6

THE DYNAMICS OF FEEDBACK SYSTEMS

6.0 Introduction

Adding feedback around an existing system, $K(s)$, yields a new system that not only may be less sensitive to variations and distortions of $K(s)$ (if, as we saw in Chapter 5, the loop gain is large), but also in general has different pole-zero locations than $K(s)$ and thus a different dynamic behavior. The effect of feedback on system dynamics was extensively studied in the 1940's and 1950's as an approach to the design of *control systems*. Indeed, Norbert Wiener* and his followers made feedback control under the name "cybernetics" one of the cornerstones of a virtually complete philosophical system embracing everything from automation, to physiological and psychological processes, the behavior of economic and social systems, and even the discovery of principles of ethics. Time and the development of alternative perspectives have somewhat diminished this glorious vision, but feedback control remains both an important concept and a useful design technique.

6.1 Inverse Systems

The dynamic effect of feedback depends on the magnitude of the loop gain, $\beta(s)K(s)$, over the interesting range of frequencies. If the amount of feedback is "small" ($|\beta(s)K(s)| \ll 1$), then

$$H(s) = \frac{K(s)}{1+\beta(s)K(s)} \approx K(s) \qquad (6.1\text{–}1)$$

and the dynamic effect of feedback is also small; the poles and zeros of $H(s)$ will usually be in nearly the same locations as the poles and zeros of $K(s)$. On the other hand, if the amount of feedback is "large" ($|\beta(s)K(s)| \gg 1$), then

$$H(s) = \frac{K(s)}{1+\beta(s)K(s)} \approx \frac{1}{\beta(s)}. \qquad (6.1\text{–}2)$$

*N. Wiener, *Cybernetics, or Control and Communication in the Animal and the Machine* (New York, NY: John Wiley, 1948).

The dynamic effect of feedback in this case can be enormous, since the natural frequencies of $H(s)$ are approximately the poles of $1/\beta(s)$ rather than the poles of $K(s)$.

In the case of a large loop gain, the system function of the feedback system is approximately the inverse of the system function of the feedback path. Employing a feedback system with a large loop gain to realize a system function inverse to that of a given system is often a useful design technique. An inverse system is typically desired to help overcome or compensate for the dynamic deficiencies of some communication, measurement, or control device. For example, suppose a sensor responds sluggishly to the changes in some quantity $x(t)$. In such cases, the transform of the sensor output can often be described as the product, $X(s)H_0(s)$, rather than simply the transform, $X(s)$, of the desired signal $x(t)$. The system function $H_0(s)$ thus describes the corrupting effect of the sluggish sensor. Suppose, however, we could arrange to connect the sensor in cascade with a system whose system function is $1/H_0(s)$; then the transform of the overall response would be $X(s)H_0(s)(1/H_0(s)) = X(s)$, that is, the inverse system $1/H_0(s)$ would recover the uncorrupted signal $x(t)$ from the distorted sluggish output waveform of the sensor.* Feedback is a particularly effective way to realize an inverse system if the details of $H_0(s)$ are unknown or if they change with time or environmental conditions; a replica of the original system in the feedback path should presumably behave similarly under similar conditions. Feedback may also be useful for synthesizing an inverse system if a system with system function $H_0(s)$ is easier to build for some reason than a system with system function $1/H_0(s)$. The following examples illustrate several applications of this idea.

Example 6.1–1

It is easy to show by impedance manipulations that the open-circuit voltage transfer ratio of the *bridged-T network* in Figure 6.1–1 is as given with frequency response and pole-zero locations as shown (for large values of a). The inverse of $H_0(s)$ can be realized approximately by placing the bridged-T network in the feedback path around an amplifier, as shown in Figure 6.1–2. This circuit will have the system function $H_2(s) \approx 1/H_0(s)$, with pole-zero locations and frequency response as shown. Since the poles of $H_2(s)$ are located at complex frequencies, a passive circuit realizing $H_2(s)$ would have to contain inductors. Several other circuits using feedback to achieve complex poles with RC elements have been described previously (see Examples 1.3–2, 1.4–2, 4.2–1, and Problems 1.6, 3.11, 4.9, 4.15).

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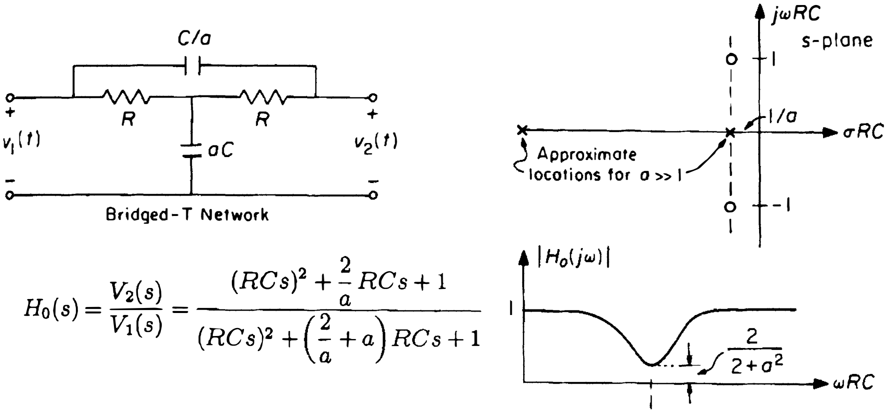
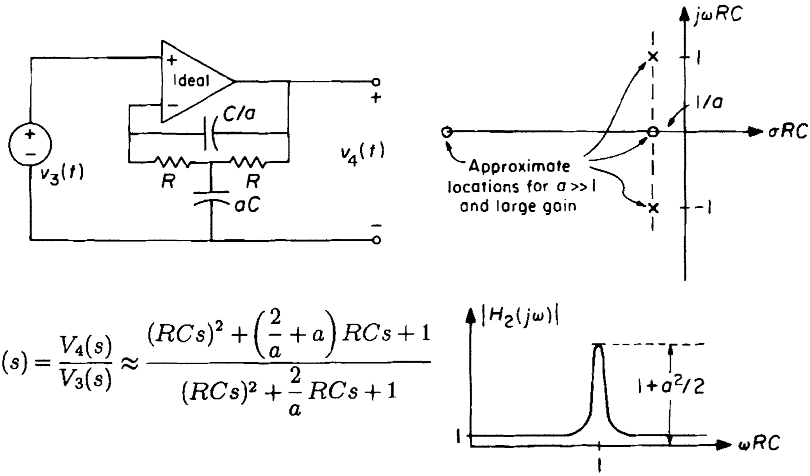
*The process of operating on the output of an LTI system to recover its input is also called *deconvolution*, for reasons to be described in Chapter 10. Note that, for reasons discussed in Chapter 3, an inverse system cannot usually be constructed by simply driving the output terminals of the original system and observing the signal at the input terminals.



$$H_0(s) = \frac{V_2(s)}{V_1(s)} = \frac{(RCs)^2 + \frac{2}{a}RCs + 1}{(RCs)^2 + \left(\frac{2}{a} + a\right)RCs + 1}$$

Figure 6.1–1. Bridged-T network.



$$H_2(s) = \frac{V_4(s)}{V_3(s)} \approx \frac{(RCs)^2 + \left(\frac{2}{a} + a\right)RCs + 1}{(RCs)^2 + \frac{2}{a}RCs + 1}$$

Figure 6.1–2. Approximate inverse of bridged-T transfer function.

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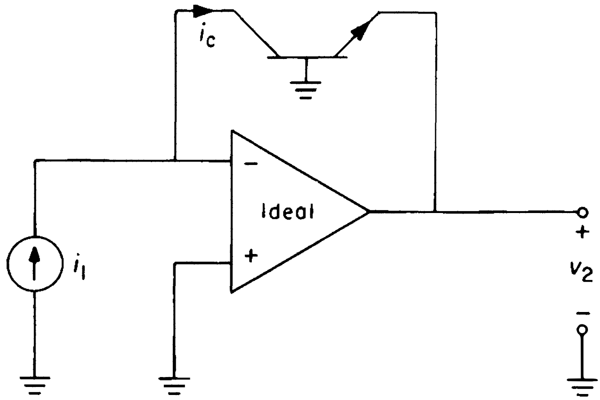
Note that, since the zeros of a passive transfer function may be in the right half-plane, attempts to realize the inverse of such a system may lead to an unstable structure with right half-plane poles. Instabilities in feedback systems can arise in many ways and set limits on the benefits that can be achieved with feedback, as we shall discuss in Section 6.3.

Example 6.1–2

The use of feedback to construct inverse systems is not limited to linear systems. Thus many types of transistors show exponential dependence of collector current on base-emitter voltage near zero collector-base voltage. The exponential law holds for many decades. (A similar relationship holds for diodes, but over a smaller dynamic range.) The circuit shown in Figure 6.1–3 then yields an accurate logarithmic characteristic that can be exploited in analog multipliers and many other devices.



$$i_1 = i_C = I_s e^{-qv_2/kT} \quad \text{or} \quad v_2 = -\frac{kT}{q} \ln \frac{i_1}{I_s}$$

Figure 6.1–3. A logarithmic amplifier.

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6.2 Feedback Effects on Bandwidth and Response Time

If the magnitude of the loop gain $|\beta(s)K(s)|$ is not much greater than 1, then the overall feedback system response is not $1/\beta(s)$, independent of $K(s)$. Nevertheless, the overall response may still be dramatically different from the response of the feedforward path alone, as the following examples illustrate.

Example 6.2–1

Figure 6.2–1 shows the open-loop gain of a typical 741 op-amp. This is a Bode plot with low-frequency gain $\approx 2 \times 10^5$, 6 dB/octave high-frequency slope, and a half-power bandwidth of about 6 Hz $\approx$ 40 rad/sec. We may thus represent the op-amp up to fairly high frequencies by the equivalent circuit shown in Figure 6.2–2 with $K(s) = \dfrac{8 \times 10^6}{s + 40}$.

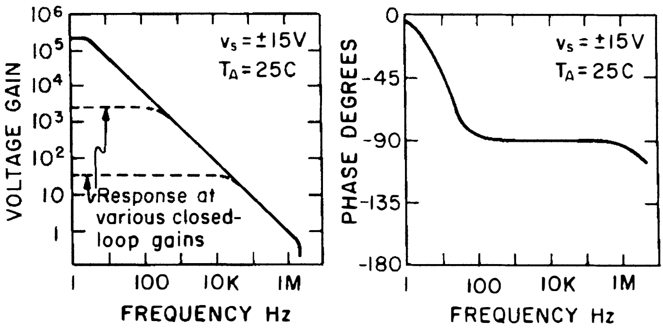

Figure 6.2–1. Open-loop voltage gain and phase response of a 741 op-amp as functions of frequency.

If the op-amp is now connected in the non-inverting amplifier circuit shown in Figure 6.2–2 with

$$\beta = \frac{R_1}{R_1 + R_2}$$

then the closed-loop system function is

$$H(s) = \frac{V_2(s)}{V_0(s)} = \frac{K(s)}{1 + \beta K(s)} = \frac{8 \times 10^6}{s + (40 + 8 \times 10^6 \beta)}.$$

The half-power bandwidth has thus been raised to $(40 + 8 \times 10^6 \beta)$ rad/sec, which for typical values of β is very much larger than the open-loop bandwidth of 40 rad/sec.

It is interesting and useful to observe that the product of the low-frequency gain,

$$H(0) = \frac{8 \times 10^6}{40 + 8 \times 10^6 \beta}$$

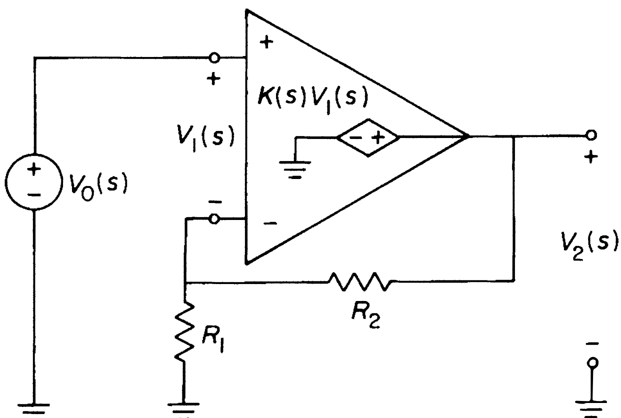

Figure 6.2–2. Non-inverting amplifier.

and the half-power bandwidth is a constant,

$$\text{gain} \times \text{bandwidth (Hz)} = \frac{8 \times 10^6}{2\pi} \approx 1.2 \times 10^6 \text{ Hz}$$

independent of β. (The value of this constant is often given in op-amp specification listings as the *unit-gain bandwidth.*) The Bode plots for the closed-loop frequency response thus have the form shown by the dotted lines in the preceding figure—the smaller the required gain, the greater the resulting bandwidth. For example, a 741 op-amp in a feedback circuit yielding a closed-loop gain of 100 will have a half-power bandwidth equal to $1.2 \times 10^6/100 = 12$ kHz. For further discussion of the frequency response effects of op-amps, see Problem 6.9.

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Example 6.2–2

As a second example we shall study the effect of feedback on the sluggish response of a motor speed controller. Mathematically, this system is almost identical to the feedback amplifier examined in the last example, but the context is different and we shall emphasize the step response rather than the frequency response.

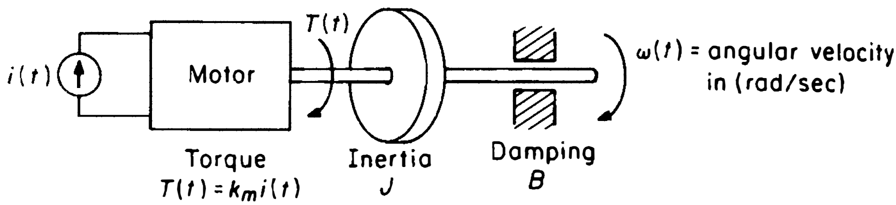

Figure 6.2–3. Description of an electric motor and its load.

