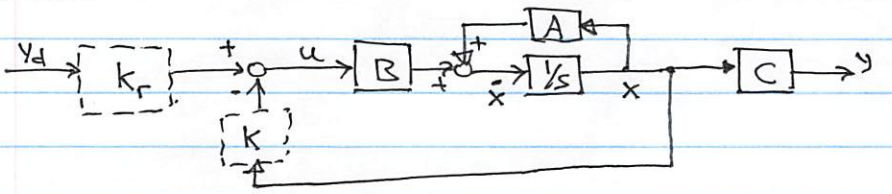
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STATE ESTIMATION : MOTIVATING EXAMPLES

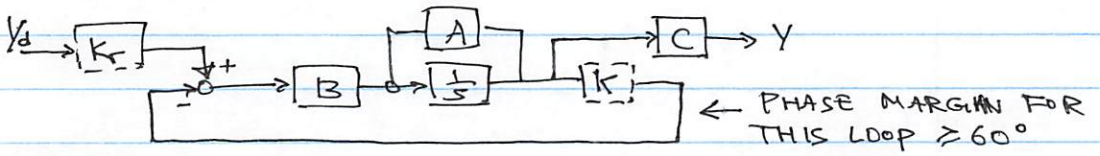
SO FAR : LEARNED TO DESIGN FULL STATE FEEDBACK USING LQR



1. FIGURE OUT A, B, C (MODEL)
2. CHOOSE COST OUTPUT $e = C_e x + D_e u$
3. `K, S, E = ct.lqr(A, B, Ce.T@Ce, De.T@De, Ce.T@De)` # lqr(A, B, $C_e^T C_e$, $D_e^T D_e$, ...
4. `Kr = -1/(C@np.linalg.solve(A-B@K, B))` # $-\frac{1}{C(A-BK)^{-1}B}$

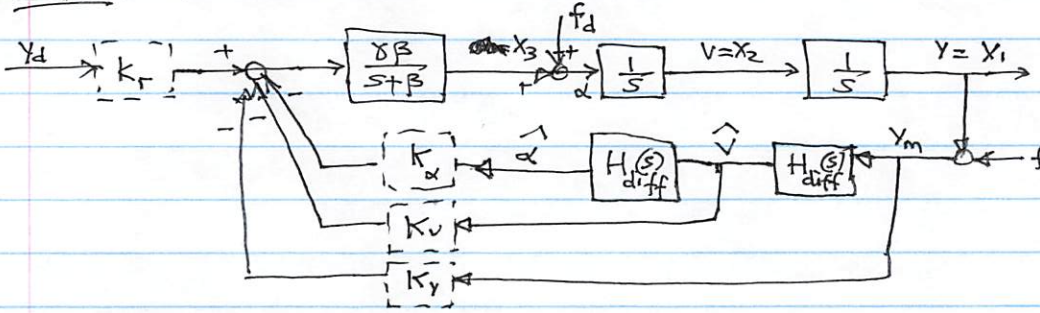
LQR BRINGS MANY BENEFITS :

* GUARANTEED STABILITY W/ GOOD ROBUSTNESS MARGINS (E.G., PHASE MARGIN $\geq 60°$)
* INTUITIVE INTERFACE : CHOOSE WHAT TO "PUSH DOWN"



HOWEVER, THERE IS A CATCH : WE USUALLY DO NOT MEASURE "THE WHOLE x"

Ex.0 (PROP. SYSTEM)



WE ALREADY ESTIMATE $\hat{V}$ BY "NUMERICAL DIFFERENTIATION"

$$\hat{V}(t) = \frac{y_m(t) - y_m(t - M\Delta_T)}{M\Delta_T} \quad \text{CORRESPONDS TO} \quad H_{diff}(s) = \frac{1 - e^{-M\Delta_T s}}{M\Delta_T} \approx \begin{cases} s, & |s| \ll M\Delta_T \\ O\left(\frac{1}{M\Delta_T}\right), & |s| \gg M\Delta_T \end{cases}$$

A (SOMEWHAT) SIMPLER NUMERICAL DIFFERENTIATOR:

$$H_{diff}(s) = \frac{LS}{s+L} \approx \begin{cases} s, & |s| \ll L \\ L, & |s| \gg L \end{cases}$$

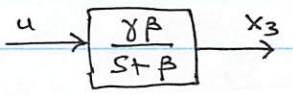
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THE ISSUE WITH "NUMERICAL DIFFERENTIATION" :

IT AMPLIFIES SENSOR NOISE TOO MUCH

(WHICH IN TURN REDUCES ROBUSTNESS TO MODELING ERRORS)

IS THERE A BETTER WAY?

Ex. 1: 

WE KNOW "u" PERFECTLY!

CAN WE USE THIS TO ESTIMATE X_3?

SURE! JUST "COPY" THE $\frac{\gamma\beta}{s+\beta}$ BLOCK!

