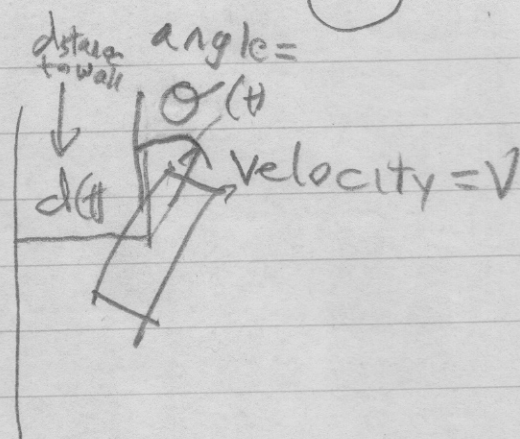


6.310 10/13/26

Wall Follower



$$\lim_{\Delta t \rightarrow 0} \frac{d[n] - d[n-1]}{\Delta t} = \frac{d}{dt} d(t) \approx V \theta(t)$$

$$\frac{d}{dt} \theta(t) \approx \gamma c(t)$$

↑
commanded
rotation
rate

Prop.
Feedback

$$c(t) = K_p (d_d(t) - d(t))$$

desired
wall
distance

The CTx
matrix
approach
we are not
taking (Now)

In D.T.

$$\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta t V \\ \Delta t K_p & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}}_X + \underbrace{\begin{bmatrix} 0 \\ \Delta t K_p \end{bmatrix}}_B d_d[n]$$

$$X[n] = A^n X[0] + \sum_{m=0}^{n-1} A^{n-m-1} B U[m]$$

if $|\lambda_i(A)| < 1 \forall i$

as $n \rightarrow \infty$

zero

bounded

convolution
sum

In C.T.

$$\frac{d}{dt} \begin{bmatrix} d(t) \\ \theta(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & V \\ K_p & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} d(t) \\ \theta(t) \end{bmatrix}}_X + \underbrace{\begin{bmatrix} 0 \\ K_p \end{bmatrix}}_B d_d(t)$$

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} B U(\tau) d\tau$$

if $\text{Re}(\lambda_i(A)) < 0 \forall i$

as $t \rightarrow \infty$

zero

bounded

convolution
in integral

Modeling Blocks with Frequency Responses.