The basic system to be controlled is shown in Figure 6.2–3. For simplicity, the electric motor will be idealized as producing a torque $T(t)$ proportional to armature current. Mechanically, the motor and load together will be represented by an inertia J and a damping term B. Let us first study the way in which the shaft speed $\omega(t)$ responds to changes in the current $i(t)$. Balancing drive and reaction torques yields

$$T(t) = k_m i(t) = J\frac{d\omega(t)}{dt} + B\omega(t).$$

The system function relating shaft speed and current is then

$$H_m(s) = \frac{\Omega(s)}{I(s)} = \frac{k_m}{Js + B}.$$

If a step of current $i(t) = Iu(t)$ is applied from rest, then $I(s) = I/s$ and the ZSR is

$$\Omega(s) = H_m(s)I(s) = \frac{k_m I/J}{s(s + B/J)} = \frac{k_m I/B}{s} - \frac{k_m I/B}{s + B/J}$$

or

$$\omega(t) = \omega_\infty(1 - e^{-t/t_0})u(t)$$

where the final velocity is $\omega_\infty = k_m I/B$ and the time constant is $t_0 = J/B$. The step drive and motor response are shown in Figure 6.2–4. If the inertia J of the system is large, the step response will be sluggish. Moreover, both the response time constant and the final speed depend directly on the mechanical damping B, which is frequently an unreliable parameter, sensitive to aging, temperature, etc.

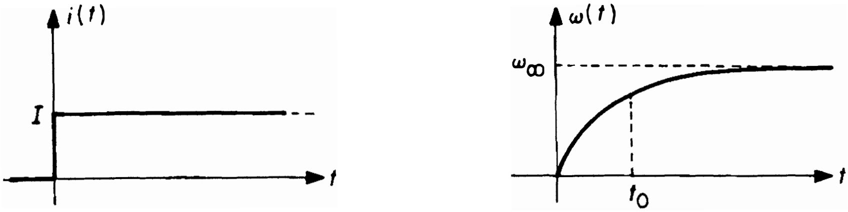

Figure 6.2–4. Motor speed response to a step current input.

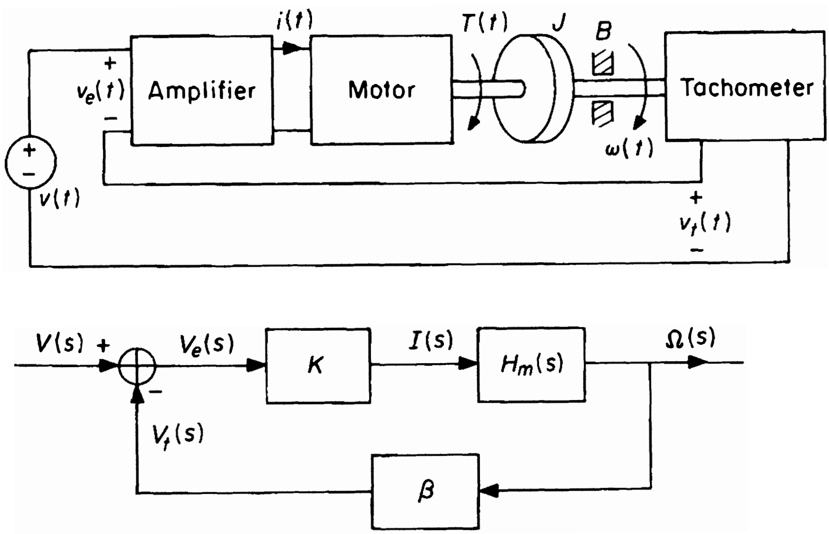

Figure 6.2–5. Motor controller with velocity feedback.

A more rapid and reliable response can be obtained by adding feedback to the motor as shown in the upper part of Figure 6.2–5. A tachometer is a measuring device (basically an electrical generator) whose output voltage is proportional to input shaft velocity:

$$v_t(t) = \beta\omega(t).$$

The amplifier is described by the equation

$$i(t) = K v_e(t).$$

Hence the system can be represented by the block diagram in the lower part of Figure 6.2–5. The form of the overall system function $H(s)$ is the same as for the unfedback system $H_m(s)$, but the pole is at $-(B+\beta K k_m)/J$ instead of $-B/J$:

$$H(s) = \frac{\Omega(s)}{V(s)} = \frac{KH_m(s)}{1+\beta K H_m(s)} = \frac{Kk_m/J}{s+(B+\beta K k_m)/J}.$$

Consequently, for $\beta K > 0$ the bandwidth is larger and the step response

$$\omega(t) = \hat{\omega}_\infty(1-e^{-t/\hat{t}_0})u(t) \qquad \text{where} \qquad \hat{\omega}_\infty = \frac{Kk_mV}{B+\beta K k_m},\ \hat{t}_0 = \frac{J}{B+\beta K k_m}$$

is faster than before, as shown in Figure 6.2–6. Note also that both the time constant of the response and the final motor speed in the feedback system are largely independent of the damping B if βK is large enough.

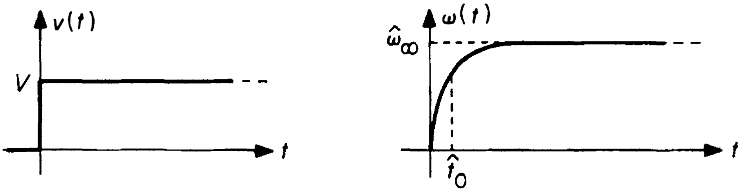

Figure 6.2–6. Response of the feedback system to a step input.

As $\beta K \to \infty$, the response time of the feedback system to an input step change in desired speed theoretically goes to zero, and the final speed depends only on the tachometer calibration β. What limits the improvements that can be achieved in practice? One problem is that as βK grows, so does the peak input current to the motor. To see this, treat $I(s)$ as the output of the feedback system and compute

$$\begin{aligned} I(s) &= \frac{KV(s)}{1+\beta K H_m(s)} = \frac{K(s+B/J)V}{s(s+B/J+\beta K k_m/J)} \\ &= \frac{\hat{\omega}_\infty B/k_m}{s} + \frac{\hat{\omega}_\infty \beta K}{s+B/J+\beta K k_m/J}. \end{aligned}$$

The corresponding time function $i(t) = \hat{\omega}_\infty[B/k_m + \beta K e^{-t/\hat{t}_0}]u(t)$ is sketched in Figure 6.2–7.

Physically, a large initial current is necessary to provide sufficient torque to quickly overcome the inertia of the motor and load. However, burnout and saturation effects

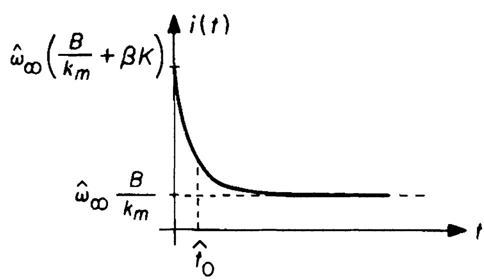

Figure 6.2–7. Armature current for a step input to the feedback controller.

limit the applicable range of our simplified representation of the motor to input currents less in magnitude than some maximum $I_{\max}$. Assuming that the worst transient results from a sudden reversal from maximum velocity $\omega_{\max}$ in one direction to the same speed in the opposite direction, we leave it as a simple exercise to show that the loop gain must not exceed

$$\beta K < \frac{I_{\max}}{2\omega_{\max}} - \frac{B}{k_m}.$$

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Example 6.2–3

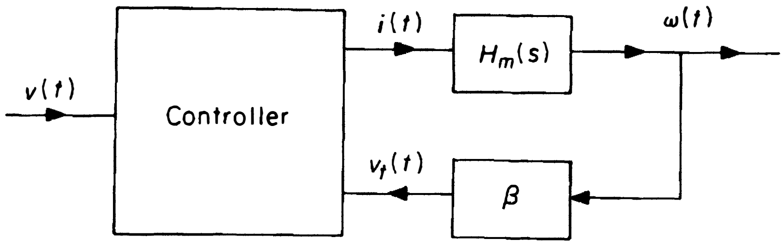


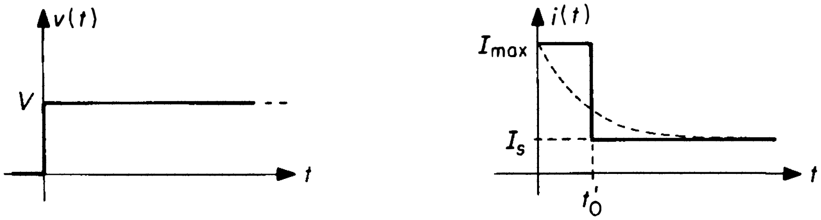
Figure 6.2–8. A non-linear motor controller.

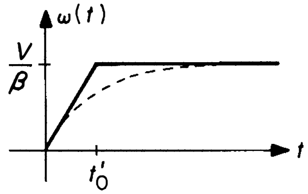
In control systems of the kind described in Example 6.2–2, where certain internal variables are constrained, improved performance can often be obtained by replacing the linear feedback system with one containing a non-linear *controller*, as illustrated in Figure 6.2–8. The properties of the controller are described by the following equation:

$$i(t) = \begin{cases} +I_{\max}, & v(t) > v_t(t) \\ \dfrac{Bv(t)}{\beta k_m}, & v(t) = v_t(t) \\ -I_{\max}, & v(t) < v_t(t). \end{cases}$$

Whenever the speed differs from that desired, maximum current (torque) is applied to correct the error as soon as possible. As soon as the speed reaches the desired value, the current is set equal to the value required to maintain that speed in the steady state.* The performance of this system in response to a step change from rest is readily calculated as in Example 6.2–2 and is shown in Figure 6.2–9.

*In practice, such an extreme example of a "bang-bang" controller would probably not be very satisfactory—drift in the value of $B/\beta k_m$ or small rapid fluctuations (noise) in either the input or the output in the steady state would yield large rapid "hunting" swings in $i(t)$. A more practical scheme would include a region where the error, $v(t) - v_t(t)$, is small but not zero and in which $i(t)$ is proportional to the error, as in the linear feedback system previously proposed.





$$I_s = \frac{BV}{\beta k_m}$$

$$t_0' = \frac{J}{B} \ln\left[1 \Big/ \left(1 - \frac{I_s}{I_{\max}}\right)\right]$$

$$\approx \frac{J}{B}\frac{I_s}{I_{\max}} \qquad \text{for } I_s \ll I_{\max}$$

$$\omega(t) = \begin{cases} \dfrac{k_m I_{\max}}{B}(1 - e^{-Bt/J}), & 0 < t < t_0' \\ \dfrac{V}{\beta}, & t > t_0' \end{cases}$$

Figure 6.2–9. Step response of non-linear controller.

For comparison, the response of the linear feedback system is shown dotted in Figure 6.2–9, assuming β and K are adjusted to give the same final velocity and the same maximum current. Under these circumstances the time constant of the linear response is $\hat{t}_0 = JI_s/BI_{\max}$. Thus, provided that $I_s \ll I_{\max}$, the time t_0' that the non-linear controller takes to reach the desired final speed is equal to the time $\hat{t}_0$ that the linear feedback controller takes to get within $1/e$ of the desired final value. The initial rate of change of $\omega(t)$ is the same for both systems. However, the non-linear-controller system maintains (almost) this same initial rate for some time, and thus reaches the desired speed in substantially less time than is required for the linear feedback system to approach this speed. And the comparison would be even more favorable to the non-linear controller if the gain of the linear system had not been optimally chosen for precisely this speed change. It is because of improvements of this kind that "controller" approaches to control-system design have today largely replaced the "feedback" approach popular a few decades ago.

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6.3 Stability

The available dynamic range of certain internal system variables often limits the practical improvements in performance that can be achieved with linear feedback—as Example 6.2–2 showed. But another difficulty is perhaps even more common. As the loop gain is increased, poles of the closed-loop system may be moved into the right half-s-plane, yielding an unstable system.

In general, a physical system is said to be *stable* if any small perturbation in the conditions under which it is being operated (in the drives or initial state) produces only a small perturbation in the behavior (response) of the system. For linear systems, an equivalent statement is that Bounded Inputs in the zero state yield Bounded Outputs (BIBO).* Evidently, a necessary condition for BIBO stability (as mentioned in Chapter 4) is that the natural frequencies (the poles of the system function) must lie in the left half-plane (l.h.p.); otherwise a step function (a bounded input) would initiate an unbounded response. For circuits such as we have been considering—composed of a finite number of elements and having system functions that are well-behaved at $s = \infty$—the l.h.p. condition is also sufficient. (We shall discuss a generalization in Chapter 10.) Circuits composed of positive R, L, and C elements (passive circuits) are always stable, as noted in Chapter 4. (Pure lossless circuits—containing L's and C's but no R's— are perhaps marginally stable, since such circuits have poles on the $j\omega$ axis and will give unbounded responses to bounded sinusoidal excitation at their natural frequencies; see Example 10.3–1.) Circuits containing controlled sources, however, may be either stable or unstable; typically they will be stable if the controlled-source gains are small enough (since for zero gains the circuit is composed of positive R's, L's, and C's only) but may become unstable as the gain(s) are increased beyond some critical value(s). Thus it is not surprising that stability problems often limit the advantages of feedback, since such improvements depend, as we have seen, on large loop gains.

For a simple feedback system with system function

$$H(s) = \frac{K(s)}{1 + \beta(s)K(s)}$$

the natural frequencies are the zeros of $1 + \beta(s)K(s)$.† An important part of the feedback system design process is to explore the zero locations of $1 + \beta(s)K(s)$ as a function of various design parameters (such as the magnitude of the forward gain or the fraction of the output fed back) and to discover (if possible) ways to modify the design if the desired performance is not realized with a stable system.

