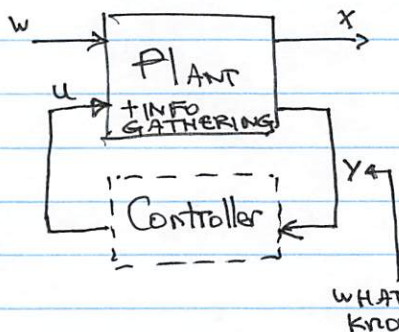


PD AND PID CONTROLLERS / PROPERTIES

$$P: \begin{cases} x[n+1] = A_{ol} x[n] + B_{ctr} u[n] + B_{ext} w[n] \\ y[n] = Cx[n] + Dw[n] \end{cases}$$

$$C: u[n] = Ky[n]$$

Ex. 1

$$d[n+1] = d[n] + \Delta_r V \theta[n] + \Delta_r f_d[n]$$

$$\theta[n+1] = \theta[n] + \Delta_r \omega[n] + \Delta_r f_\theta[n]$$

$r[n]$ - desired value of $d[n]$

PROPORTIONAL FEEDBACK: $w[n] = K_p(r[n] - d[n])$

Here: $x[n] = \begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix}$, $w[n] = \begin{bmatrix} r[n] \\ f_d[n] \\ f_\theta[n] \end{bmatrix}$, $y[n] = \begin{bmatrix} r[n] \\ d[n] \end{bmatrix}$, $u[n] = w[n]$

i.e., $A_{ol} = \begin{bmatrix} 1 & \Delta_r V \\ 0 & 1 \end{bmatrix}$, $B_{ctr} = \begin{bmatrix} 0 \\ \Delta_r \end{bmatrix}$, $B_{ext} = \begin{bmatrix} 0 & \Delta_r & 0 \\ 0 & 0 & \Delta_r \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$K = [K_p, -K_p]$$

DOES THIS WORK? NEVER! BECAUSE $A_{cl} = \begin{bmatrix} 1 & \Delta_r V \\ -\Delta_r K_p & 1 \end{bmatrix}$ HAS AT LEAST ONE EIG λ WITH $|\lambda| \geq 1$ FOR ALL K_p

Ex. 1A

TRYING TO DO BETTER: $w[n] = K_p(r[n] - d[n]) - K_\theta \theta[n]$

NEED TO MEASURE θ

WHAT HAS CHANGED?

$$y[n] = \begin{bmatrix} r[n] \\ d[n] \\ \theta[n] \end{bmatrix}, \text{ i.e., } C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, K = [K_p, -K_p, -K_\theta]$$

(A_{ol}, B_{ctr}, B_{ext} REMAIN THE SAME)

THIS WORKS BETTER:

$$A_{cl} = \begin{bmatrix} 1 & \Delta_r V \\ -\Delta_r K_p & 1 - \Delta_r K_\theta \end{bmatrix} \text{ CAN BE MADE TO HAVE ALL EIGS } \lambda \text{ WITH } |\lambda| < 1$$

(EXTREME: $\Delta_r K_\theta = 2$, $\Delta_r^2 K_p V = 1$ MAKES BOTH EIGS 0)

MORE PRACTICAL: USE $K_p, K_\theta > 0$, $\Delta_r \ll 1$ SMALL ENOUGH

WE USUALLY CARE ABOUT MORE THAN "JUST" STABILITY!

? HOW ABOUT THE "EXACTNESS" OF STEADY STATE RESPONSE TO

? $r[n] = \text{const}$

$$\begin{array}{l} r[n] = r_{\infty} = \text{const} \\ f_d[n] = f_{d,\infty} = 0 \end{array} \xRightarrow[\text{STABILITY}]{\text{ASSUMING}} d[n] \rightarrow d_{\infty}$$

IS THERE AN ASSURANCE THAT $d_{\infty} = r_{\infty}$ WHENEVER $f_{d,\infty} = f_{\theta,\infty} = 0$?

OR "STABILITY IMPLIES"

!!! SINCE $d[n+1] - d[n] = \Delta_T v \theta[n]$, WE MUST HAVE $\theta_{\infty} = 0$

--- SINCE $\theta[n+1] - \theta[n] = \Delta_T \omega[n]$, WE MUST HAVE $\omega_{\infty} = 0$

Now $\omega_{\infty} = K_p(r_{\infty} - d_{\infty}) - K_{\theta}\theta_{\infty}$ IMPLIES $r_{\infty} = d_{\infty}$ (SINCE $K_p \neq 0$)

(WHAT WE ACTUALLY DID ~~WAS~~ IS SOLVED $(I - A_{cl})x_{\infty} = B_{cl}r_{\infty}$ "ON THE CHEAP")

?? HOW ABOUT ZERO STEADY STATE RESPONSE TO $f_{d,\infty} \neq 0$??