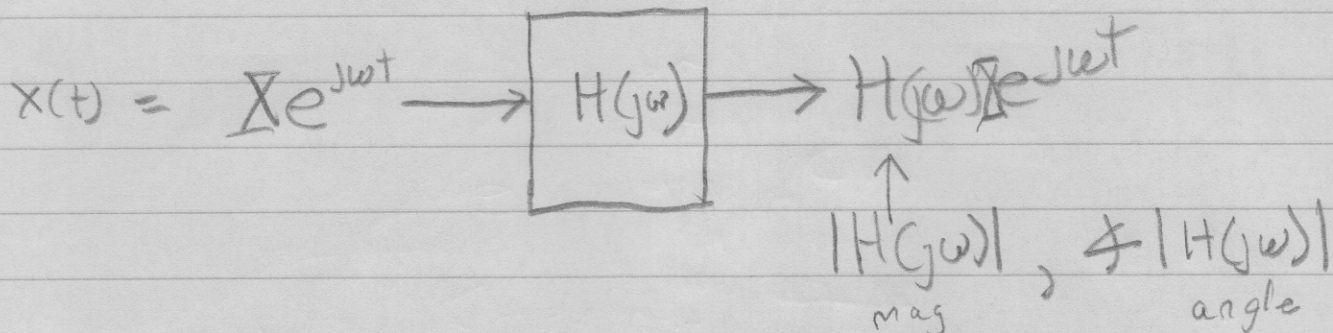
2

Last Time

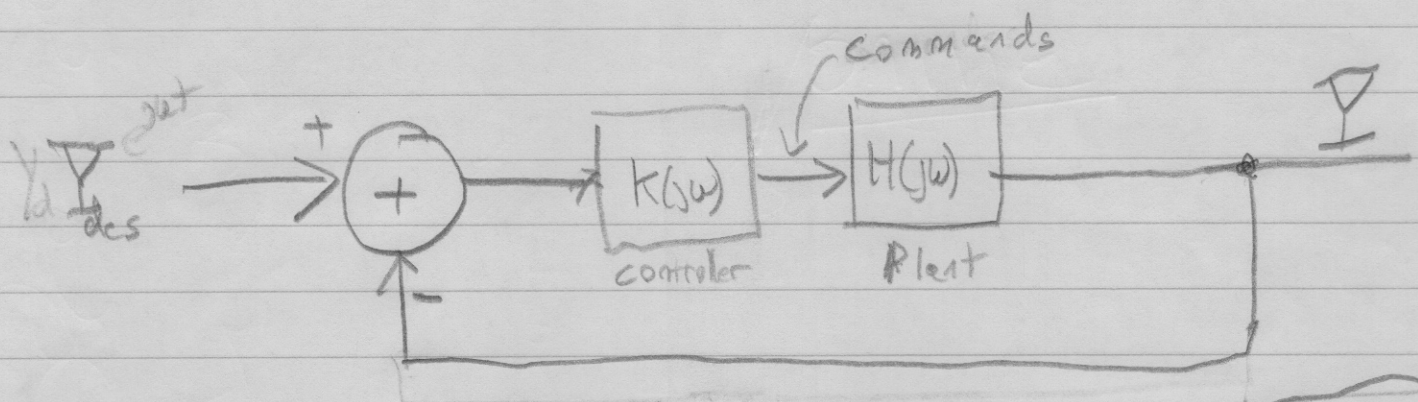
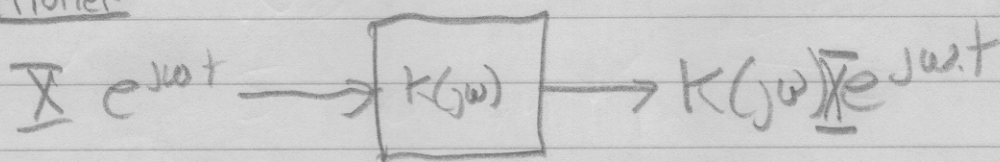
System

$$\omega = 2\pi f$$

f = frequency



Controller



$\lim_{\omega \rightarrow \infty} K(j\omega) = \infty$
usually causes problems

$$\underline{Y} = H(j\omega) K(j\omega) (\underline{Y}_{des} - \underline{Y})$$

$$\underline{Y} = \frac{K(j\omega) H(j\omega)}{1 + K(j\omega) H(j\omega)} \underline{Y}_{des}$$

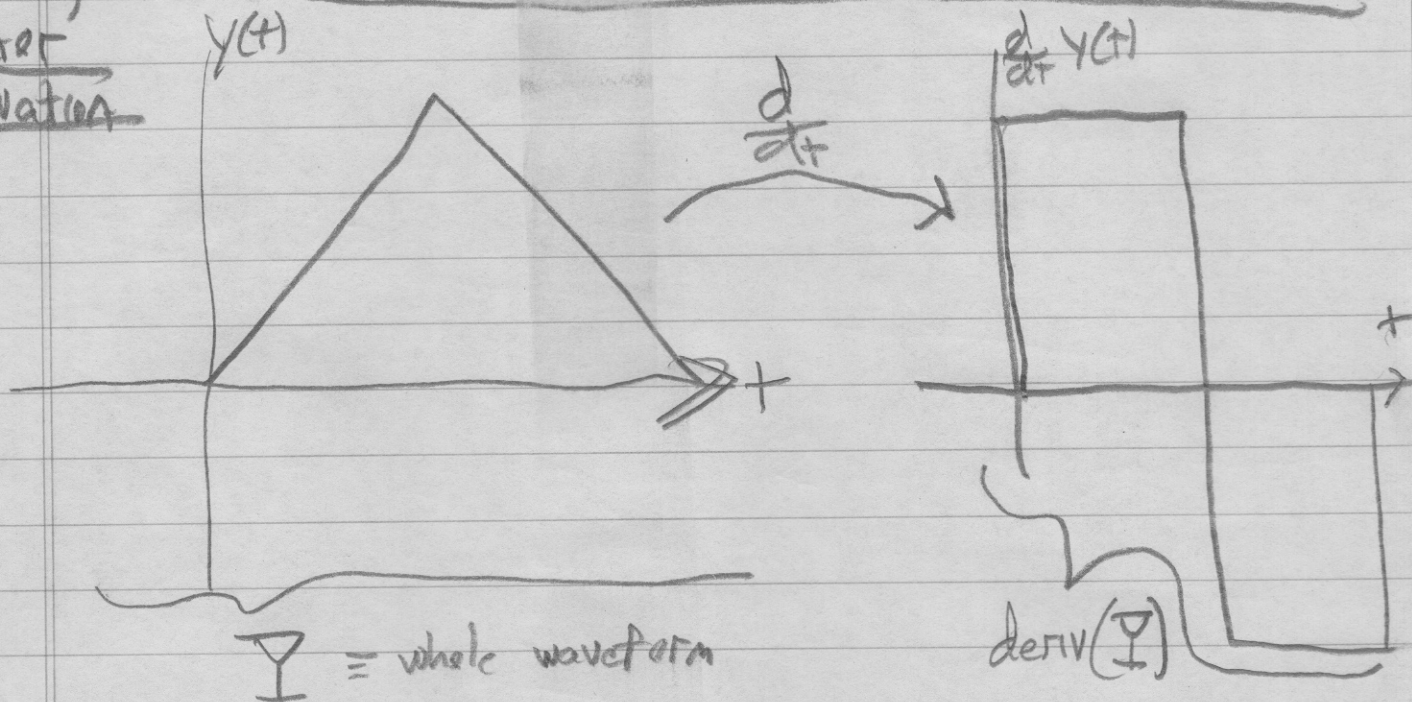
$$\text{If } |K(j\omega) H(j\omega)| \gg 1 \Rightarrow \underline{Y} \approx \underline{Y}_{des}$$