Example 6.3–1

Let $\beta(s)K(s) = \mu/(\tau s + 1)^3$, which might describe the moderate-frequency behavior of the loop gain of a 3-stage feedback amplifier. The quantity μ is the loop gain at low frequencies; to reduce output distortion we wish μ to be as large as possible. Stability requires that the zeros of $1 + \beta(s)K(s) = [(\tau s + 1)^3 + \mu]/(\tau s + 1)^3$ have negative real

*More fundamentally, we should demonstrate for a stable system that all internal variab well as the designated outputs remain bounded. It is enough to show that the state variables remain bounded; this is called *stability in the sense of Liapunov* as contrasted with BIBO stability. A non-linear system may be stable for some inputs and initial states but not for others. We shall give an example of the importance of Liapunov stability in the next section.

†Since $K(s)$ appears in both numerator and denominator, poles of $K(s)$ will not be poles of $H(s)$ unless they are also zeros of $\beta(s)$. It is also possible for a zero of $1 + \beta(s)K(s)$ not to be a natural frequency of $H(s)$ if $K(s)$ and $\beta(s)$ have a zero and a pole, respectively, at the zero of $1 + \beta(s)K(s)$.

parts. There are general formulas for finding the roots of a cubic polynomial, but they are clumsy. In the present case, it is clear that the numerator is zero when

$$\tau s + 1 = (-\mu)^{1/3}.$$

That is, the roots of $(\tau s + 1)^3 + \mu = 0$ are

$$s = \frac{1}{\tau}\left(-1 - \mu^{1/3}\right),\ \frac{1}{\tau}\left(-1 + \mu^{1/3} e^{j\pi/3}\right),\ \frac{1}{\tau}\left(-1 + \mu^{1/3} e^{-j\pi/3}\right).$$

The loci of the roots, as functions of μ, are shown in Figure 6.3–1. For $-1 < \mu < 8$ the roots all lie in the l.h.p.; for $\mu < -1$ there is one real zero in the r.h.p., and for $\mu > 8$ there are two conjugate zeros in the r.h.p. If the system is to be stable, then, the low-frequency loop gain cannot exceed 8. In practice, a value substantially less than 8 would probably be chosen—to give a stability *margin*, and to yield a transient or step response without the substantial *ringing* or *overshoot* that often results from a system pole pair close to the $j\omega$-axis.

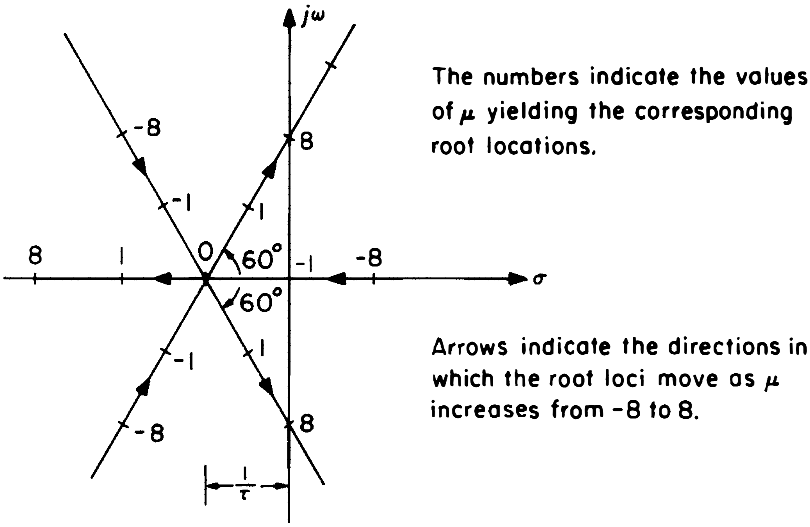

Figure 6.3–1. Root loci of $(\tau s + 1)^3 + \mu$.

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Sketching the root loci of $1 + \beta(s)K(s) = 0$ as above can be an important aid to feedback system design. Fortunately, simple techniques exist for finding the approximate pattern of the root loci without actually finding the roots analytically—see, for example, J. K. Roberge, *Operational Amplifiers: Theory and Practice* (New York, NY: Wiley, 1975) Chapter IV, or almost any book on linear control theory.

A test, called the *Routh test*, is also available to determine whether all the roots of a polynomial lie in the l.h.p. The Routh test proceeds directly from the coefficients of the various terms in the polynomial and does not require finding the roots analytically. Unfortunately, the Routh test gives no information about the location of the roots except the half-plane in which they lie; it thus provides no insight into relative measures of stability that depend on the proximity of the poles to the $j\omega$-axis, or the balancing of time and frequency responses. Nevertheless, it is frequently useful in conjunction with other procedures.

Certain special aspects of the Routh test, however, have broad applicability and can be summarized as follows:

1. Necessary conditions for a polynomial to have all its roots in the l.h.p. are:
   i) All of the terms must have the same sign;
   ii) All of the powers between the highest and the lowest must have nonzero coefficients, unless all even-power or all odd-power terms are missing.
2. For quadratic polynomials these conditions are also sufficient.
3. For a cubic polynomial, $s^3+\alpha s^2+\beta s+\gamma$, necessary and sufficient conditions (NASC) are $\alpha, \beta, \gamma > 0$ and $\beta > \gamma/\alpha$.

The following example illustrates the application of these rules.

Example 6.3–2

1. $s^2 - 3s + 2$ does not have all its roots in the l.h.p. because two of the terms have + signs and one has a − sign. Indeed, $s^2 - 3s + 2 = (s-2)(s-1)$.
2. $s^5 + s^3 + 10s^2$ does not have all its roots in the l.h.p. because the s^4 term is missing. Indeed, $s^5+s^3+10s^2 = s^2(s+2)(s-1+j2)(s-1-j2)$. Also we note that $s^5+s^3+10s^2 = s^2(s^3+s+10)$ and the cubic factor does not satisfy the NASC for cubics in the third rule above.
3. $s^3 + s^2 + 4s + 30$ satisfies the necessary conditions under the first rule above, but in fact $s^3 + s^2 + 4s + 30 = (s+3)(s-1+j3)(s-1-j3)$ so these conditions are not sufficient. Of course, this polynomial does not satisfy the NASC in the third rule.
4. $s^4 + 13s^2 + 36$ satisfies all the necessary conditions and in fact has all of its roots on the $j\omega$ axis: $s^4 + 13s^2 + 36 = (s+j2)(s-j2)(s+j3)(s-j3)$. Note that the roots of the polynomial $z^2+13z+36$ are negative real, as is clearly required if the s-roots are to be pure imaginary.
5. $s^4 + 6s^2 + 25$ satisfies all the necessary conditions but has its roots in both half-planes at mirror-image points: $s^4+6s^2+25 = (s-1+j2)(s-1-j2)(s+1+j2)(s+1-j2)$. Note that the polynomial $z^2+6z+25$ has complex roots.

▸▸▸

Example 6.3–3

The characteristic polynomial in Example 6.3–1 was

$$(\tau s+1)^3+\mu = \tau^3\left(s^3+\frac{3}{\tau}s^2+\frac{3}{\tau^2}s+\frac{1+\mu}{\tau^3}\right).$$

The NASC from the third rule for cubics above gives

$$\frac{3}{\tau^2} > \frac{1+\mu}{\tau^3}\frac{\tau}{3}$$

or

$$\mu < 8$$

when μ is positive. When μ is negative, we must satisfy $1 + \mu > 0$ or the last term will have a negative coefficient. Hence the stable region is

$$-1 < \mu < 8$$

as previously determined by plotting the locus of the roots.
▶ ▶ ▶

Example 6.3–4

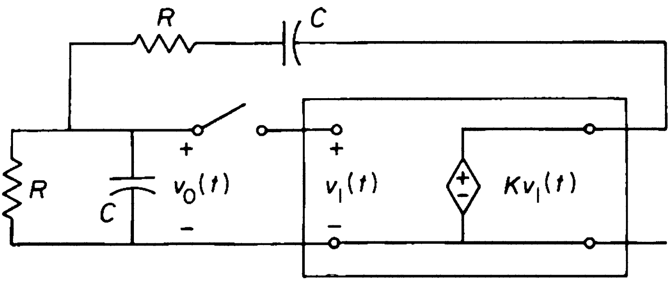

Figure 6.3–2. A simple *RC* audio oscillator.

Figure 6.3–2 is a much simplified diagram of one of the first successful products of the Hewlett-Packard Company, an *RC audio oscillator.* To determine the condition for oscillation, open the switch and compute the loop gain:

$$\frac{V_0(s)}{V_1(s)} = \frac{K\dfrac{R\left(\frac{1}{Cs}\right)}{R+\left(\frac{1}{Cs}\right)}}{\dfrac{R\left(\frac{1}{Cs}\right)}{R+\left(\frac{1}{Cs}\right)} + R + \left(\frac{1}{Cs}\right)} = \frac{K(RCs)}{(RCs)^2 + 3(RCs) + 1}.$$

With the switch closed, the circuit will be unstable if the values of s for which the open-loop gain $V_0(s)/V_1(s) = 1$ lie in the r.h.p. Equivalently these are the values of s for which

$$(RCs)^2 + (3 - K)(RCs) + 1 = 0.$$

If $K = 3$, the natural frequencies will lie precisely on the $j\omega$-axis, so that—once started—sinusoidal oscillations will be produced. In practice, K is designed to be slightly greater than 3 so that the poles lie inside the right half-plane and the oscillator is self-starting. As the amplitude of the oscillations builds up, a resistor (not shown) in the amplifier changes value slightly, reducing the gain; the amplitude stabilizes at the value that yields an effective K of 3. The frequency of the oscillation is then the root of

$$(RCs)^2 + 1 = 0$$

or $\omega = 1/RC$. The frequency can be changed over a very wide range by making the two resistors variable and ganging them together.

▶▶▶

The root-locus and Routh stability tests for feedback systems require that $\beta(s)$ and $K(s)$ be known rational functions of s. But often information about $\beta(s)$ and $K(s)$ may be given in other forms, such as non-rational functions or experimentally determined magnitude and angle plots of open-loop frequency response. In such cases a graphical stability test due to Nyquist is most useful, as discussed in the appendix to this chapter.

If we examine the behavior of a closed-loop system in the sinusoidal steady state, it is easy to explain physically (rather than just mathematically) why large values of loop gain may lead to instabilities. Thus suppose there is a frequency $s = j\omega$ for which the loop gain $\beta(j\omega)K(j\omega)$ has an angle of 180°. Then (including the minus sign in the comparator) the signal returned and added to the input will—at this frequency—be precisely in phase with the input. If, furthermore, the magnitude of the loop gain at this frequency is equal to unity, then the amplitude of this returned signal will be equal to the input amplitude. Once such a returned signal is established, the input could be set equal to zero and the signal in the loop and at the output would maintain itself! For larger values of loop gain, any incidental disturbance will start an oscillation at this frequency that will grow; that is, the system will be unstable. To avoid instabilities, then, we must ensure that the magnitude of the loop gain is less than unity at any frequency at which the angle of the loop gain is 180°. (There are some interesting exceptions to this rule—see the appendix.)

The value of the loop gain at lower frequencies can be increased—and the effectiveness of feedback at these frequencies enhanced—if a *compensating network* is included in the loop to reduce the loop gain at higher frequencies where the phase angle of the loop gain approaches 180°. This scheme is explored further in the appendix and in Problem 6.6.

6.4 Feedback Stabilization of Unstable Systems

Feedback around a stable system can produce instabilities by in effect moving poles from the left to the right half-plane. On the other hand, feedback can also be used to stabilize an inherently unstable system, as the following example illustrates.

Example 6.4–1

Consider an inverted pendulum attached by a hinge to a car free to move along a horizontal track. Our objective is to design a driving system for the car that will use signals derived from the car and pendulum motions to stabilize the pendulum in its inverted position. The whole scheme may be considered a one-dimensional simulation of the common parlor trick of balancing a ruler or broom stick on end on your palm by moving your hand appropriately. The details of what we have in mind are shown in Figure 6.4–1.

It is straightforward to derive from the elementary mechanics of rigid bodies that

$$mg\ell \sin\theta - m\ell\ddot{x}\cos\theta = I\ddot{\theta}$$

where the dots imply differentiation and the variables and parameters have the meanings defined in Figure 6.4–1. For small angles this equation may be linearized to

$$I\ddot{\theta} - mg\ell\theta = -m\ell\ddot{x}$$

which implies that $\theta(t)$ and $x(t)$ are related by the system function

$$H(s) = \frac{\Theta(s)}{X(s)} = \frac{-m\ell s^2}{Is^2 - mg\ell}$$

which of course is unstable.

As a first attempt at stabilization, connect a rotary potentiometer as shown in Figure 6.4–1 to the pivot shaft of the inverted pendulum from which a voltage proportional to $\theta(t)$ can be derived. Apply the difference between $\theta(t)$ and the desired value $\theta_0(= 0)$ through an amplifier to drive the motor in such a direction as to increase $x(t)$ if $\theta(t) - \theta_0$ is positive. Assume that the motor and pulley system can be described by the system function

$$M(s) = \frac{X(s)}{V(s)} = \frac{k_m}{s(s+\alpha)}.$$

That is, the motor produces in the steady state a velocity $\dot{x}(t)$ proportional to the applied voltage $v(t)$ (independent of load) and the velocity responds to changes in $v(t)$ as a first-order system with time constant $1/\alpha$.

The block diagram for this closed-loop system is shown in Figure 6.4–2. If we take θ_0 as the input, the system function is

$$\hat{H}(s) = \frac{\Theta(s)}{\Theta_0(s)} = \frac{-KM(s)H(s)}{1 - KM(s)H(s)} = \frac{Kk_m\frac{m\ell}{I}s}{s^3 + \alpha s^2 + \frac{m\ell}{I}(Kk_m - g)s - \frac{m\ell}{I}g\alpha}.$$

Because of the minus sign in the constant term of the denominator, the system of Figure 6.4–2 remains unstable for any value of K. Physically, the trouble is that a steady angular error in the system of Figure 6.4–2 leads only to a steady velocity of the car. But movement of the car at constant velocity has no effect on the angular motion of the pendulum; to produce a restoring torque, the car must be accelerated.