?? I.E., IF $r_{\infty} = f_{\theta,\infty} = 0$, DOES THIS IMPLY $d_{\infty} = 0$??

$$\left. \begin{array}{l} d[n+1] - d[n] = \Delta_T (v \theta[n] + f_d[n]) \\ \theta[n+1] - \theta[n] = \Delta_T \omega[n] \\ \omega[n] = -K_p d[n] - K_{\theta} \theta[n] \end{array} \right\} \Rightarrow \left. \begin{array}{l} v \theta_{\infty} = -f_{d,\infty} \\ \omega_{\infty} = 0 \\ \omega_{\infty} = -K_p d_{\infty} - K_{\theta} \theta_{\infty} \end{array} \right\} \Rightarrow d_{\infty} = \frac{f_{d,\infty}}{K_p}$$

WE "KNOW" THIS

SIMILAR CALCULATION WITH $f_{\theta,\infty} \neq 0$, $f_{d,\infty} = 0$, $r_{\infty} = 0$:

$$\left. \begin{array}{l} d[n+1] - d[n] = \Delta_T v \theta[n] \\ \theta[n+1] - \theta[n] = \Delta_T (\omega[n] + f_{\theta}[n]) \\ \omega[n] = -K_p d[n] - K_{\theta} \theta[n] \end{array} \right\} \Rightarrow \left. \begin{array}{l} \theta_{\infty} = 0 \\ \omega_{\infty} + f_{\theta,\infty} = 0 \\ \omega_{\infty} = -K_p d_{\infty} - K_{\theta} \theta_{\infty} \end{array} \right\} \Rightarrow d_{\infty} = \frac{f_{\theta,\infty}}{K_p}$$

CAN WE DO "BETTER" IN REJECTING CONSTANT DISTURBANCES ?

Ex. 1B PD (PROPORTIONAL-DERIVATIVE) CONTROLLER:

$$\omega[n] = K_p(r[n] - d[n]) - K_d \underbrace{\frac{d[n] - d[n-1]}{\Delta_T}}_{\text{AN ESTIMATE OF } \dot{d}}$$

THIS REQUIRES A "LARGER" STATE $x[n]$, FOR EXAMPLE,

$$x[n] = \begin{bmatrix} d[n-1] \\ d[n] \\ \theta[n] \end{bmatrix}, \text{ so } A_{ol} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & \Delta_T v \\ 0 & 0 & 1 \end{bmatrix}, B_{ctrl} = \begin{bmatrix} 0 \\ 0 \\ \Delta_T \end{bmatrix}, B_{ext} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta_T & 0 \\ 0 & 0 & \Delta_T \end{bmatrix}$$

$$y[n] = \begin{bmatrix} r[n] \\ d[n] \\ d[n-1] \end{bmatrix}, K = \begin{bmatrix} -K_p & -\frac{K_d}{\Delta_T} & \frac{K_d}{\Delta_T} \end{bmatrix}$$

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(3)

$$\text{Now } A_{CL} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & \Delta_T V \\ K_d & -K_p \Delta_T - K_d & 1 \end{bmatrix}$$

CAN WE MAKE IT STABLE? YES, BUT THIS TAKES MORE EFFORTTYPICAL APPROACH: FIX $K_d > 0$, SEARCH THROUGH K_p USING NUMERICAL SOFTWAREDO WE GUARANTEE $d_\infty = r_\infty$ WHENEVER $f_{d,\infty} = f_{\theta,\infty} = 0$? YES

CALCULATIONS / ARGUMENTS ARE VERY SIMILAR:

$$\left. \begin{aligned} d[n+1] - d[n] &= \Delta_T V \theta[n] \\ \theta[n+1] - \theta[n] &= \Delta_T \omega[n] \\ \omega[n] &= -K_p (r[n] - d[n]) - K_d \frac{d[n] - d[n-1]}{\Delta_T} \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \theta_\infty &= 0 \\ \omega_\infty &= 0 \\ \omega_\infty &= K_p (r_\infty - d_\infty) \end{aligned} \right\} \Rightarrow r_\infty = d_\infty$$

HOW ABOUT PERFECT "REJECTION" OF f_d (WHEN IT IS CONSTANT)?