$(K(j\omega) H(j\omega)) \gg 1$
 $\underline{Y} \approx \underline{Y}_{des}$
 $(K(j\omega) H(j\omega)) \ll 1$
 $\underline{Y} \approx 0$
 $K(j\omega) H(j\omega) \approx -1$

(3)

The Transfer Function Approach

"Operator"
Derivation



• $\frac{d}{dt}$ is an operator mapping I to its deriv

• Use ' s ' to denote this operator

So $\frac{d}{dt} d(t) = Y(t) \Rightarrow sD = Y(t)$

Some identities

Scalars $s^2 I = s(sI) \Leftrightarrow \frac{d}{dt} \left(\frac{d}{dt} Y(t) \right) = \frac{d^2}{dt^2} Y(t)$

$(\beta s + \alpha) I \Leftrightarrow \beta \frac{d}{dt} Y(t) + \alpha Y(t)$

$(s + \alpha_1)(s + \alpha_2) I = (s^2 + (\alpha_1 + \alpha_2)s + \alpha_1 \alpha_2) I$

$\Rightarrow \frac{d^2}{dt^2} Y(t) + (\alpha_1 + \alpha_2) \frac{d}{dt} Y(t) + \alpha_1 \alpha_2 Y(t)$

Using "S" for Line Follower

$$s D = V \Phi$$

$$D = \frac{V}{s} \Phi \quad \leftarrow d(t) = \int v(t) dt$$

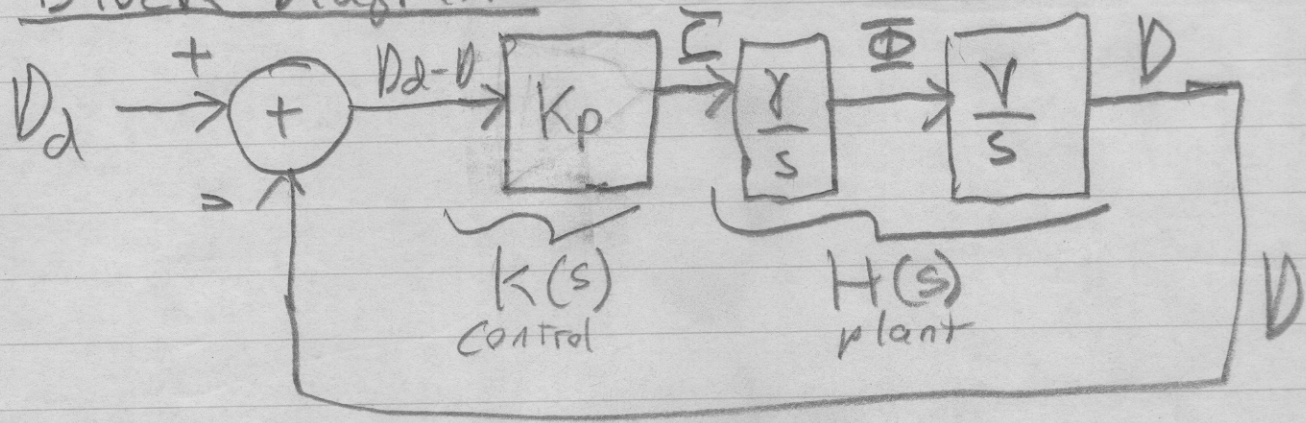
$$s \Phi = \gamma \Sigma$$

$$\Phi = \frac{\gamma}{s} \Sigma \quad \leftarrow \phi(t) = \int \gamma(t) dt$$

$$\Sigma = K_p (D_d - D)$$

$$\Sigma = K_p (D_d - D)$$

Block Diagram



$$D = \overbrace{K_p (D_d - D)}^{\Sigma} \left(\frac{\gamma}{s} \frac{V}{s} \right)$$