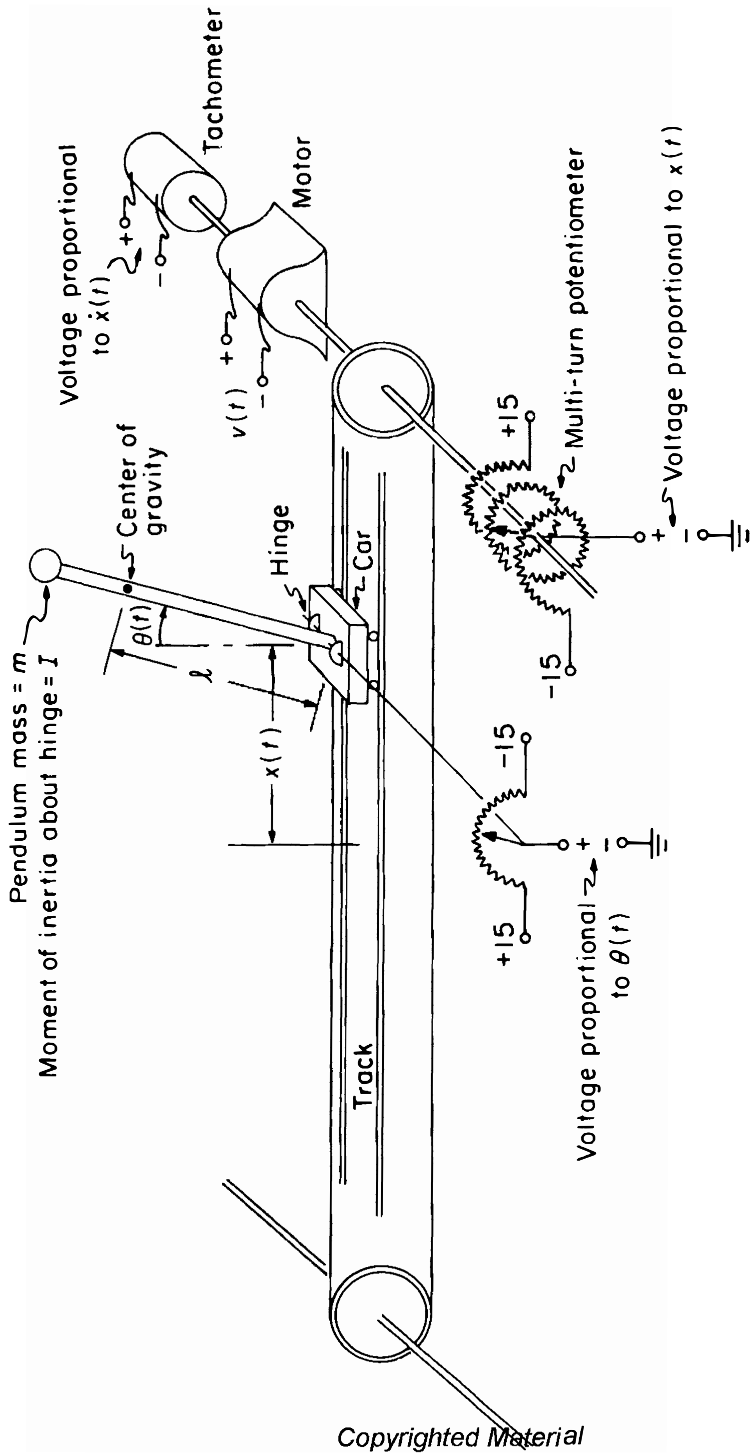

Figure 6.4–1. Inverted pendulum simulation.

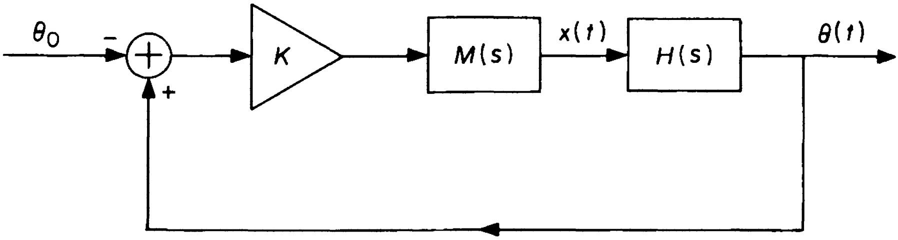

Figure 6.4–2. Block diagram for a first attempt at stabilization.

Such an effect can be obtained by adding an additional feedback path proportional to the integral of $\theta(t)$. The resulting block diagram has the form shown in Figure 6.4–3 and is characterized by the system function

$$\hat{H}(s) = \frac{\Theta(s)}{\Theta_0(s)} = \frac{-KM(s)H(s)}{1 - \left(1 + \frac{a}{s}\right)KM(s)H(s)}$$

$$= \frac{Kk_m \frac{m\ell}{I} s}{s^3 + \alpha s^2 + \frac{m\ell}{I}(Kk_m - g)s + \frac{m\ell}{I}(Kk_m a - g\alpha)}.$$

From the special case of the Routh test given earlier for the cubic, it follows that this system will be stable if $a < \alpha$ and $Kk_m a > g\alpha$, conditions that obviously can be met by appropriate choices of a and K.

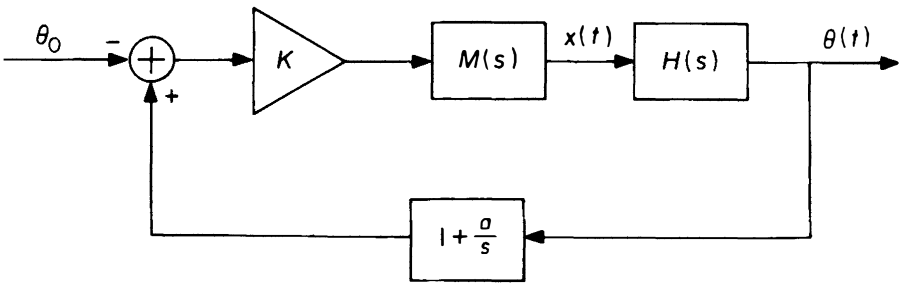

Figure 6.4–3. An improved stabilization system.

However, despite the fact that the feedback system of Figure 6.4–3 has theoretically "stabilized" the inverted pendulum, the scheme would not work very well in practice. There are two difficulties. First, inserting practical numbers for the various parameters will show that, although the system poles can be placed in the left half-s-plane, it will probably be difficult to move them very far from the $j\omega$-axis. As a result, transient disturbances of the pendulum would be only slowly corrected, and in an oscillatory manner—a series of modest but rapid disturbances could lead to failure by driving the system beyond its linear range. The factor primarily responsible for this effect is the probably small value of α, describing the frequency response of the motor. But this difficulty is not fundamental since, as in Example 6.2–2, the response of the motor can be speeded up by connecting a tachometer to provide velocity feedback around the motor, as shown in Figure 6.4–4. It is easy to show that the system function of this block diagram is $\hat{H}(s)$ of Figure 6.4–3 with α replaced by $\alpha' = \alpha + Kk_m b$. The "settling time" of the new system can consequently be significantly reduced.

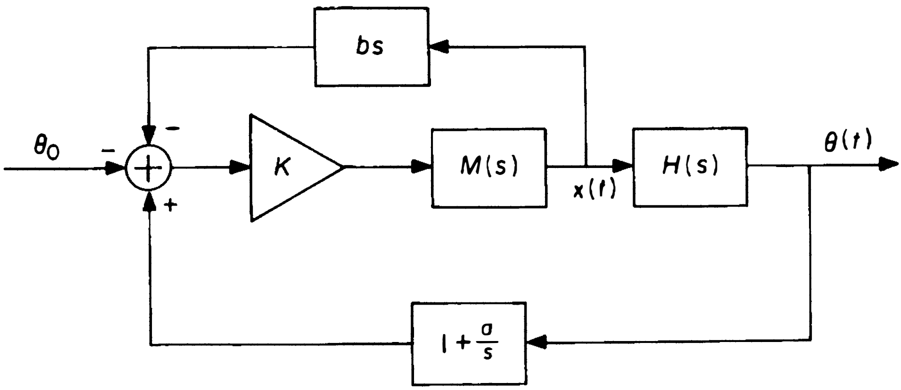

Figure 6.4–4. Further improvements.

A second difficulty with the system of Figure 6.4–3 (or Figure 6.4–1) is more subtle. Consider the system function describing the displacement $x(t)$ of the car, which is

$$\frac{X(s)}{\Theta_0(s)} = \frac{\hat{H}(s)}{H(s)} = \frac{-Kk_m\left(s^2 - g\frac{m\ell}{I}\right)}{s\left[s^3 + \alpha' s^2 + \frac{m\ell}{I}(Kk_m - g)s + \frac{m\ell}{I}\left(Kk_m a - g\alpha'\right)\right]}.$$

This system has a pole at $s = 0$ and is thus only marginally stable—a succession of random disturbances will induce a "random walk" in the car's position that will sooner or later cause it to go off one end or the other of the track. This can be avoided by adding still another feedback path, as in Figure 6.4–5.

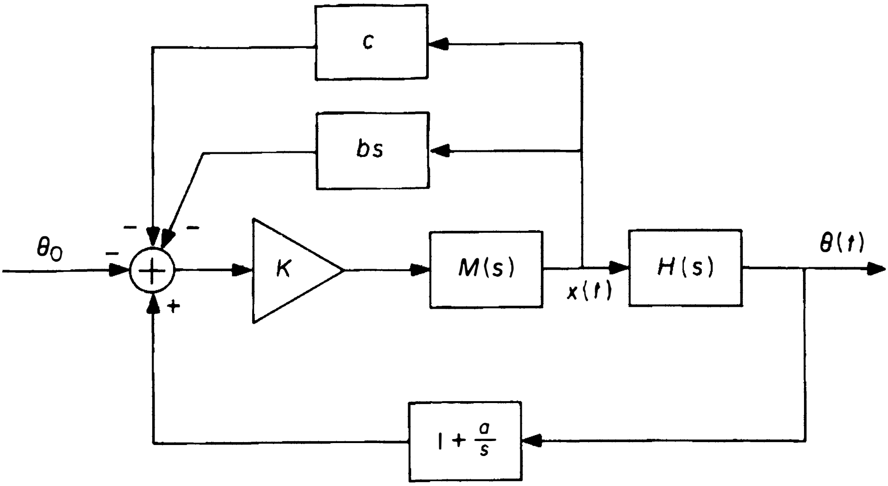

Figure 6.4–5. The final stabilization system.

The input-output system function now becomes

$$H(s) = \frac{\Theta(s)}{\Theta_0(s)}$$

$$= \frac{Kk_m \frac{m\ell}{I} s^2}{s^4 + \alpha' s^3 + \left[\frac{m\ell}{I}(Kk_m - g) + Kk_m c\right] s^2 + \frac{m\ell}{I}\left(Kk_m a - g\alpha'\right)s - cgKk_m \frac{m\ell}{I}}.$$

It is apparent from the sign of the last term in the denominator that stability has been upset unless c is negative—which implies that deviations of $x(t)$ from its zero position will induce motor inputs in a direction that makes the error worse. But a little reflection on how you move your hand balancing a pointer will make it clear that this counterintuitive result is indeed correct. To achieve an ultimate motion of your hand to the right, you must first move it sharply to the left, displacing the pendulum angle to the right so that you can then steadily move your hand to the right under the pendulum. A full study of the Routh conditions for this 4^{th}-order polynomial shows that stability now requires

$$-\frac{m\ell}{Ic}\left(1 - \frac{a}{\alpha'}\right)\left(1 - \frac{g\alpha'}{Kk_m a}\right) > 1$$

in addition to the conditions $a < \alpha'$ and $Kk_m a > g\alpha'$ already given for the earlier cubic. The coefficient c, in addition to being negative, must thus not be too big. Examination of the system function for $X(s)/\Theta_0(s)$ shows that the pole at $s = 0$ has in fact now been removed, so that with appropriate parameter choices our final design (Figure 6.4–5) should give satisfactory behavior for both angle $\theta(t)$ and car position $x(t)$.

This example illustrates the importance of looking at the stability of internal variables as well as input-output variables—which, as mentioned earlier, is called stability in the sense of Liapunov.

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6.5 Summary

Feedback can both reduce the sensitivity of a system to various parameter changes and loading effects, and also dramatically change the dynamic response of the system by altering the locations of the system poles. The latter effect can be useful, for example, in speeding up the response (or increasing the bandwidth) of a control system or amplifier, or in stabilizing an inherently unstable system such as the inverted pendulum. But feedback can also move system poles into the right half-s-plane and thereby change a stable system into an unstable one. Often stability considerations set limits on the sensitivity reductions that are realizable with feedback.

APPENDIX TO CHAPTER 6

The Nyquist Stability Criterion

The *Nyquist stability criterion* is based on an examination of the locus or polar plot (called the *Nyquist plot*) of the complex number $\beta(s)K(s)$ as s ranges along a contour following the $j\omega$-axis and around the right half-plane. The Nyquist plot, as shown in Figure 6.A–1, is thus a mapping into the βK-plane of the closed contour C shown in the s-plane.

