lab data:

1. Hold $u = u_a$ until y settles; record y_a.
2. Step to $u = u_b$; record the transient and the new level y_b.
3. Compute $\dfrac{\gamma}{1-\lambda} = \dfrac{y_b - y_a}{u_b - u_a}$.
4. Find M, the number of steps to get halfway from y_a to y_b: $\lambda \approx 0.5^{1/M}$.
5. Then $\gamma = (1-\lambda)\dfrac{y_b - y_a}{u_b - u_a}$ (and, if needed, $c = (1-\lambda)\, y_a - \gamma u_a$).

Don't rely on absolute output values or transients "from zero": trust only differences.

Why Do the Transients Look Alike?

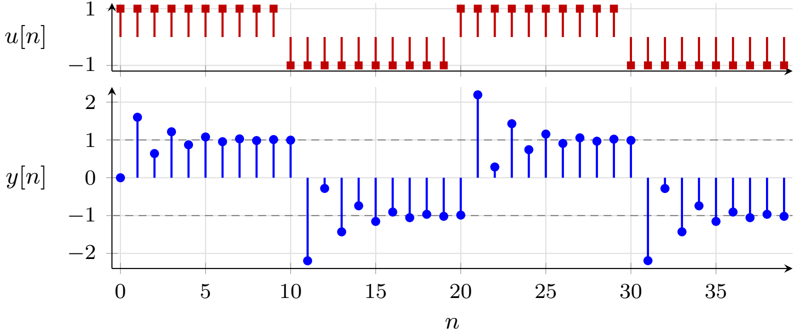
Warm-up system again ($\lambda = 0.6$, $\gamma = 0.4$, $y[0] = 0$), but the input now switches between $+1$ and -1 every 10 steps.



Question: each transient looks like a scaled, time-shifted copy of the step response. Why? **Answer:** Linearity and Time Invariance.

Why Do the Transients Look Alike?

Same input, but now $\lambda = -0.6$, $\gamma = 1 - \lambda = 1.6$.



Question: why does the output overshoot and ring? How could you tell $\lambda < 0$? **Answer:** λ^n alternates in sign when $\lambda < 0$. And again: Linearity and Time Invariance.

Property I: Decomposition of General Solution

Recall our two special cases:

- Zero Input Response: If $u[n] = 0$ for all n: $y[n] = \lambda^n y[0]$
- Zero State Response: If $y[0] = 0$: $y[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$

Then we can decompose $y[n]$ into its ZIR and ZSR:

$$y[n] = \lambda^n y[0] + \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$$

$$\text{(I): } y[n] = y_{ZIR}[n] + y_{ZSR}[n]$$

Property II: Linearity & Time Invariance of ZSR

Suppose that I have two different input functions $u_A[n], u_B[n]$. Then,

$$y_{A,ZSR}[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u_A[m],$$
$$y_{B,ZSR}[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u_B[m].$$

Assume $u_A[n] = u_B[n] = 0$ for all $n < 0$, and $N_A, N_B \geq 0$.

If $u[n] = a\, u_A[n - N_A] + b\, u_B[n - N_B]$, then

$$\text{(II): } y_{ZSR}[n] = a\, y_{A,ZSR}[n - N_A] + b\, y_{B,ZSR}[n - N_B].$$

Property III: An Aspect of Time Invariance of ZIR

If $u[n] = 0$ for all $n \geq N$, then

$$\text{(III): } y[n] = \lambda^{n-N} y[N] \quad \text{for all } n \geq N.$$

Proof sketch:

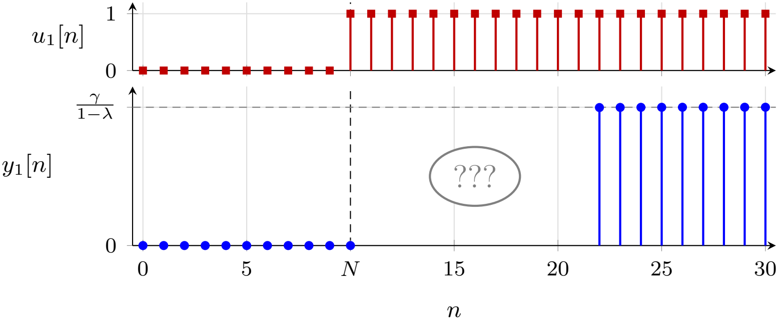
$$\begin{aligned} y[N+1] &= \lambda y[N] + \gamma u[N] = \lambda y[N] \qquad (u[N] = 0) \\ y[N+2] &= \lambda y[N+1] + \gamma u[N+1] = \lambda^2 y[N] \\ y[N+3] &= \lambda^3 y[N] \\ &\vdots \end{aligned}$$

Transient to Steady State (Using (I)–(III))

Suppose that $|\lambda| < 1$ and the input is a unit step that turns on at $n = N$:

$$u_1[n] = 0 \text{ for } n < N, \qquad u_1[n] = 1 \text{ for } n \geq N,$$

with initial state $y_1[0] = 0$. What is the output $y_1[n]$?