$$\left. \begin{aligned} d[n+1] - d[n] &= \Delta_T V \theta[n] + \Delta_T f_d[n] \\ \theta[n+1] - \theta[n] &= \Delta_T (\omega[n] + f_{\theta,\infty}) \\ \omega[n] &= -K_p d[n] - K_d \frac{d[n] - d[n-1]}{\Delta_T} \end{aligned} \right\} \Rightarrow \left. \begin{aligned} V \theta_\infty &= -f_{d,\infty} \\ \omega_\infty &= -f_{\theta,\infty} \\ \omega_\infty &= -K_p d_\infty \end{aligned} \right\} \Rightarrow \underline{d_\infty = 0} \text{ GREAT!}$$

HOW ABOUT PERFECT "REJECTION" OF f_θ (WHEN IT IS CONSTANT)?

$$\left. \begin{aligned} d[n+1] - d[n] &= \Delta_T V \theta[n] \\ \theta[n+1] - \theta[n] &= \Delta_T (\omega[n] + f_\theta[n]) \\ \omega[n] &= -K_p d[n] - K_d \frac{d[n] - d[n-1]}{\Delta_T} \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \theta_\infty &= 0 \\ \omega_\infty &= -f_{\theta,\infty} \\ \omega_\infty &= -K_p d_\infty \end{aligned} \right\} \Rightarrow d_\infty = \frac{f_{\theta,\infty}}{K_p}$$

"CLOSE, BUT NO CIGAR"

??? CAN WE STILL DO BETTER ???

!!! IF WE WANT $r_\infty = d_\infty$ MAKE SURE THAT!!! $r[n] - d[n]$ IS AN INCREMENT OF A STATESIMPLY DEFINE $x_i[n]$ BY $x_i[n+1] - x_i[n] = \Delta_T (r[n] - d[n])$ JUST FOR BETTER SCALING WITH Δ_T PID CONTROLLER: $\omega[n] = K_p (r[n] - d[n]) - K_d \frac{d[n] - d[n-1]}{\Delta_T} = K_i x_i[n]$ NOW WE NEED 4 STATES: $x[n] = \begin{bmatrix} d[n-1] \\ d[n] \\ \theta[n] \\ x_i[n] \end{bmatrix}$, $y[n] = \begin{bmatrix} d[n] \\ d[n-1] \\ x_i[n] \end{bmatrix}$

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OPEN LOOP FOR PID:

$$x[n+1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & \Delta_T V & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\Delta_T & 0 & 1 \end{bmatrix} x[n] + \begin{bmatrix} 0 \\ 0 \\ \Delta_T \\ 0 \end{bmatrix} w[n] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta_T & 0 \\ 0 & 0 & \Delta_T \\ \Delta_T & 0 & 0 \end{bmatrix} w[n]$$

CLOSED LOOP FOR PID:

$$x[n+1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & \Delta_T V & 0 \\ K_d & -K_d \Delta_T K_p & 1 & -\Delta_T K_i \\ 0 & -\Delta_T & 0 & 1 \end{bmatrix} x[n] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta_T & 0 \\ 0 & 0 & \Delta_T \\ \Delta_T & 0 & 0 \end{bmatrix} w[n]$$

NEED ALL EIGS λ
TO HAVE $|\lambda| < 1$

ONCE STABILITY IS ASSURED, WE GUARANTEE THAT $d_\infty = [1 \ 0 \ 0] w_\infty = r_\infty$
WHEN $w[n] \rightarrow w_\infty$ AS $n \rightarrow +\infty$

QUESTION: WHEN $w[n]$ IS EXTENDED TO INCLUDE "d-SENSING NOISE"
 $f_s[n]$, SO THAT $d_{\text{meas}}[n] = d[n] + f_s[n]$, AND A STABILIZING
PID CONTROLLER IS

$$w[n] = K_p(r[n] - d_{\text{meas}}[n]) - K_d \frac{d_{\text{meas}}[n] - d_{\text{meas}}[n-1]}{\Delta_T} - K_i x_i[n], \text{ WHERE}$$

$$x_i[n+1] - x_i[n] = \Delta_T (r[n] - d_{\text{meas}}[n]),$$

WHAT IS THE VALUE OF $d_\infty - r_\infty$ IN TERMS OF w_∞

$$(\text{ASSUMING } w[n] = \begin{bmatrix} r[n] \\ f_d[n] \\ f_\theta[n] \\ f_s[n] \end{bmatrix} \rightarrow w_\infty = \begin{bmatrix} r_\infty \\ f_{d,\infty} \\ f_{\theta,\infty} \\ f_{s,\infty} \end{bmatrix} \text{ AS } n \rightarrow +\infty ?)$$

ANSWER: SINCE $0 = r_\infty - d_{\text{meas},\infty} = r_\infty - d_\infty - f_{s,\infty}$, WE HAVE
 $d_\infty - r_\infty = f_{s,\infty}$ (NO REJECTION OF SENSOR OFFSET $f_{s,\infty}$!)