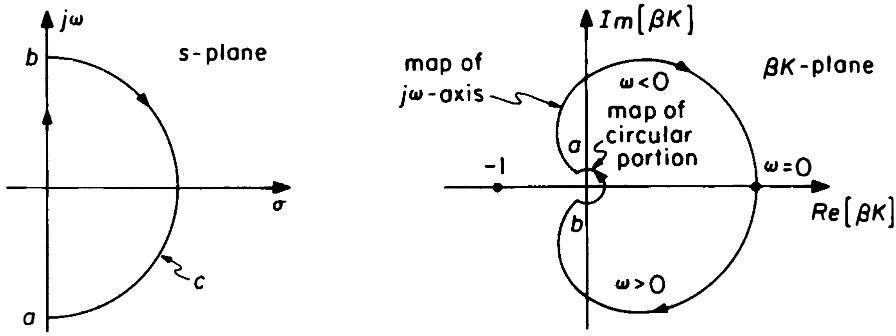


Figure 6.A–1. Contours in the s-plane and βK-plane.

The foundation for the Nyquist criterion is a theorem from function theory called Cauchy's Principle of the Argument. This theorem in general compares the contour generated by some function $X(s)$ in the X-plane as s traces clockwise around any simple contour in the s-plane. It states that the net number of clockwise encirclements of the point $X(s) = 0$ by the closed contour in the X-plane is equal to $Z - P$, where Z is the number of zeros of $X(s)$ and P the number of poles of $X(s)$ enclosed by the contour in the s-plane. Cauchy's Principle of the Argument is easily demonstrated for a rational function

$$X(s) = A\frac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots}$$

by recalling from Chapter 4 that the angle (or argument) of $X(s)$ (assuming A is real) is the sum of the angles of the vectors $(s - z_i)$ minus the sum of the angles of the vectors $(s - p_i)$, as illustrated in Figure 6.A–2. To be specific, suppose the contour in the s-plane is the contour C shown to the left in Figure 6.A–1. Consider a zero (or pole)

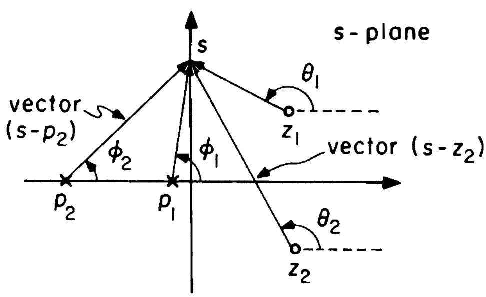


Figure 6.A–2. $\angle X(s) = \theta_1 + \theta_2 - \phi_1 - \phi_2$.

outside the contour C, as shown to the left in Figure 6.A–3. As the point s traverses the contour, the angle of $(s - z_1)$ (or $(s - p_1)$) varies, but it returns to its initial value when the contour is completely traced. If, however, z_1 (or p_1) lies inside the contour, as shown to the right in Figure 6.A–3, there is a net decrease in angle of 2π radians as the contour is traversed. Since the contributions of the individual terms add (subtract), the desired result follows immediately.*

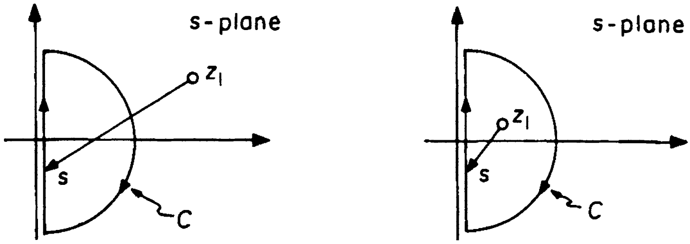


Figure 6.A–3. Zeros outside and inside a contour.

To derive Nyquist's criterion, set $X(s) = 1 + \beta(s)K(s)$ and require that the radius of the semicircular part of the contour C be very large. Then the number of clockwise encirclements of the origin by the $X(s)$ contour, or equivalently the number of clockwise encirclements of the point $\beta(s)K(s) = -1$ by the $\beta(s)K(s)$ contour, is equal to the difference between the number of zeros and the number of poles of $X(s)$ in the right half-plane. Since zeros of $X(s)$ are poles of the closed-loop feedback system and poles of $X(s)$ are poles of the loop gain $\beta(s)K(s)$, the closed-loop system will be unstable if the number of clockwise encirclements of $\beta(s)K(s) = -1$ by the $\beta(s)K(s)$ contour is greater than the number of poles of $\beta(s)K(s)$. This is Nyquist's criterion.

To apply this test, it is often helpful to carry out the following imaginary experiment. Pretend that there is a nail driven into the βK-plane at $\beta(s)K(s) = -1$ and that a string is tied to the nail so tightly that it cannot slip. The other end of the string is tied to a pencil. Starting anywhere, trace with the pencil entirely around the contour in the direction implied by a clockwise encirclement of the right half-s-plane. The net number of times that the string ends up wrapped around the nail in the clockwise direction is equal to the number of -1's of $\beta(s)K(s)$ in the right half-plane (minus the number of poles of $\beta(s)K(s)$ in the right half-plane, if any). In the sketches in

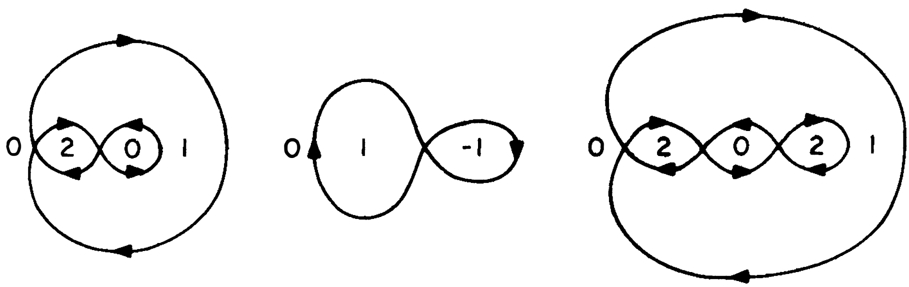


Figure 6.A–4. The numeral is the number of clockwise encirclements.

*The theorem actually applies to a wider function class than rational functions. For a more complete discussion see H. W. Bode, *Network Analysis and Feedback Amplifier Design* (New York, NY: Van Nostrand, 1945) p. 147ff.

Figure 6.A–4, the numeral indicates the number of clockwise encirclements if the point $\beta(s)K(s) = -1$ lies in the corresponding region.

Example 6.A–1

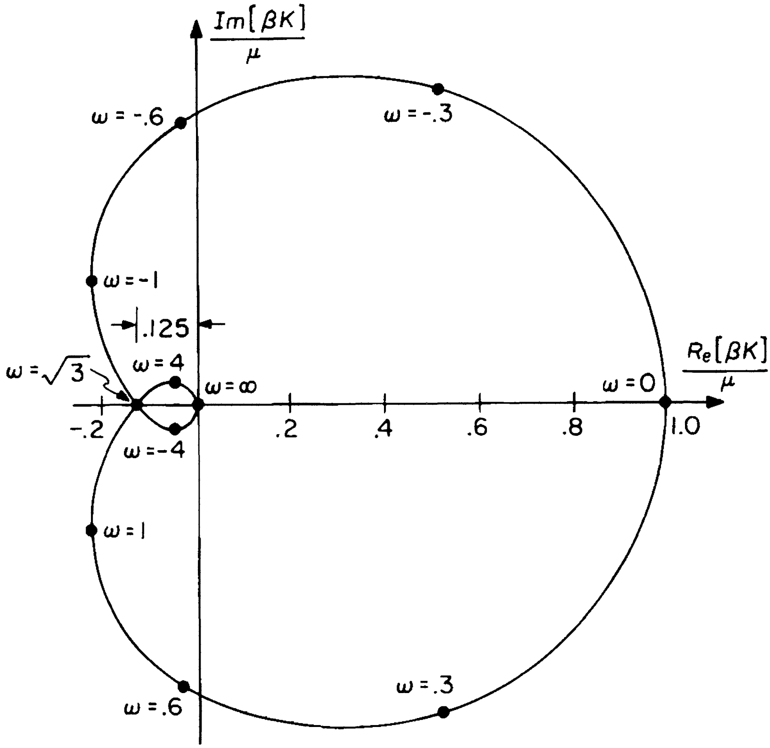


Figure 6.A–5. Polar plot of $\mu/(s+1)^3$.

Continuing Example 6.3–1, Figure 6.A–5 shows a polar plot of

$$\beta(s)K(s) = \frac{\mu}{(s+1)^3}.$$

For $\mu = 1$, the -1 point is well to the left outside the contour. For $\mu = 8$, the scales of ordinate and abscissa are multiplied by 8, and the -1 point coincides with the point on the contour labelled $\omega = \sqrt{3}$; for large μ the point -1 lies inside the contour and in fact is encircled twice, implying two zeros in the right half-plane. For negative μ, the contour is flipped over right for left; the point -1 will lie inside the contour, encircled once (implying one zero in the r.h.p.) for $\mu < -1$. Thus the closed-loop system will be stable for $-1 < \mu < 8$, as was previously demonstrated directly. Note that although we used an analytic formula for $\beta(s)K(s)$ to calculate the values of the loop gain for various frequencies, measured sinusoidal steady-state values would have worked as well. This point is illustrated in the next example.

▶ ▶ ▶

Example 6.A–2

Figure 6.A–6 is a sketch of $|\beta K|/\mu$ and $\angle\beta K/\mu$ vs. ω as might have been measured as the loop gain of some feedback system. (Actually these curves are the Bode plot for

$$\frac{\beta(s)K(s)}{\mu} = \frac{\left(1+\frac{s}{30}\right)^2}{(1+s)^3}$$

as constructed in Example 4.4–2.)

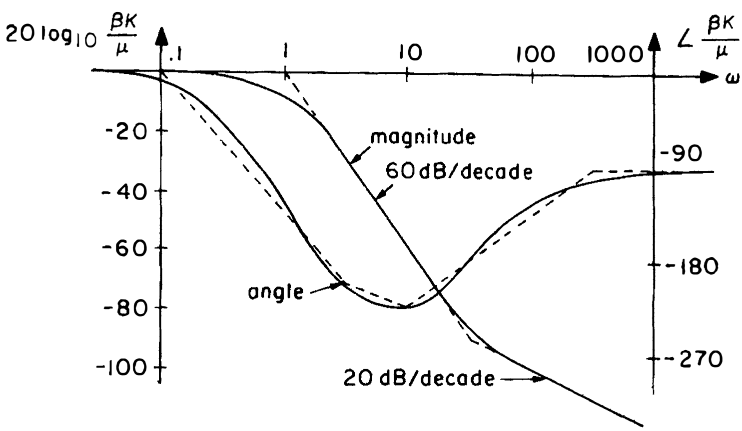


Figure 6.A–6. Measured loop gain.

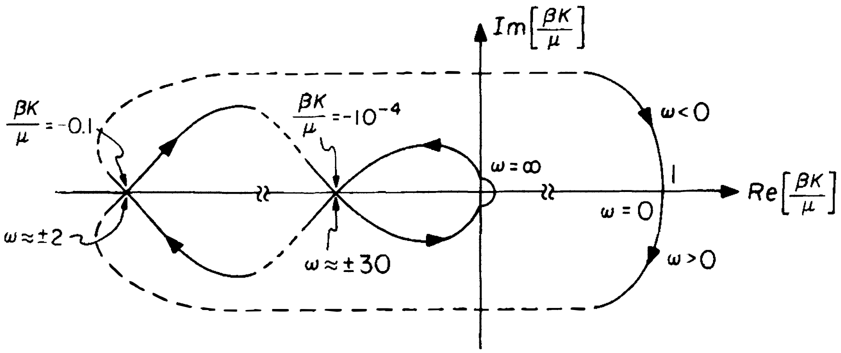


Figure 6.A–7. The shape of the Nyquist plot for $\beta K/\mu$ of Figure 6.A–6.

For stability analysis, interest centers on the Nyquist plot of $\beta K/\mu$ near the real axis, as shown schematically in Figure 6.A–7. Clearly for $\mu < 10$ (approximately) the system is stable, whereas for $10 < \mu < 10^4$ (again approximately) it is unstable. But, remarkablly, if $\mu > 10^4$, the system is again stable! This type of *conditional* (or *Nyquist*) *stability* is occasionally observed. Prior to Nyquist no one had understood how it was possible for an amplifier to be stable in spite of the fact that there was a frequency at which the returned sinusoidal signal was in phase with and larger than the sinusoidal input. Conditional stability may help explain the observation that

overdriving an amplifier will sometimes cause it to oscillate—despite the fact that (because the dominant non-linearity is usually a saturation) the effect of overdriving is to a first approximation often an effective reduction in gain.
▶ ▶ ▶

Example 6.A–3

Continuing Example 6.3–1, it should be clear from the Nyquist plot of Example 6.A–1 (reproduced in Figure 6.A–8 below) that performance would be improved if a compensating network could be inserted in cascade with either $\beta(s)$ or $K(s)$ that would add a positive phase shift to the loop gain for frequencies around $\omega = 1$ without substantially changing the magnitude for $\omega < 1$—for example, as shown by the dashed contour in Figure 6.A–8.