Before N: nothing happens. Long after N: steady state. In between? The key tool already told us; let's rederive it using Properties (I)–(III).

Transient to Steady State (Cont.)

Consider a system that starts at steady state, and whose input turns *off* at $n = N$:

$$y_2[0] = \frac{\gamma}{1-\lambda}, \qquad u_2[n] = 1 \text{ for } n < N, \qquad u_2[n] = 0 \text{ for } n \geq N.$$

What is $y_2[n]$?
Directly from the system equation (#1):

$$\begin{aligned} y_2[n] &= \frac{\gamma}{1-\lambda} \text{ for } n \leq N & &\text{(input still 1)} \\ y_2[N+1] &= \lambda y_2[N] + \gamma u_2[N] = \lambda \frac{\gamma}{1-\lambda} & &(u_2[N] = 0) \\ y_2[n] &= \lambda^{n-N} \frac{\gamma}{1-\lambda} \text{ for } n \geq N & &\text{(repeating the same step)} \end{aligned}$$

Transient to Steady State (Cont.)

Consider a system with

$$y_3[0] = \frac{\gamma}{1-\lambda}, \qquad u_3[n] = 1 \text{ for all } n.$$

What is $y_3[n]$?
Since we initialized at steady state, and the input function $u_3[n]$ does not change, we will remain in steady state:

$$y_3[n] = \frac{\gamma}{1-\lambda} \text{ for all } n.$$

Transient to Steady State: Putting It Together

Back to $u_1[n]$ (unit step turning on at N, with $y_1[0] = 0$). Notice that

$$u_1[n] = u_3[n] - u_2[n] \text{ for all } n, \qquad \text{and} \qquad y_1[0] = y_3[0] - y_2[0] = 0.$$

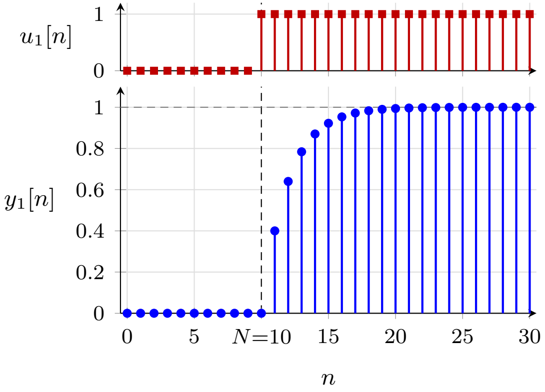
Both the inputs *and* the initial states subtract, so by linearity (Properties (I) and (II)):

$$\begin{aligned} y_1[n] &= y_3[n] - y_2[n] \\ &= \frac{\gamma}{1-\lambda} - \lambda^{n-N}\frac{\gamma}{1-\lambda}, \quad n \geq N \\ &= \frac{\gamma}{1-\lambda}\left(1 - \lambda^{n-N}\right), \quad n \geq N, \end{aligned}$$

and $y_1[n] = 0$ for $n \leq N$.

Now, We Can Fill in the Gap: $0 < \lambda < 1$

$$y_1[n] = \frac{\gamma}{1-\lambda}\left(1 - \lambda^{n-N}\right), \quad n \geq N$$



Here $\frac{\gamma}{1-\lambda} = 1$.

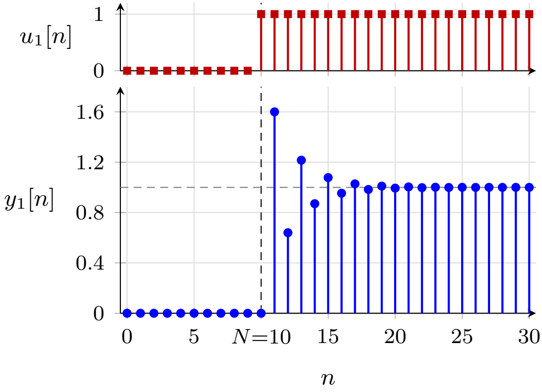
Note $y_1[N] = 0$: the output first responds at $N+1$ (the one-step delay in (#1)).

Question: roughly, what is λ?

The gap to steady state goes $1 \to 0.6 \to 0.36 \to \cdots$, shrinking by a factor λ each step, so $\lambda \approx 0.6$.

Now, We Can Fill in the Gap: $-1 < \lambda < 0$

$$y_1[n] = \frac{\gamma}{1-\lambda}\left(1 - \lambda^{n-N}\right), \quad n \geq N$$



Again $\frac{\gamma}{1-\lambda} = 1$.

Question: roughly, what is λ?

The gap $y_1[n] - 1$ goes $-1 \to +0.6 \to -0.36 \to \cdots$: it flips sign and shrinks by 0.6 each step, so $\lambda \approx -0.6$.

Negative $\lambda \Rightarrow$ overshoot and oscillation.

Closing Thoughts

How can we generate the previous plots?

- We get to pick which controller we use (and set the gains)!
- We can find an expression for λ which will (probably) be a function of the gain(s) of our controller.
- With proportional control and heat loss ($\beta \neq 0$), the error is zero only when the target equals the nominal temperature T_0. Can we design a controller that removes it for *any* target?