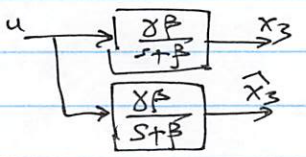
$$\left.\begin{aligned}\frac{d}{dt}X_3(t) &= \beta(\gamma u(t) - X_3(t))\\ \frac{d}{dt}\hat{X}_3(t) &= \beta(\gamma u(t) - \hat{X}_3(t))\end{aligned}\right\} \Rightarrow \frac{d}{dt}\left[X_3(t) - \hat{X}_3(t)\right] = -\beta\left[X_3(t) - \hat{X}_3(t)\right]$$

SINCE $\beta > 0$,

$$\lim_{t\to+\infty} X_3(t) - \hat{X}_3(t) = 0$$ (FAST, WHEN $\beta = 25$)

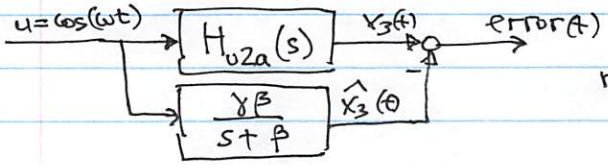
IS $\hat{X}_3(t) \equiv X(t)$? NOT REALLY, BECAUSE OF INTIAL CONDITIONS MISMATCH

(ALSO, OUR KNOWLEDGE OF γ, β IS IMPERFECT)

STILL, 

GENERATES $\hat{X}_3$ AS A "PERFECT" ESTIMATE OF X_3

WHAT IF $H_{u2a}(s) \neq \frac{\gamma\beta}{s+\beta}$?



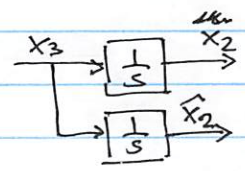
$$\max_t |\text{error}(t)| = \left|H_{u2a}(j\omega) - \frac{\gamma\beta}{j\omega+\beta}\right|$$

GOOD MEASURE OF APPROXIMATION QUALITY

THE IDEA WORKED WELL FOR X_3 ESTIMATION. HOW ABOUT X_2?

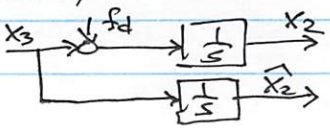
Ex. 2: 

"KIND OF" KNOWN, AS $\hat{X}_3$ IS (NEAR) PERFECT

CAN WE USE 

NO! $\frac{d}{dt}(\hat{X}_2 - X_2) = 0$ (IDEALLY), HENCE ERRORS DO NOT DIE OUT AS $t \to +\infty$

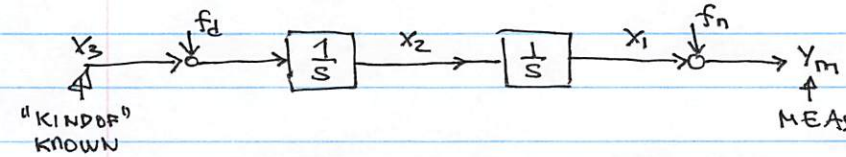
ACTUALLY, WITH FORCE DISTURBANCE FACTORED IN, THE OUTCOME IS WORSE



$$\frac{d}{dt}(X_2 - \hat{X}_2) = f_d \Rightarrow X_2 - \hat{X}_2$$ GROWS BY WAYS OF A "RANDOM WALK"

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HERE IS A BETTER IDEA FOR ESTIMATION:



ESTIMATING X_2 USING JUST X_3 FAILS
ESTIMATING X_2 USING JUST Y_m IS NOISY
} WHY NOT USING BOTH X_3 AND Y_m (INPUT AND OUTPUT) ???

THE "DYNAMICS" OF $X_2(t)$ IS UNSTABLE
CAN WE FIND L (A COEFFICIENT) SUCH THAT $g(t) = X_2(t) - LX_1(t)$ is "STABLE"

$$\left.\begin{aligned}\dot{X}_2 &= X_3 + f_d \\ \dot{X}_1 &= X_2\end{aligned}\right\} \quad \frac{d}{dt}\underbrace{[X_2 - LX_1]}_{g} = X_3 + f_d - LX_2 = -L[X_2 - LX_1] + X_3 + f_d - L^2X_1$$

$$= \underbrace{-L}_{\text{INDICATES STABILITY, WHEN } L>0}\underbrace{[X_2 - LX_1]}_{g} + \underbrace{X_3 - L^2 Y_m}_{\text{"KNOWN"}} + \underbrace{f_d + L^2 f_n}_{\text{UNKNOWN "NOISE"/ "DISTURBANCE"}}$$

$$\frac{d}{dt}g(t) = -Lg(t) + X_3 - L^2 Y_m + f_d + L^2 f_n$$

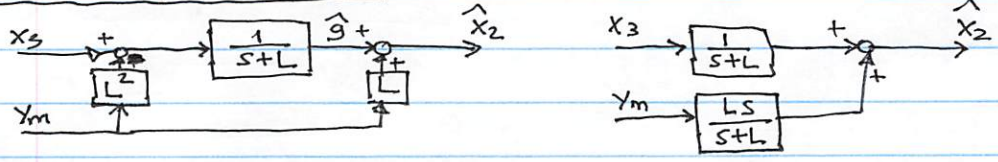
$$\frac{d}{dt}\hat{g}(t) = -L\hat{g}(t) + X_3 - L^2 Y_m \quad \longleftarrow \text{"ESTIMATOR" FOR } g$$

