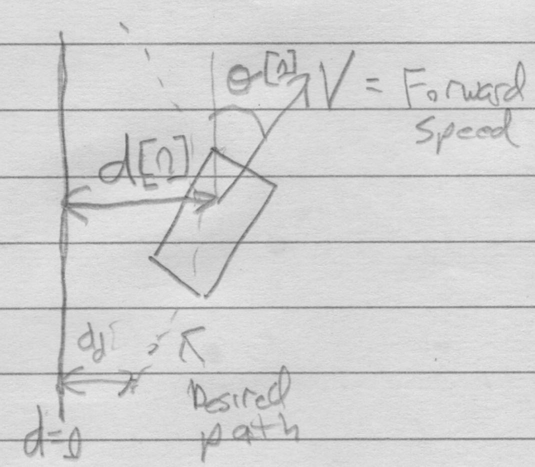
6.3100/2 9/28/26 1

More General Path Following Robot

Matrix Form (assumes $\theta \simeq \sin\theta$)



$$\underbrace{\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix}}_{X[n]} = \underbrace{\begin{bmatrix} 1 & V\Delta T \\ -K_p\Delta T & 1-\Delta T K_\theta \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}}_{X[n-1]} + \underbrace{\begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix}}_{B} \underbrace{d_d[n-1]}_{u[n-1]}$$

Model

$$d[n] = d[n-1] + \Delta T\, V\, \theta[n-1]$$

$$\theta[n] = \theta[n-1] + \Delta T \underbrace{\omega[n-1]}_{\text{control}}$$

Aside: Not general 2D Position control!

Generically $X[n] = \overset{n}{A}\, X[n-1] + \overset{1 \text{ if S.I.}}{B}\, u[n-1]$

Last Week Stability & Oscillations

$$\max_i |\lambda_i(A)| < 1$$

$$\lambda_i^n = M e^{j\theta n} = M^n \cos\theta n + j M^n \sin\theta n$$

Eigen condition Implies

$$x[n] = \underbrace{\sum A^n x[0]}_{\to 0 \text{ if } \max_i |\lambda_i(A)| < 1 \text{ as } n\to\infty} + \underbrace{\sum_{m=0}^{n} A^{m-1} B\, u[n-m]}_{\text{is finite as } n\to\infty}$$

Useful Definition Spectral radius

$$sp(A) \equiv \max_i |\lambda_i(A)|$$

$$\max_i |\lambda_i(A)| < 1 \equiv sp(A) < 1$$

Today - Disturbance steady-state and response

Steady-State

Assuming $sp(A) < 1$ & $\underbrace{u[n] = u[\infty]}_{\text{constant input}}$

what is $\lim_{n \to \infty} x[n]$?

$$\underbrace{\lim_{n\to\infty} x[n]}_{} = A \underbrace{\lim_{n\to\infty} [n-1]}_{} + B u[\infty]$$

$$x[\infty] \equiv A x[\infty] + B u[\infty]$$

or

$$(I - A) x[\infty] = B u[\infty]$$

or

$$x[\infty] = \underbrace{(I - A)^{-1}}_{} B u[\infty]$$

Assumes $I - A$ is invertible

* Stability implies $(I-A)^{-1}$ exists!

SMT for polynomials of order p

Spectral Mapping theorem (SMT)

If $f(x) = \sum_{j=1}^{p} \alpha_j x^j$ then $f(A) \equiv \sum_{j=1}^{p} \alpha_j A^j$ (e.g. those with taylor series)

and

$$\lambda(f(A)) = A(f(\lambda))$$

By SMT $\lambda(I - A) = 1 - \lambda(A)$ $(f(x) = 1 - x)$

If $\underbrace{\max_i |\lambda_i(A)|}_{sp(A)} < 1$

Then $1 - \lambda_i(A) \neq 0 \quad \forall i$

and $(I - A)^{-1}$ exists

Back to Robot

$$A = \begin{bmatrix} 1 & \Delta T \gamma \\ -\Delta T \gamma K_p & 1 - \Delta T K_\sigma \end{bmatrix}$$

$$\text{eig}(A) \Rightarrow \lambda_{1,2} = 1 - \frac{\Delta T K_\sigma}{2} \pm \Delta T \sqrt{\frac{K_\sigma^2}{4} - K_p \gamma}$$