$$(D + D K_p \frac{\gamma}{s} \frac{V}{s}) = K_p \frac{\gamma V}{s^2} D_d$$

$$(1 + K_p \frac{\gamma V}{s^2}) D = K_p \frac{\gamma V}{s^2} D_d$$

Is this justified
Is $\frac{s^2}{s^2}$
really the P identity.

$$D = \frac{s^2 K_p \frac{\gamma V}{s^2}}{s^2 + K_p \frac{\gamma V}{s^2}} D_d$$

$$D = \frac{K_p \gamma V}{s^2 + K_p \gamma V} D_d$$

$G(s)$

(5)

Cross Multiply $\Rightarrow (s^2 + K_p \gamma V) D = K_p \gamma V D_d$

I/O DFF. Eqn $\Rightarrow \left(\frac{d^2}{dt^2} + K_p \gamma V \right) d(t) = K_p \gamma V d_d(t)$

We can use operator calculus to determine I/O differential eqn

Determining Natural Frequencies

Zero Input Response $d_d(t) = 0$ $d(0) \neq 0$

$\frac{d^2}{dt^2} d(t) + K_p \gamma V d(t) = 0$

Suppose $d(t) = \alpha e^{\lambda t} \Rightarrow \frac{d}{dt} d(t) = \alpha \lambda e^{\lambda t}$
 $\frac{d^2}{dt^2} d(t) = \alpha \lambda^2 e^{\lambda t}$

$\Rightarrow \alpha \lambda^2 e^{\lambda t} + K_p \gamma V \alpha e^{\lambda t} = 0$

$\lambda^2 = -K_p \gamma V \Rightarrow \lambda = \pm j \sqrt{K_p \gamma V}$

Or generally

$d(t) = \alpha_1 e^{j \sqrt{K_p \gamma V} t} + \alpha_2 e^{-j \sqrt{K_p \gamma V} t}$

If $d(0)$, $\dot{d}(0)$ are given, then

$\frac{d}{dt} d(t) = V \sigma(t) \Rightarrow \left(\frac{d}{dt} d(t) \right) \Big|_{t=0} = V \sigma(0) \Big|_{t=0}$

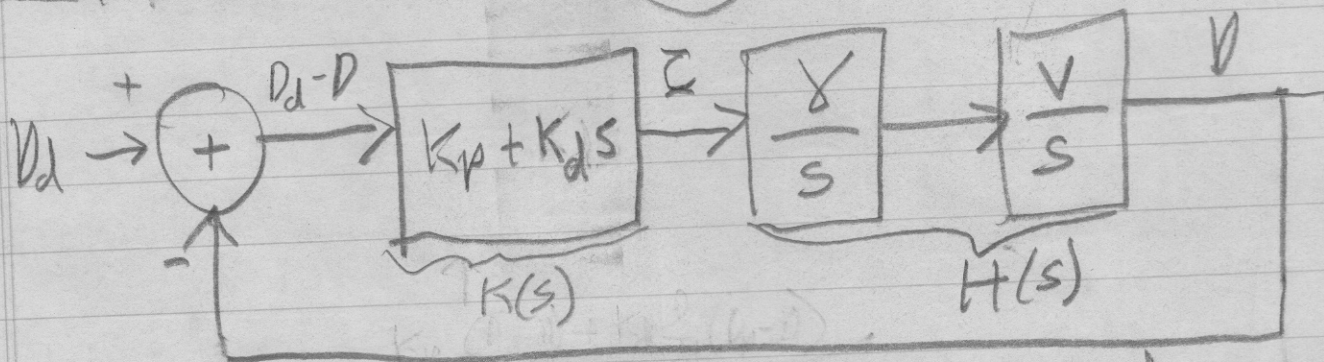
Given $d(0)$, $\frac{d}{dt} d(t) \Big|_{t=0}$ can find α_1, α_2