ERROR DYNAMICS: $\frac{d}{dt}[g(t) - \hat{g}(t)] = -L[g(t) - \hat{g}(t)] + f_d + L^2 f_n$: STABLE

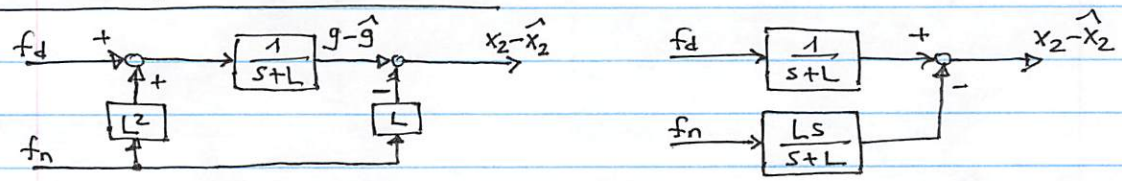
HOW TO GET $\hat{X}_2(t)$ FROM $g(t)$? NATURALLY,

$$\hat{X}_2(t) = g(t) + LY_m(t)$$

RESULTING ESTIMATOR:



RESULTING ERROR DYNAMICS:



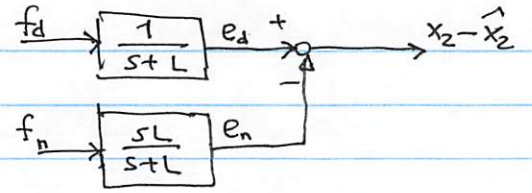
DOES THIS MAKE SENSE?

① WITH NO NOISES/DISTURBANCES: $X_3 = s^2 Y$, $Y_m = Y \Rightarrow X_2 = \left[\frac{s^2}{s+L} + \frac{Ls}{s+L}\right]Y = sY$

② WHEN L IS LARGE, AND X_3-TERM IS DROPPED, WE GET "NUMERICAL DIFFERENTIATOR" $H_{diff}(s) = \frac{Ls}{s+L}$

BUT WHAT IS THE BEST VALUE OF "L"?

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L IS THE "DESIGN PARAMETER"

HOW TO OPTIMIZE "SENSITIVITY" OF $\overbrace{x_2-\hat{x}_2}^{\text{ESTIMATION ERROR}}$ TO f_d, f_n ?

THIS DEPENDS ON OUR MODEL FOR f_d, f_n, AND A CHOSEN MEASURE OF SENSITIVITY

ONE APPROACH (OF MANY POSSIBLE):

(a) ALLOW f_d, f_n TO BE SINUSOIDS, OF ARBITRARY (AND POSSIBLY DIFFERENT) FREQUENCIES, AND MAGNITUDES BOUNDED BY F_d, F_n, RESPECTIVELY (F_d, F_n REPRESENT OUR KNOWLEDGE OF MAXIMAL "INTENSITY" OF f_d, f_n)

(b) MEASURE SENSITIVITY AS THE MAXIMAL MAGNITUDE OF $x_2-\hat{x}_2$ SUBJECT TO THE CONSTRAINTS FROM (a)

$f_d(t) = A_d \cos(\omega_d t + \varphi_d)$, $|A_d| \le F_d$, ω_d, φ_d ARBITRARY

$f_n(t) = A_n \cos(\omega_n t + \varphi_n)$, $|A_n| \le F_n$, ω_n, φ_n ARBITRARY

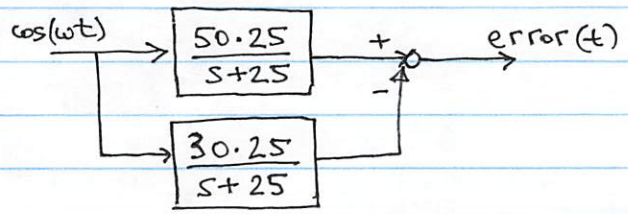
THEN $|e_d| \le \max_{\omega_d} \left|\frac{1}{j\omega_d + L}\right| F_d = \frac{F_d}{L}$

$|e_n| \le \max_{\omega_n} \left|\frac{L\omega_n}{j\omega_n + L}\right| F_n = LF_n$

$|x_2(t) - \hat{x}_2(t)| \le \frac{F_d}{L} + LF_n \longrightarrow \min$

MINIMUM ACHIEVED AT $L = \sqrt{\frac{F_d}{F_n}}$, AS $\frac{d}{dL}\left(\frac{F_d}{L} + LF_n\right) = -\frac{F_d}{L^2} + F_n$

QUIZ:



WHAT IS THE LARGEST (OVER ω) MAGNITUDE OF THE "error", IN STEATY STATE?

ANSWER SINCE $\max_{\omega} \left|\frac{50.25}{j\omega+25} - \frac{30.25}{j\omega+25}\right| = \max_{\omega} \frac{20.25}{\sqrt{\omega^2+25^2}} = 20$

THE ANSWER IS "20"