$|\lambda_i| < 1$ for "good" K_p, K_σ pairs

Assuming $d_d[n] = d_d[\infty] \ \forall n$

$$\begin{bmatrix} d[\infty] \\ \sigma[\infty] \end{bmatrix} = \underbrace{\left(I - A \right)}^{-1} \underbrace{B} \, d_d[\infty]$$

$$= \begin{bmatrix} 0 & -\Delta T \gamma \\ \Delta T K_p & \Delta T K_\sigma \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix} d_d[\infty]$$

$$= \underbrace{\left(\frac{1}{\Delta T} \begin{bmatrix} \frac{K_\sigma}{\gamma K_p} & \frac{1}{K_p} \\ -\frac{1}{\gamma} & 0 \end{bmatrix} \right)}_{(I - A)^{-1}} \underbrace{\begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix}}_{B} d_d[\infty]$$

3A

How Fast does $x[n] \to x[\infty]$?

$$x[n] = A\,x[n-1] + B\,u[n-1] \leftarrow \text{Assume } u[n]=u[\infty]\ \forall n$$

$$x[\infty] = A\,x[\infty] + B\,u[\infty]$$

$$\underbrace{x[n]-x[\infty]}_{e[n]} = A\underbrace{\left(x[n-1]-x[\infty]\right)}_{e[n-1]} + B\underbrace{\left(u[n-1]-u[\infty]\right)}_{=0}$$

$$e[n] = A\,e[n-1] = A^n e[0]$$

OR

$$e_j[n] = C_{j1}\lambda_1^n + \dots + C_{jN}\lambda_N^n$$

$j \in [1, \dots N]$

$e[0]$ and A dependent constants

$$\Rightarrow \quad e[n] \to 0 \quad \text{like } \max_i |\lambda_i|^n = (sp(A))^n$$

Idea Pick K_p & K_θ to minimize $sp(A)$

Robot Example with PD

$$\operatorname{eig}\left(\begin{bmatrix} 1 & \gamma\Delta T \\ \Delta T K_p & 1-\Delta T K_\theta \end{bmatrix}\right) = \underbrace{1 - \frac{\Delta T K_\theta}{2} \pm \Delta T\sqrt{\frac{K_\theta^2}{4} - K_p\gamma}}_{\text{By Determinant Formula}}$$

Fastest $e[n] \to 0$?

Impractical $\left\{ \begin{array}{l} \lambda_1 = \lambda_2 = 0 \\ e[1] = 0! \end{array} \right.$ $\Rightarrow K_\theta = \dfrac{2}{\Delta T} \quad K_p = \dfrac{1}{\gamma}\left(\dfrac{1}{\Delta T}\right)^2$ $\left\{ \begin{array}{l} \text{If } \Delta T = \frac{1}{100} \\ K_p = \frac{10{,}000}{\gamma} \\ \text{Huge!} \end{array} \right.$

$$\overbrace{\begin{bmatrix} d[\infty] \\ \\ \theta[\infty] \end{bmatrix}}^{X[\infty]} = \overbrace{\begin{bmatrix} d_d[\infty] \\ \\ 0 \end{bmatrix}}^{(I-A)^{-1}BU[\infty]}$$

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$d[\infty] = d_d[\infty]$ ✓ $\theta[\infty] = \theta$ ✓

<u>Perfect</u> for any K_p, K_θ

(as long as K_p, K_θ stable)

<u>But</u> Our Model does <u>not</u> include disturbances (Like the fan blowing on the encoder in Lab 1).

<u>No disturbance system</u>

(I) $$X[n] = A X[n-1] + B U[n-1]$$

<u>Add another input to model disturbance</u>

(II) $$X_{dist}[n] = A X_{dist}[n-1] + B U[n-1] + B_d U_d[n-1]$$

How disturbance effects equations

Disturber

<u>Subtract II - I</u>

change in x due to disturbance

$$X_{dist}[n] - X[n] \equiv \tilde{e}[n] = A\,\tilde{e}[n-1] + B_d U_d[n-1]$$

Steady-State Disturbance Response

Assume $\underbrace{u[n] = u[\infty], \quad u_d[n] = u_d[\infty]}_{\forall n}$

$$\underline{x_{dist}[\infty] - x[\infty] \equiv \tilde{e}[\infty]} \leftarrow \text{steady-state disturbance response}$$