Stable?
Does the ZIR $\rightarrow 0$ as $t \rightarrow \infty$

(6)

$$e^{j\sqrt{K_p \gamma V} t} = \cos(\sqrt{K_p \gamma V} t) + j \sin(\sqrt{K_p \gamma V} t)$$

Control with Prop. Fdbk is Not stable!
 $\lim_{t \rightarrow \infty} \neq 0$



P.D. Control $C(t) = K_p(d_d(t) - d(t)) + K_d \frac{d}{dt}(d_d(t) - d(t))$
 $\Rightarrow K_p(D_d - D) + K_d s(D_d - D)$

$$D = D = K(s) H(s) (D_d - D)$$

$$(1 + K(s) H(s)) D = K(s) H(s) D_d$$

Black's Formula: $D = \frac{K(s) H(s)}{1 + K(s) H(s)} D_d$

$$K(s) H(s) = \frac{(K_p + K_d s) \gamma V}{s^2} = \frac{n(s)}{d(s)}$$

numerator poly in s
denom poly in s

$$D = \frac{\frac{n(s)}{d(s)}}{1 + \frac{n(s)}{d(s)}} D_d = \frac{n(s)}{d(s) + n(s)} D_d$$

$$D = \frac{(K_p + K_d s) Y V}{s^2 + (K_p + K_d s) Y V} \quad Dd$$

$$(s^2 + K_d Y V s + K_p Y V) D = (K_p + K_d s) Dd$$

Unnecessary

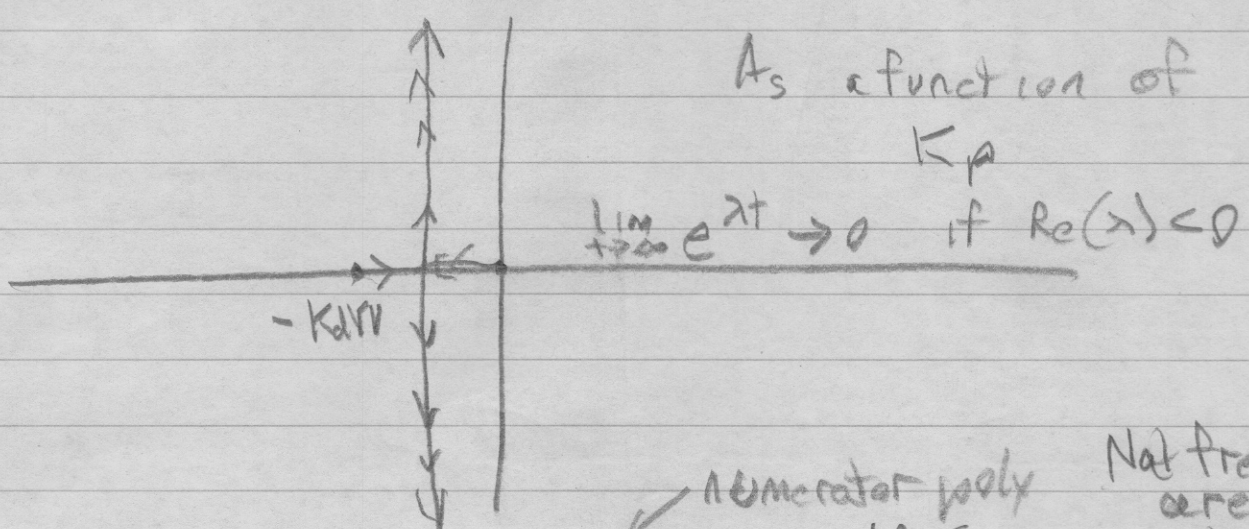
$$\left(\frac{d^2}{dt^2} + K_d Y V \frac{d}{dt} + K_p Y V \right) d(t) = 0 \quad \frac{Z I A}{Vd=0}$$

Assume exponential solutions

$$(\lambda^2 + K_d Y V \lambda + K_p Y V) e^{\lambda t} = 0 \Rightarrow$$

roots are nat freqs!

$$\lambda_1 \lambda_2 = -\frac{K_p Y V}{2} \pm \sqrt{\frac{(K_d Y V)^2}{4} - K_p Y V}$$



If $G(s) = \frac{n_g(s)}{d_g(s)}$

$n_g(s)$ ← numerator poly in s

$d_g(s)$ ← denom poly in s

Nat freqs are roots of $d_g(s)$!

Solns of $d_g(s) = 0$