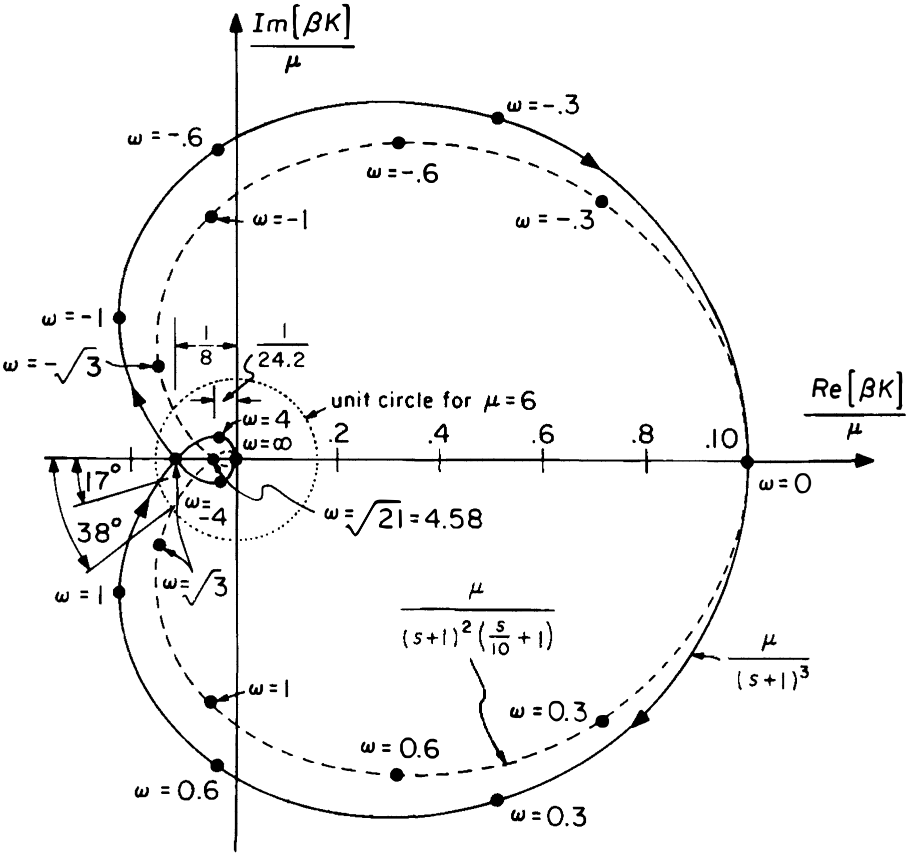


Figure 6.A–8. Nyquist plot of Example 6.A–1 with compensation (shown dashed).

The improvement in performance that could be provided by compensation can be measured in several ways:

a) The maximum value of μ yielding stable performance would be increased from just less than $1/0.125 = 8$ to just less than $1/0.041 = 24.2$.

b) For a gain less than the maximum allowed stable value, the *gain margin* is defined as the ratio of the maximum allowed stable value to the actual gain. (That is, for any selected value of μ, the gain margin is the reciprocal of the actual gain at a phase angle of $-180°$.) Thus if the chosen value of gain were 6, the gain margin would go from $8/6 = 1.33$ without compensation to $24.2/6 = 4.03$ with compensation. As a rule of thumb for electronic amplifiers, gain margins of 3 or more are considered desirable.

c) Again for a gain less than the maximum allowed stable value, the *phase margin* is defined as the difference between $-180°$ and the angle of the loop gain $\beta(s)K(s)$ at the frequency ω where the magnitude of the loop gain is 1. For a chosen gain of 6, the phase margin goes, as shown, from about $17°$ without compensation to about $38°$ with compensation. Again as a rule of thumb, phase margins of $30°$ to $60°$ are considered desirable.

As we shall see in Chapter 16, the magnitude and angle of the sinusoidal frequency response of a causal network are not independently selectable. Hence arbitrary adjustments of Nyquist plots are not possible. The compensation shown in the dashed line of Figure 6.A–8 was obtained by cascading $\beta(s)K(s)$ with a *lead network* whose system transfer function is

$$G(s) = \frac{s+1}{\frac{s}{10}+1}.$$

A Bode plot of $G(j\omega)$ is shown in Figure 6.A–9; the name "lead network" comes from the fact that the positive phase angle of $G(j\omega)$ implies that a sinusoidal output waveform "leads" (reaches its peak value before) the input. The key to the success of the lead network is that it produces substantial phase shift ($\sim 40°$ at $\omega = 1$) before it produces a significant increase in gain ($|G(j1)| \approx 1.4$).

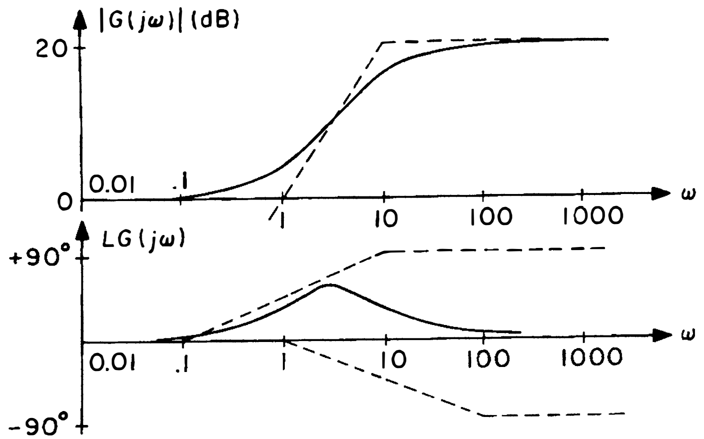


Figure 6.A–9. Magnitude and angle of the lead network response $G(j\omega)$.

With the lead network inserted in the loop, the loop gain becomes

$$\beta(s)K(s)G(s) = \frac{\mu}{(s+1)^2\left(\frac{s}{10}+1\right)}$$

and

$$1+\beta(s)K(s)G(s) = \frac{s^3+12s^2+21s+10(1+\mu)}{(s+1)^2(s+10)}.$$

Stable positive values of μ (from the Routh test) thus satisfy

$$21 > \frac{10(1+\mu)}{12} \quad \text{or} \quad \mu < 24.2.$$

For the maximum value of μ,

$$s^3 + 12s^2 + 21s + 10(1+\mu) = (s+12)(s^2+21).$$

These calculations determine the magnitude and frequency of the Nyquist plot at the point where it crosses the negative real axis. For further consideration of this compensation scheme see Problem 6.5.

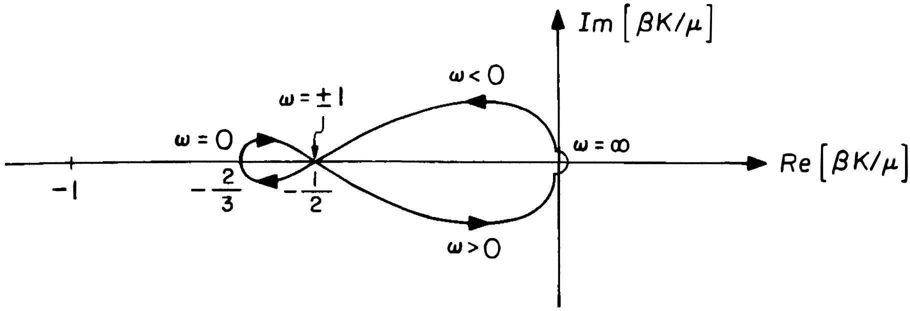
▶ ▶ ▶

Example 6.A-4

Suppose that

$$\beta(s)K(s) = \frac{\mu(s+2)}{(s+1)(s-3)}.$$

The loop-gain system function is unstable because of the pole at $s = +3$. However, the Nyquist diagram appears as shown in Figure 6.A–10. The values in the table in Figure 6.A–10 take into account the fact that βK has a pole in the right half-s-plane. Hence, for $1+\beta K$ not to have zeros in the right half-plane, the Nyquist contour must have one counterclockwise encirclement of -1. This occurs for $\mu > 2$. For these values of gain the overall amplifier with the loop closed is stable, in spite of the fact that if the loop were opened the amplifier would oscillate. This is another example of how feedback can be used to stabilize an unstable system.



Range of μ	Number of CW encirclements of $\beta K = -1$	Number of r.h.p. closed-loop poles
$0 < \mu < 3/2$	0	1
$3/2 < \mu < 2$	1	2
$\mu > 2$	-1	0

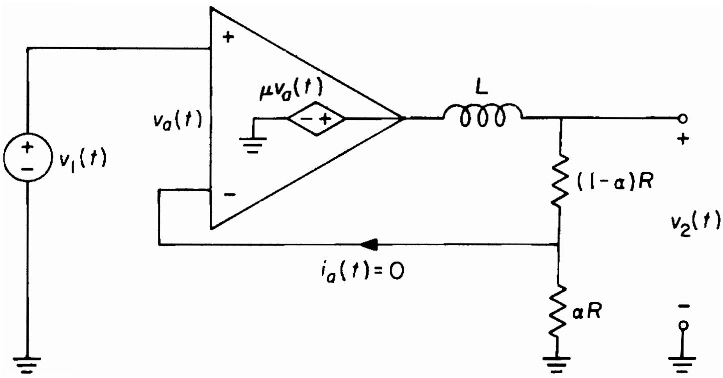
Figure 6.A–10. Nyquist plot for Example 6.A–4.

▶ ▶ ▶

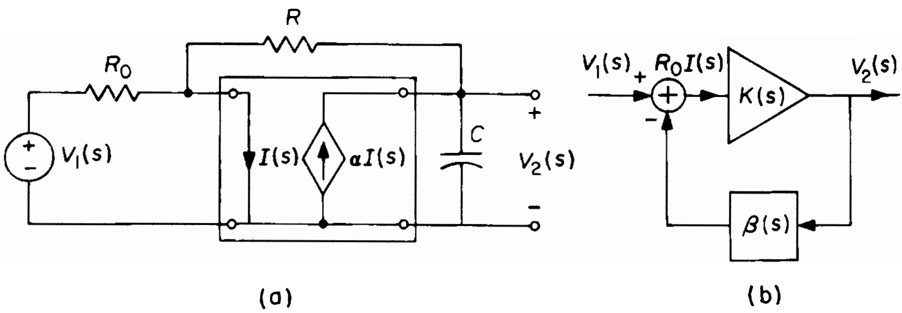
EXERCISES FOR CHAPTER 6

Exercise 6.1

For the circuit below, find the values of μ and α such that the response, $v_2(t)$, to a unit step, $v_1(t) = u(t)$, has the same final value but rises ten times faster than the step response that would be obtained with $\alpha = 0$ and $\mu = 1$. Answers: $\alpha = 0.9$, $\mu = 10$.



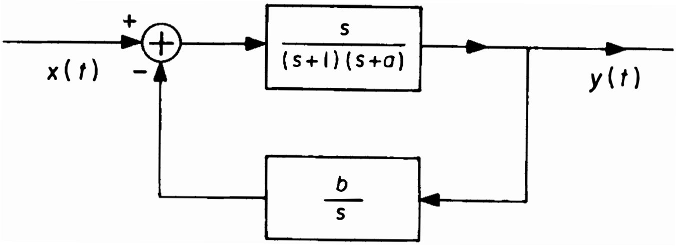
Exercise 6.2



a) Show that the feedback amplifier of (a) can be represented by the block diagram of (b) with $K(s) = (\alpha/R_0C)/(S + 1/RC)$ and $\beta(s) = -R_0/R$. (Note that the output of the summation box in (b) is specified to be the quantity $R_0I(s)$.)

b) Show that the amplifier is stable if $-\infty < \alpha < 1$.

Exercise 6.3



a) Determine a and b for the system above so that the overall system function becomes

$$H(s) = \frac{s}{(s+2)(s+3)}.$$

b) If $a = 2$, what is the range of values of b for which the system is input-output stable?

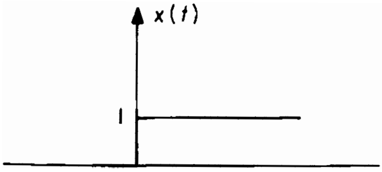
c) Determine the unit step response of this system if the system function is as in (a).

Answers: (a) $a = 4,\ b = 2$; (b) $b > -2$; (c) $y(t) = \left(e^{-2t} - e^{-3t}\right)u(t)$.

Exercise 6.4

a) An (unstable) LTI system has the response $y(t)$ to a unit step input $x(t)$ as shown below. Show that the system function for this system is

$$H(s) = \frac{1}{s-\alpha}.$$



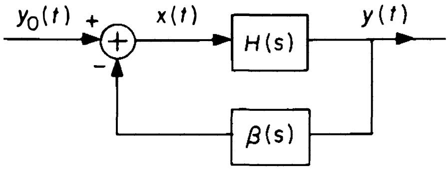
$x(t) = u(t)$

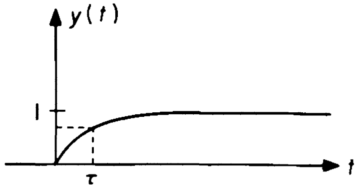
$y(t) = \frac{1}{\alpha}\left(e^{\alpha t} - 1\right)u(t),\ \alpha > 0$

b) To stabilize this system, apply feedback of the form

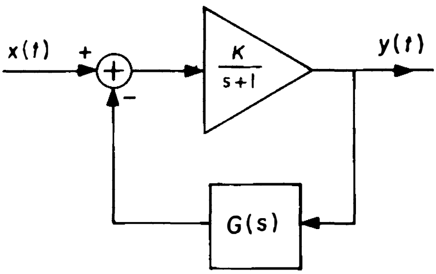
$$\beta(s) = (1+\alpha) + s.$$

Show that the overall system now has the unit step response shown on the next page with $\tau = 2$ sec.





Exercise 6.5



For what values of $K, -\infty < K < \infty$, will the overall system be stable if $G(s)$ has the following forms:

a) $G(s) = \dfrac{s^2}{s+1}$.

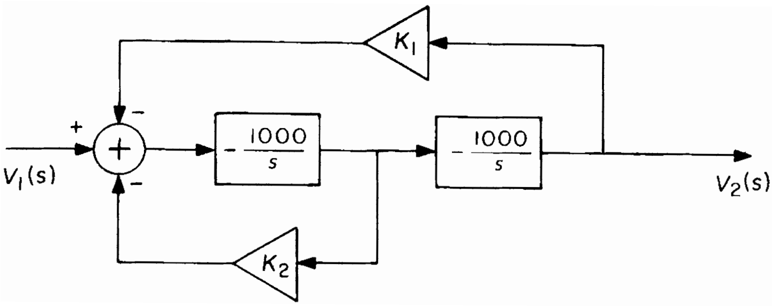
b) $G(s) = \dfrac{s^3}{(s+1)^2}$.

c) $G(s) = \dfrac{s+2}{s}$.

Answers: (a) $K > -1$; (b) $8 > K > -1$; (c) $K > 0$.

