3/10/20 6.302 (Preliminary) 1

A Quick Reminder About Complex Exponentials

$e^{j\omega t} = \cos\omega t + j\sin\omega t$

$Re(e^{j\omega t}) = \cos\omega t$

$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t} = -\omega\sin\omega t + j\omega\cos\omega t$

Real Part

$Re(\frac{d}{dt}e^{j\omega t}) = -\omega\sin\omega t$

$\frac{d^2}{dt^2}e^{j\omega t} = (j\omega)^2 e^{j\omega t} = -\omega^2\cos\omega t - j\omega^2\sin\omega t$

$Re(\frac{d^2}{dt^2}e^{j\omega t}) = -\omega^2\cos\omega t$

$e^{j(\omega t+\theta)} = e^{j\theta}e^{j\omega t}$

$= (\cos\theta + j\sin\theta)(\cos\omega t + j\sin\omega t)$

$= \cos(\omega t+\theta) + j\sin(\omega t+\theta)$

$Re(e^{j(\omega t+\theta)}) = \cos(\omega t+\theta)$

<u>Note</u> any complex number

$a+bj = Me^{j\theta}$ $M=\sqrt{a^2+b^2}$

A Quick Reminder about $H(s)$

If $$\underbrace{Y e^{st} = H(s)}_{\text{system function}} X e^{st} \quad -\infty < t < \infty$$

Then for $t > 0$

$$y(t) = \underbrace{\sum_{i=1}^{L} A_i e^{\lambda_i t}}_{\text{Homogeneous}} + \underbrace{H(s)e^{st}}_{\text{particular}}$$

A_i's Picked to match $y(0), \ldots \left.\frac{d^{L-1}y(t)}{dt^{L-1}}\right|_{t=0}$

$\lambda_i \in \text{roots}(\text{den}(H(s)))$
$\in$ poles of $H(s)$

If $s = j\omega \Rightarrow e^{st} = e^{j\omega t}$

And if $Re(\lambda_i) < 0$

Then for t large enough

$$y(t) = H(j\omega)e^{j\omega t}$$

Proof

$$y(t) = \underbrace{\sum A_i e^{\lambda_i t}}_{\substack{\to 0 \text{ as } t \to \infty \\ re(\lambda_i) < 0}} + \underbrace{H(j\omega)e^{j\omega t}}_{\substack{\text{never goes} \\ \text{to zero}}}$$

2B

A Quick Reminder about $H(j\omega)$

If $Y(s) = H(s)X(s)$

And $H(s)$ is strictly stable
(poles in left half plane)

Then

if $x(t) = e^{j\omega t}$

eventually $(t \to \infty)$ $y(t) = H(j\omega)e^{j\omega t}$

Proof Partial fraction Expansion in rational function unique poles case

$$\mathcal{L}(e^{j\omega t}) = \frac{1}{s - j\omega}$$

$$Y(s) = \frac{n(s)}{(s-p_1)(s-p_2)\ldots(s-p_L)} \frac{1}{s-j\omega}$$

$$= \sum_{i=1}^{L} \frac{A_i}{(s-p_i)} + \frac{H(j\omega)}{s-j\omega}$$

$H(j\omega)e^{j\omega t}$

In Time Domain Decays to zero if $\mathrm{Real}(p_i) < 0$

$$A_i = \left.\frac{(s-p_i)H(s)}{(s-j\omega)}\right|_{s=p_i}$$

$\underline{\text{If}}$ $\quad x(t) = e^{j\omega t}$

strictly stable

$\underline{\text{And}}$ $\quad Y(s) = H(s)\, X(s)$

$\underline{\text{Then}}$ in the limit as $t \to \infty$

$$
\begin{aligned}
y(t) &= H(j\omega)\, e^{j\omega t} \\
&= |H(j\omega)|\, e^{j \measuredangle H(j\omega)}\, e^{j\omega t} \\
&= |H(j\omega)|\, e^{j(\omega t + \measuredangle H(j\omega))}
\end{aligned}
$$

$\underline{\text{If}}$ $\quad \measuredangle H(j\omega) = -90^\circ$

$$
\begin{aligned}
e^{j(\omega t - 90^\circ)} &= \cos(\omega t - 90) + j \sin(\omega t - 90) \\
&= \sin \omega t - j \cos \omega t
\end{aligned}
$$