$$\tilde{e}[\infty] = A\,\tilde{e}[\infty] + B_d\, u_d[\infty]$$

or

$$\tilde{e}[\infty] = (I - A)^{-1} B_d\, u_d[\infty]$$

For the path-following robot

$$\begin{bmatrix} \tilde{e}_1[n] \\ \tilde{e}_2[n] \end{bmatrix} = \begin{bmatrix} d_{dist}[\infty] - d[\infty] \\ \theta_{dist}[\infty] - \theta[\infty] \end{bmatrix} = \underbrace{\frac{1}{\Delta T}\begin{pmatrix} \frac{K_\theta}{V K_p} & \frac{1}{K_p} \\ -1/V & 0 \end{pmatrix}}_{(I-A)^{-1}} \underbrace{\begin{bmatrix} B_{d_1} \\ B_{d_2} \end{bmatrix}}_{\text{impact on distance \& angle equation}} \underbrace{u_d[\infty]}_{\text{disturbance scale factor}}$$

Suppose wind blows robot

$$d_{dist}[n] = d_{dist}[n-1] + \Delta T V \theta[n-1] + \Delta T \underbrace{Wind_{vel}[n-1]}_{\text{wind velocity}}$$

$$\underbrace{\theta_{dist}[n] = \theta_{dist}[n-1] + \Delta T\, \omega[n-1]}_{\text{wind does not change angle eqn}}$$

robot distance including disturbances

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Wind Blowing Disturbance B_d

$$B_d = \begin{bmatrix} \Delta T \\ 0 \end{bmatrix}$$

only affects distance equation

$$\begin{bmatrix} d_{dist}[\infty] - d[\infty] \\ \theta_{dist}[\infty] - \theta[\infty] \end{bmatrix} = \frac{1}{\Delta T} \begin{pmatrix} \frac{K_\theta}{\gamma K_p} & \frac{1}{K_p} \\ -\frac{1}{\gamma} & 0 \end{pmatrix} \begin{bmatrix} \Delta T \\ 0 \end{bmatrix} U_d[\infty]$$

scales the Wind

$$= \begin{bmatrix} \frac{K_\theta}{\gamma K_p} U_d[\infty] \\ -\frac{U_d[\infty]}{\gamma} \end{bmatrix}$$

As $K_p \to \infty$ $d_{dist}[\infty] - d[\infty] \to 0$

Bigger K_p, Smaller Distance-Disturbance Response!

But

Robot angle Disturbance response is unaffected by K_p

&

$\Rightarrow$ Stability limits $\frac{K_\theta}{K_p}$ (Bigger K_p's need Bigger K_θ to stabilize)

Interpretations of $(I-A)^{-1}$

I) If there are no disturbances

$$x[\infty] = (I-A)^{-1} B u[\infty]$$

$(I-A)^{-1}_{i,j}$ = sensitivity of the i^{th} steady state $\equiv (x[\infty])_i$ to changes in j^{th} eqn $\equiv (Bu[\infty])_j$

For Disturbances Best $(I-A)^{-1}$ is zero but conflicts with steady state (if nonzero)

II) When there are disturbances

$$x_{dist}[\infty] - x[\infty] = (I-A)^{-1} B_{dist} u_{dist}[\infty]$$

$(I-A)^{-1}_{i,j}$ = sensitivity of the i^{th} steady-state to disturbances in the j^{th} equation

What disturbances can you reject

No 1) Can you reject all disturbances for all states?

$$\equiv \quad (I - A)^{-1} B_{dist} = 0 \quad \text{for all } B_{dist}$$

impossible! Requires $(I - A)^{-1} = 0$ which contradicts invertibility!

No 2) Can you reject one particular disturbance for all states?

$$(I - A)^{-1} B_d = 0 \quad \text{for one } B_d$$

impossible if $(I-A)^{-1} B_d = 0$ then $(I-A)^{-1}$ has a zero eigenvalue contradicts invertibility

No 3) Can you reject all disturbances for one state?

$$\left((I - A)^{-1} B_d\right)_i = 0 \quad \text{for all } B_d$$

impossible if $\left((I-A)^{-1} B_d\right)_i = 0$ for all B_d then ith row of $(I-A)^{-1} = 0$ contradicts invertibility!

4) If $(I-A)^{-1} B = X[\infty] \neq 0$ then it is impossible to reject B parallel disturbances.