PROBLEMS FOR CHAPTER 6

Problem 6.1

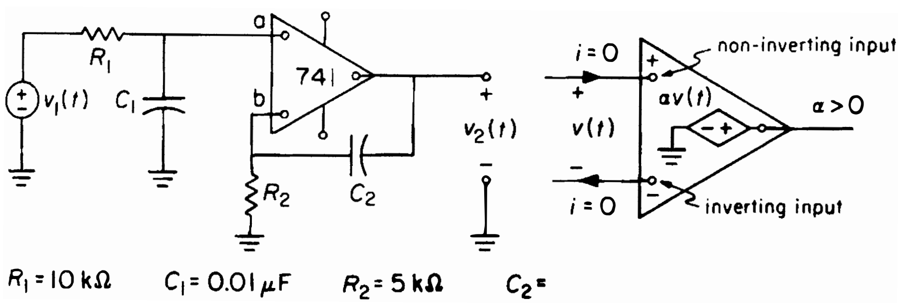


a) Find the zero-frequency gain of this system, assuming that K_1 and K_2 are picked so that it is stable.

b) For what range of (real) values of K_1 and K_2 will the system be stable?

c) Find and sketch the unit step response for $K_2 = -3$, $K_1 = 2$.

Problem 6.2



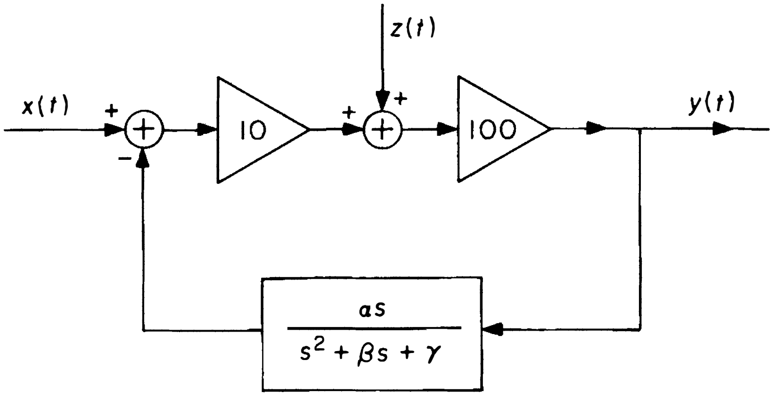
The diagram to the left above is intended to function as an integrator. Unfortunately the draftsman forgot to label the inverting and non-inverting terminals at the input to the op-amp, and he neglected to give one of the capacitor values.

a) Modelling the op-amp by the equivalent circuit shown to the right above with wire "a" connected to the non-inverting terminal of the op-amp, find the system function of the integrator circuit.

b) Should wire "a" be connected to the inverting or to the non-inverting terminal to ensure stable performance? Explain your answer. Does your answer depend on the magnitude of α?

c) Assuming that $\alpha \to \infty$ in the stable connection of the op-amp, determine the value of the unlabelled capacitor to yield ideal integrator behavior.

Problem 6.3

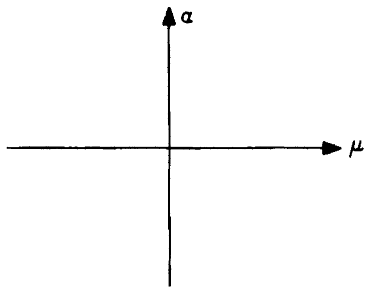


a) Find the ZSR $Y(s) = \mathcal{L}[y(t)]$ in terms of $X(s) = \mathcal{L}[x(t)]$ and $Z(s) = \mathcal{L}[z(t)]$ for the system above.

b) What conditions on α, β, and γ will ensure stability of this system?

Problem 6.4

Let $K(s) = \mu/(s+1)$ and $\beta(s) = s/(s+\alpha)$ in the conventional Black block diagram of a negative feedback system.

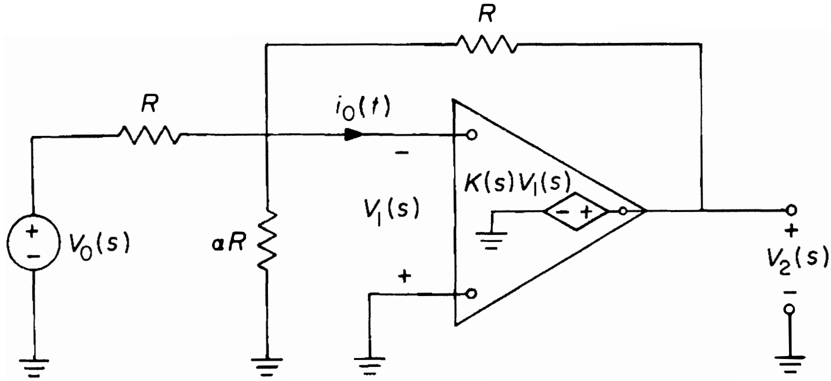


a) On a sketch of the α-μ-plane (as at the right), cross-hatch the region within which the feedback system will be stable.

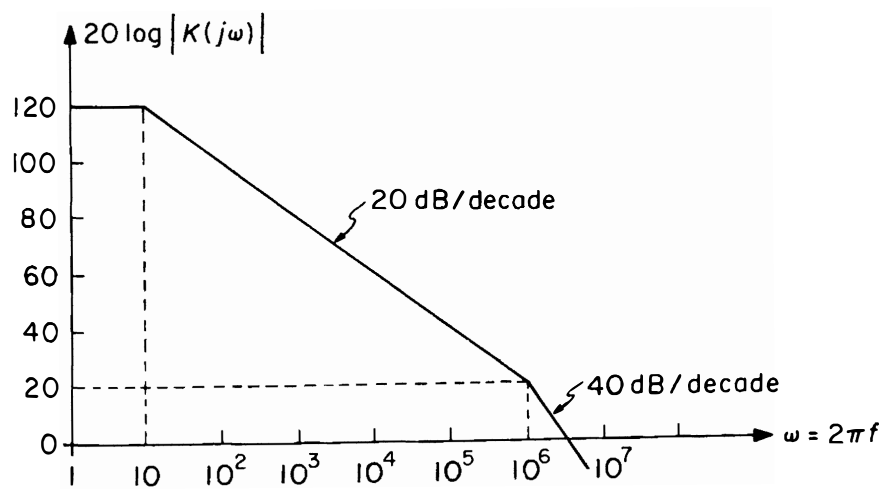
b) For $\alpha = 4$, sketch in the complex s-plane the locus of the zeros of $1 + \beta(s)K(s)$ as a function of μ, $-\infty < \mu < \infty$.

Problem 6.5

Consider the inverting op-amp circuit shown below. Assume that the op-amp is ideal to the extent that its input impedance is infinite ($i_0(t) = 0$) and its output voltage is independent of load (output impedance = 0), but that it is non-ideal in that its gain $K(s)$ is not infinite and not a constant.



a) The asymptotes of the Bode plot for $|K(j\omega)|$ are shown below. Assuming that the op-amp is stable and has a rational system function with $K(0) > 0$, find $K(s)$.

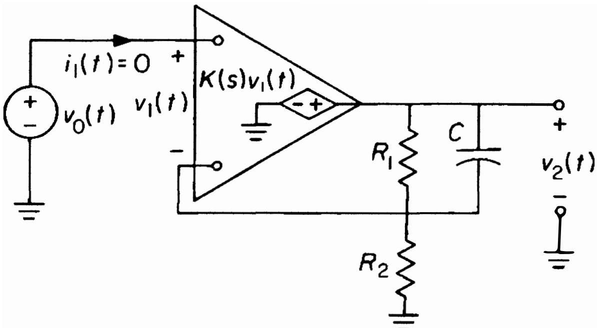


b) Find the overall system function, $H(s) = V_2(s)/V_0(s)$, for this circuit.

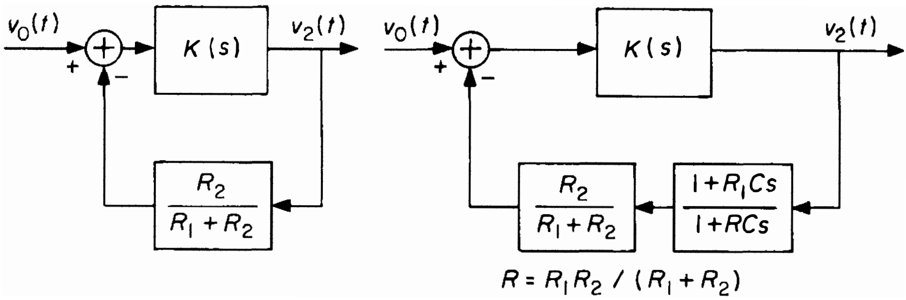
c) Find the value of α for which $H(s)$ has a second-order pole on the negative real axis. Make reasonable approximations.

Problem 6.6

The circuit below shows one way of adding *lead compensation* to a non-inverting amplifier. $K(s)$ represents the system function of the op-amp itself, reflecting primarily the high-frequency shunting effect of various internal capacitances.



a) Show that before compensation, with $C = 0$, the system can be described by the block diagram below to the left.



b) Show that with compensation the system can be described by the block diagram above to the right.

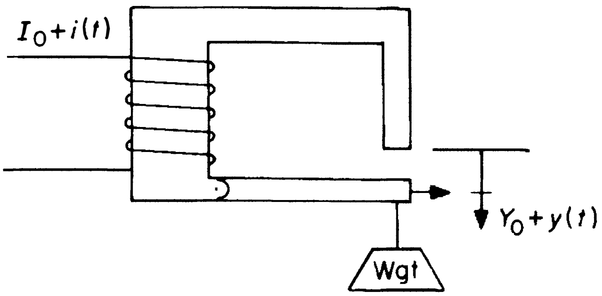
c) Let $K(s) = 60/(s+1)^3$, $R_1C = 1$ sec, $R_1 = 9R_2$. Show that the loop gains, compensated and uncompensated, are given by precisely the formulas assumed in Example 6.A–3 ($K(s)$ as given here is not, of course, a very good representation of actual op-amp behavior).

d) Show that the overall system functions for the feedback amplifier using the values in (c) are

$$\frac{V_2(s)}{V_0(s)} = H(s) = \begin{cases} \dfrac{60}{(s+2.82)(s^2+0.18s+2.49)} & \text{(uncompensated)} \\ \dfrac{60(s+10)}{(s+1)(s+10.65)(s^2+1.35s+6.62)} & \text{(compensated).} \end{cases}$$

e) Discuss the character of the unit step response (final values, rise times, overshoot, etc.) for each case in (d). (Feel free to actually find the step responses if you wish!)

Problem 6.7



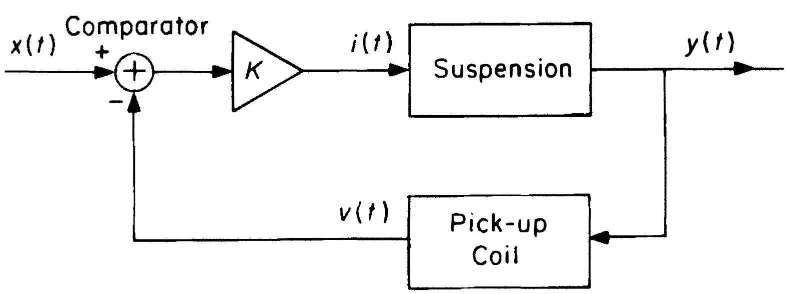
Systems to suspend a weight in a magnetic field are often inherently unstable because the magnetic force on an armature grows rapidly as the gap is reduced. For the system shown above, let Y_0 be the position at which the magnetic force due to the current I_0 just balances the gravitational force due to the weight. For small changes from this (unstable) equilibrium, assume that the incremental current and position satisfy the differential equation

$$\frac{d^2y(t)}{dt^2} - 4y(t) = -10i(t).$$

To stabilize this system, suppose we measure movement from the equilibrium point with a pickup coil whose output voltage is proportional to velocity,

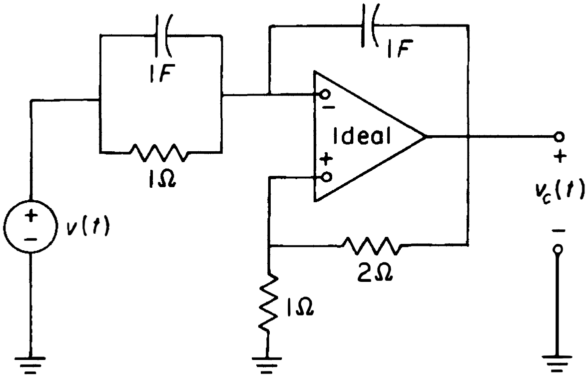
$$v(t) = 2\frac{dy(t)}{dt}.$$

We then propose to compare this signal with a desired velocity signal $x(t)$ and drive the coil with the amplified difference as shown below.



a) Show that this system is unstable for any value of K.

b) The circuit on the next page is suggested as a compensator for the system, to be inserted in the feedback path between the pickup coil and the comparator. Determine the transfer function $V_c(s)/V(s)$ of the circuit, using assumptions appropriate for an ideal op-amp.

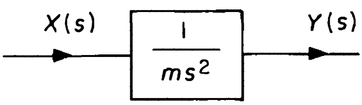


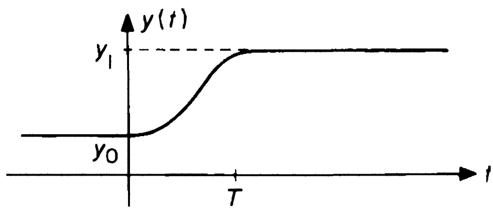
c) Compute the system function for the suspension system with the compensator in the feedback loop.

d) Find the range of values of K for which the compensated system is stable.