$\underline{\text{If}}$ $\quad \measuredangle H(j\omega) = -180^\circ$

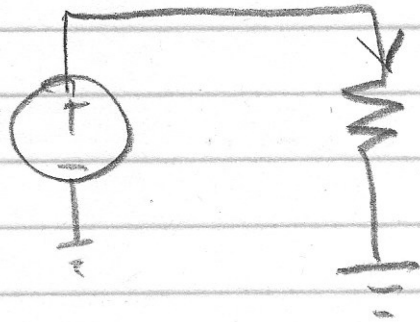
$$
\begin{aligned}
e^{j(\omega t - 180)} &= \cos(\omega t - 180) + j \sin(\omega t - 180) \\
&= -(\cos \omega t + j \sin \omega t) \\
&= -e^{j\omega t}
\end{aligned}
$$

<u>Sign</u> Flip!

3A

Circuit Example

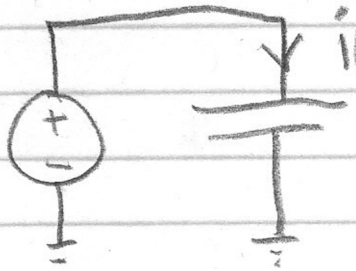
$V(t) = \mathbb{V} e^{j\omega t}$



$$i(t) = \frac{1}{R}\mathbb{V} e^{j\omega t} = \mathbb{I} e^{j\omega t}$$

$$\Rightarrow R\,\mathbb{I} = V$$

$V(t) = \mathbb{V} e^{j\omega t}$



$$i(t) = C\frac{dv(t)}{dt} = j\omega C \mathbb{V} e^{j\omega t}$$

$$= \mathbb{I} e^{j\omega t}$$

$$\Rightarrow \underbrace{\frac{1}{j\omega C}}_{Z(j\omega)} \mathbb{I}(j\omega) = \mathbb{V}$$

$Z(j\omega) \equiv$ impedance

$$|Z(j\omega)| = \left|\frac{1}{\omega C}\right| \qquad \measuredangle Z(j\omega) = -\frac{\pi}{2}$$

$$= -90^\circ$$

Why degrees? Used in practice

Examples

Velocity with friction and control input

$$\frac{d}{dt} V(t) = -\beta V(t) + \gamma F(t)$$

Approach 1 $F(t) = e^{j\omega t}$ $V(t) = ?$

Assume $V(t) = H(j\omega) e^{j\omega t}$

cancel

$$\frac{d}{dt} V(t) = j\omega H(j\omega) e^{j\omega t} = -\beta H(j\omega) e^{j\omega t} + \gamma e^{j\omega t}$$

Approach 2 Laplace $H(s) = \frac{\gamma}{s+\beta}$

Either way → **Result** $\Rightarrow H(j\omega) = \dfrac{\gamma}{j\omega+\beta} = H(s)\Big|_{s=j\omega}$

$\boxed{\omega \simeq 0}$

$|H(j\omega)|\Big|_{\omega\simeq 0} = \frac{\gamma}{\beta}$

$\measuredangle H(j\omega)\Big|_{\omega \simeq 0} = 0°$

$$|H(j\omega)| = \frac{\gamma}{\sqrt{\omega^2+\beta^2}}$$

$$H(j\omega) \simeq \frac{\gamma}{\beta} \qquad \omega \simeq 0$$

$\boxed{\lim \omega \to \infty}$

$\measuredangle H(j\omega)\Big|_{\omega\to\infty} = -90°$

$|H(j\omega)|\Big|_{\omega\to\infty} = \frac{\gamma}{\omega}$

$$H(j\omega) \simeq -\frac{j\gamma}{\omega} \qquad \omega >> \beta$$

$-j = \frac{1}{j}$ $\qquad e^{-j\frac{\pi}{2}} = -\frac{\pi}{2} = -90°$

Bode Plot (20 log|H(jω)| vs. ω on log scale
∡ H(jω) vs ω on log scale

$$H(j\omega) = \frac{\gamma}{j\omega + \beta} = H(s)\Big|_{s=j\omega}$$