Problem 6.8

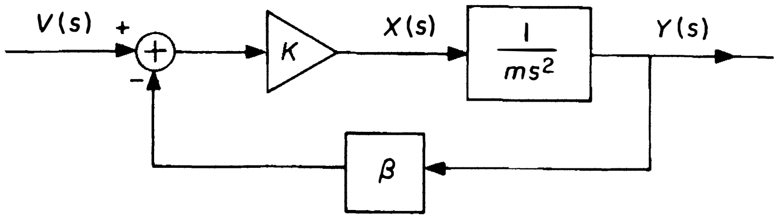
Stonewell International has hired you as a consultant to study schemes for regulating the altitude $y(t)$ of their Space Buggy by controlling the vertical thrust $x(t)$ (which can be either positive or negative). Assume that $x(t)$ and $y(t)$ can be related by the LTI model shown below to the left; m is the mass of the craft.





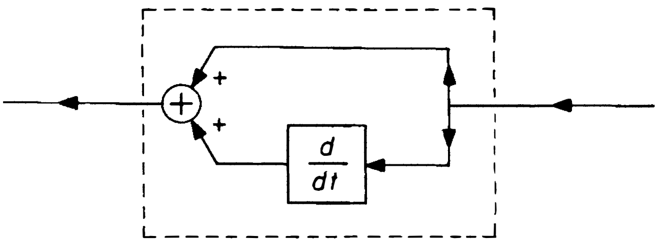
a) Sketch the thrust waveform that would be necessary to change the craft altitude smoothly but rapidly from one value to another (as shown above to the right) by direct control of vertical thrust.

b) One proposed control scheme involves adding a feedback loop and controller signal $v(t)$ as shown below. The mass m of the spacecraft changes as the fuel is used. Describe the locus of the closed-loop poles of this system assuming that β and K are positive constants and m varies from 0 to ∞.



c) Is the system BIBO stable under these conditions?

d) To improve performance, derivative feedback is added; the β-block is replaced by the dashed block shown below, where d/dt indicates an ideal differentiator. Describe the stability of this improved system as a function of the constant $K, -\infty < K < \infty$.



e) Using the improved design with very large $K > 0$, determine the controller signal $v(t)$ necessary to produce the altitude change described in (a). Do you think the "improved" design would in fact be easier or harder to control manually than direct control of the vertical thrust?

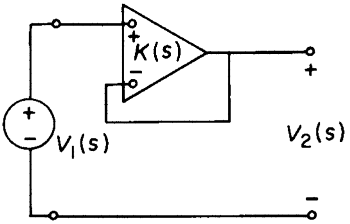
Problem 6.9

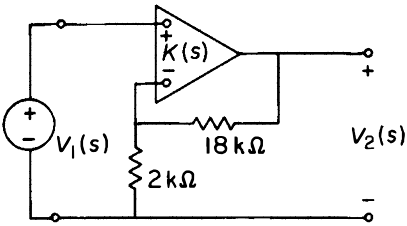
As suggested in Example 6.2–1, the frequency response effects of op-amps can be approximately described by an equivalent circuit consisting of a controlled source whose gain is a function of frequency, $K(s)$. For two popular op-amps, $K(s)$ is given approximately by

$$K(s) = \begin{cases} \dfrac{10^5}{\left(\dfrac{s}{20\pi}+1\right)\left(\dfrac{s}{2\times10^6\pi}+1\right)^2} & \text{(Type 741)} \\[2ex] \dfrac{10^5}{\left(\dfrac{s}{200\pi}+1\right)\left(\dfrac{s}{2\times10^6\pi}+1\right)^2} & \text{(Type 748).} \end{cases}$$

a) Sketch the Bode plot for each of these op-amps. Which would you say was the better op-amp? What criterion are you using for "better"?

b) Discuss the stability of each of these op-amps if used in the voltage-follower circuit shown below to the left.





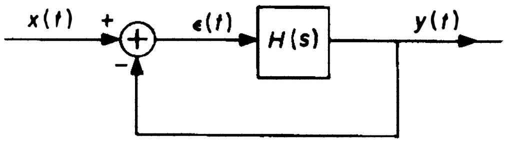
c) Discuss the stability of each of these op-amps if used in the non-inverting amplifier circuit with low-frequency gain $\approx$ 10 shown above to the right.

d) Now which op-amp do you think is better? What criterion are you now using for "better"? Why is the voltage-follower circuit a more severe test of stability than the non-inverting amplifier circuit?

The 741 and 748 are identical devices except that one of the time constants of the 741 is intentionally made larger; that is, the 741 is *internally compensated* to avoid the difficulty explored in this problem.

Problem 6.10

The simple linear feedback system shown below is intended to perform as a tracking system. Ideally the error signal, $\epsilon(t)$, should be identically zero, or equivalently $y(t)$ should be equal to $x(t)$.



a) Suppose $H(s) = 1/s$. Determine the steady-state form of $\epsilon(t)$ if the input is a unit ramp function

$$x(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Will the tracking error be zero in the steady state?

b) Repeat (a) for $H(s) = 1/s^2$. Describe the nature of the tracking error in the steady state.

c) Suppose $H(s) = A(s)/s^2$. Choose $A(s)$ so that the problems uncovered in (a) and (b) are resolved and $\epsilon(t) \to 0$ for a unit ramp input. Sketch $\epsilon(t)$, $x(t)$, and $y(t)$ for your choice on the same coordinates.

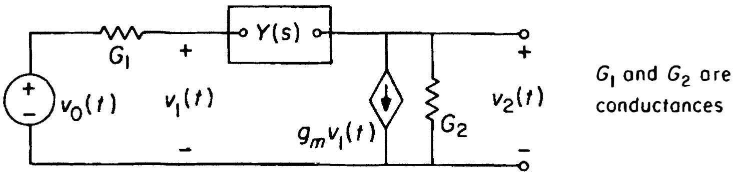
Problem 6.11

It can be shown that any two-terminal admittance, $Y(s)$, composed of positive R's, L's, and C's must satisfy two conditions:

1. The poles of $Y(s)$ cannot lie in the right half-s-plane.
2. $\Re e[Y(s)] \geq 0$ for all $\Re e[s] \geq 0$.

(For a discussion of these conditions, see Problem 4.5.)

a) Show that the following circuit is stable for any $Y(s)$ satisfying these conditions if $g_m > 0$.

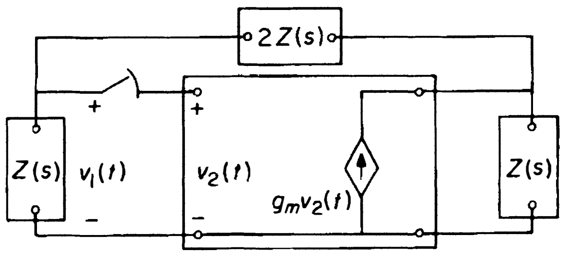


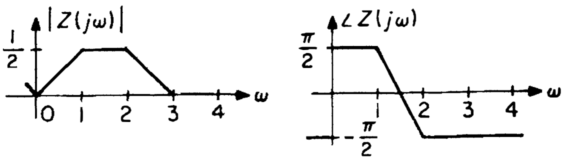
b) What is the smallest magnitude of g_m for which there might be instability for some $Y(s)$ satisfying these conditions if g_m is negative?

Problem 6.12

In the circuit shown below, the large box represents a two-stage amplifier idealized to have infinite input impedance and a current-source output as shown. The smaller boxes are one-port circuits characterized by their impedances, $Z(s)$ and $2Z(s)$.

a) With the switch open and a voltage source $v_2(t)$ applied, compute the loop gain, $V_1(s)/V_2(s)$, and show that it equals $g_m Z(s)/4$.
b) For $Z(j\omega)$ as shown in the figure, plot the Nyquist diagram (or polar plot of the loop gain).
c) If $Z(j\omega)$ is as shown, what is the maximum (positive) value that g_m can have if the circuit is to be stable when the switch is closed?





Problem 6.13

The measured sinusoidal steady-state frequency response, $K(j2\pi f)$, of a certain band-pass amplifier is shown in the polar plot on the page opposite. This amplifier is cascaded with an ideal amplifier with gain $\mu > 0$, and a fraction β of the output is fed back as shown in the insert on the opposite page.

a) What is the largest value of $\mu\beta$ for which the overall system will be stable?

b) Let $\mu\beta$ equal half the maximum value found in (a). Find μ and β such that the overall gain at $f = 1$ is 50. What are the gain and phase margin under these conditions?

c) To improve the performance, a *lag network* is inserted at the position of the dotted box with frequency response

$$G(j2\pi f) = \frac{1}{(1 + jf/10)^2}.$$

Draw directly on a copy of the polar diagram a sketch of $G(j2\pi f)K(j2\pi f)$—approximately to scale. The simplest procedure is to compute the magnitude and angle of $G(j2\pi f)$ for $f = 1$, $10/\sqrt{3}$, 10, $10\sqrt{3}$, etc. Pay particular attention to the asymptotic behavior as $f \to 0$ and $f \to \infty$.

d) Determine approximately the largest value of $\mu\beta$ for which the new system will be stable.

e) Repeat part (b) for this compensated system, letting $\mu\beta$ be the same value as before (half the value found in (a)).

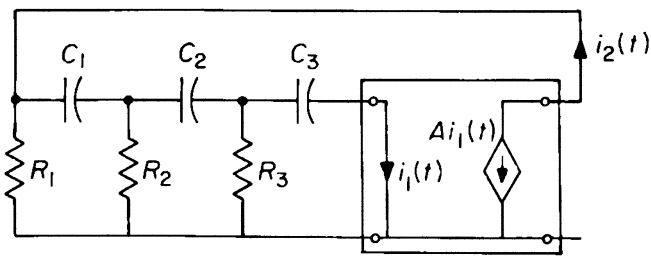
Problem 6.14

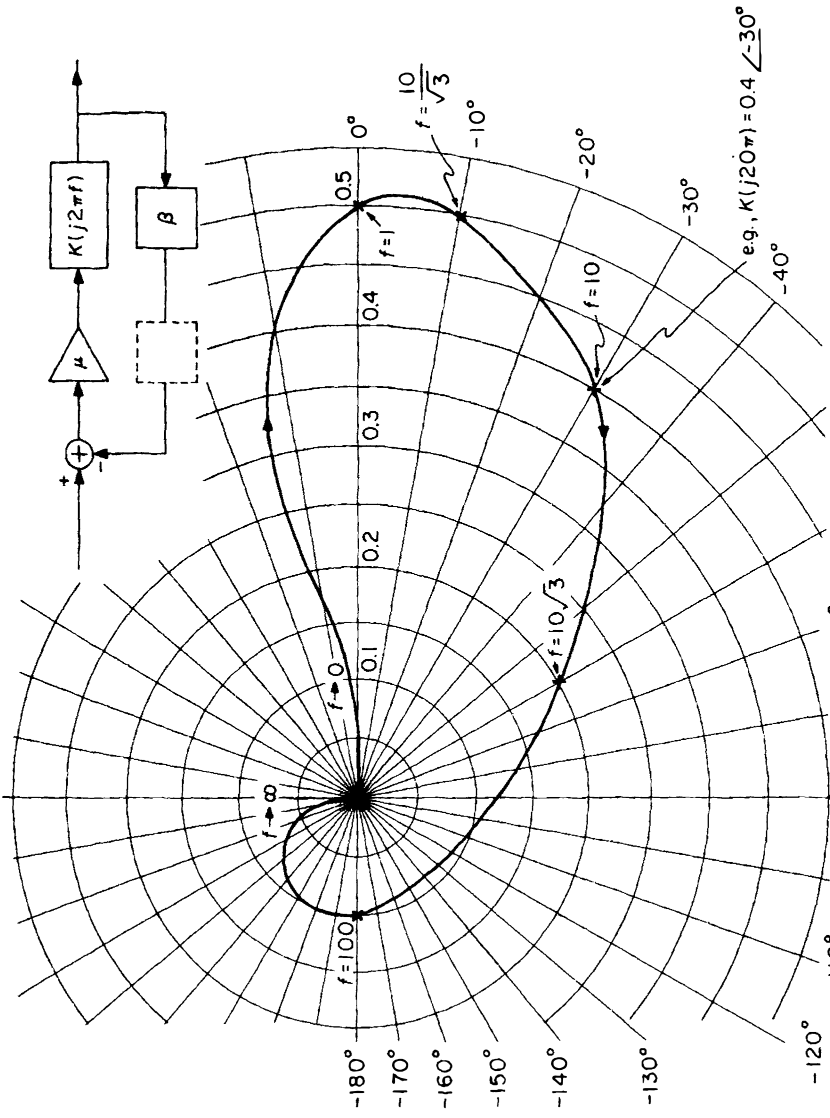
The system shown below is usually called a *phase-shift oscillator*. The transistor current amplifier A has negligibly small input impedance, high output impedance, and a current amplification $i_2/i_1 = -A$, where A is a positive constant. In practice, $C_1 < C_2 < C_3$ and $R_1 > R_2 > R_3$, but for purposes of this problem let $C_1 = C_2 = C_3 = C$ and $R_1 = R_2 = R_3 = R$.

a) Plot the Nyquist diagram of the loop transmission.

b) Find the critical value of A for which the circuit just oscillates.

c) Calculate the frequency of oscillation.





Polar Plot of $K(j2\pi f)$ for Problem 6.13.

Problem 6.15

If there were no acoustic feedback from the loudspeaker to the microphone, an outdoor public address system would have the overall frequency response $G(j\omega)$ shown below, where $G(j\omega)$ is the ratio of the sound pressure at the loudspeaker to the sound pressure at the microphone. But for a fixed angle of orientation between speaker and microphone, it is found that the microphone picks up a fraction $1/r^2$ (where r is the distance between speaker and microphone in feet) of the sound from the loudspeaker delayed by the time it takes sound to travel r feet (assume the speed of sound to be 1000 ft/sec). It is also found that the system is not stable for $r_1 \leq r \leq r_2$. Use Nyquist's criterion to calculate r_1 and r_2.