6.302 3/12/20

Propeller Arm Example

Plant with system function

$\frac{d}{dt} f(t) = p(f(t) - c(t)) \Rightarrow sF = pF - pC$

e.g. $\frac{d}{dt} V(t) = \gamma f(t) \Rightarrow sV = \gamma F$

$\frac{d}{dt} d(t) = V(t) \Rightarrow sD = V$

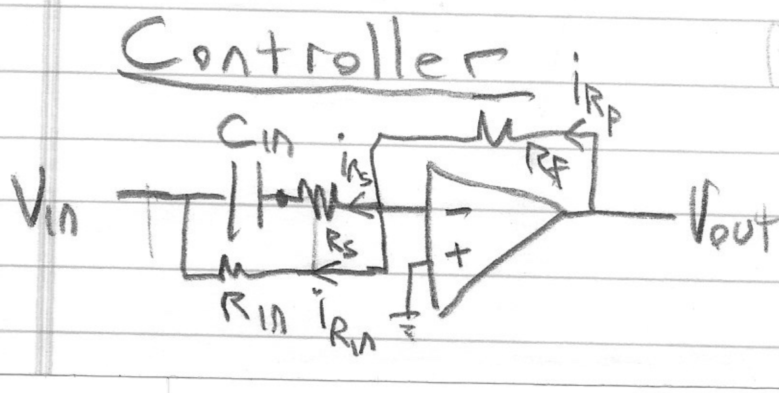
Assume

$$\boxed{\begin{aligned} d(t) &= De^{st} \\ V(t) &= Ve^{st} \\ f(t) &= Fe^{st} \\ d_d(t) &= D_d e^{st} \end{aligned}}$$

Sinusoidal Steady State
If $s = j\omega$
then
$d(t) = H(j\omega)e^{j\omega t}$
t large
if $Re(\text{poles}(H(j\omega))) < 0$

$$H(s) = \frac{1}{s} \frac{\gamma}{s} \frac{-p}{s-p}$$

Controller



$i_{R_F} = \frac{V_{out} - V_-}{R_F}$, $V_- = V_+ = 0$

$i_{R_S} = i_{R_S} + i_{R_{in}}$

$i_{R_{in}} = \frac{-V_{in}}{R_{in}}$

6.302 3/30/20 (Addendum)

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Key Ideas by example

Idea 1

Sinusoidal Steady State

if $\frac{d}{dt}y(t) = p y(t) + \gamma x(t)$

and $y(t) = Y e^{st}$ $\quad x(t) = X e^{st}$

then $sY = pY + \gamma X$

$$Y = \underbrace{\frac{\gamma}{s-p}}_{H(s)} X$$

$\underset{\sim}{X} = 1$

If $y(t)\big|_{t=0} = y_0$ & $x(t) = 1 e^{j\omega t}$

then

$$y(t) = Ae^{pt} + \underbrace{\frac{\gamma}{j\omega - p}}_{H(j\omega)} e^{j\omega t}$$

$$\underbrace{A = y_0 - \left(\frac{\gamma}{j\omega - p}\right)}$$

Matches $y(t)$ at $y(t)\big|_{t=0} = y_0$

If $Re(p) < 0$ and t is large enough

Stable systems have sinusoidal steady states

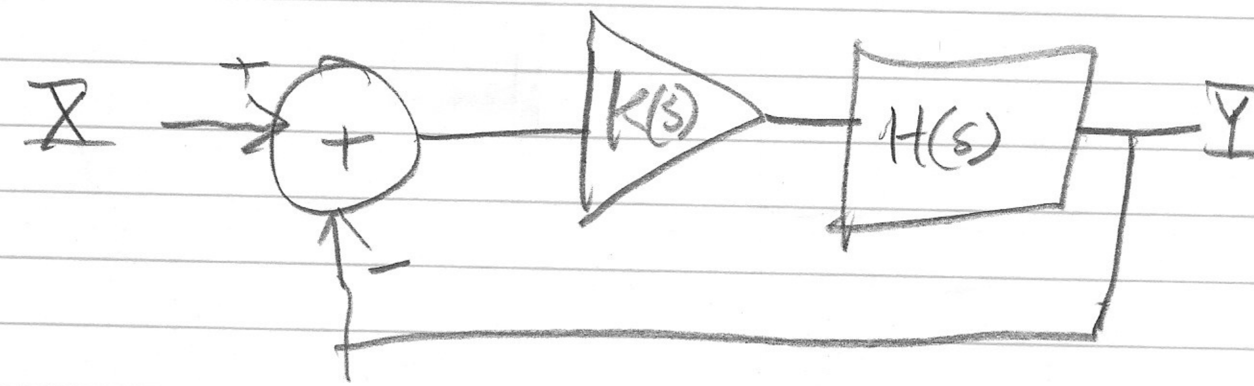
$$y(t) = \underbrace{H(j\omega)}_{H(s)|_{s=j\omega}} e^{j\omega t}$$

have a magnitude and phase!

idea 2

Black's Formula

If



$$Y = G(s)X \qquad G(s) = \frac{K(s)H(s)}{1 + K(s)H(s)}$$

idea 3

Phase Margin Heuristic!

If $K(s)H(s)\Big|_{s=j\omega} = -1$ for some ω_{unity}

$|K(j\omega_{unity})H(j\omega_{unity})| = 1$

$\measuredangle KH = \pm 180°$

Then $G(j\omega_{unity}) = \infty$

So Make sure $K(j\omega)H(j\omega)$ has angle more positive than $-180°$! when $|K(j\omega)H(j\omega)| = 1$

Huh? Why? → And don't let angle $\measuredangle K(j\omega)H(j\omega) < -180°$

$\boxed{\text{So what if } \measuredangle < -180°?}$ (4)

No! $\left\{ \begin{array}{l} \measuredangle K(j\omega)H(j\omega) = -225° \quad \omega = \omega_{unity} \\ \Rightarrow \boxed{K(j\omega)H(j\omega) \neq -1 \quad \text{is it okay?}} \end{array} \right.$ No!

Example $H(s) = \frac{1}{s^2} \cdot \frac{1}{s+100} \quad K = 10^6$

$\omega_{unity} \simeq j\omega = j100$

$$KH(j\omega) = \frac{10^6}{10^4 j} \cdot \frac{1}{100j + 100}$$

$$\simeq \frac{10^6}{-10^6 + 10^6 j}$$

close enough to 1

Exact Numbers From matlab

$$|KH(j\omega)|\Big|_{j\omega=100j} = \frac{1}{\sqrt{2}} \simeq 0.707$$

$$|KH(j\omega)| = 1 \quad \measuredangle KH(j\omega) = -221° \quad j\omega = 87j$$

"close to 1"

$$\measuredangle KH(j\omega)\Big|_{j\omega=100} = \measuredangle \frac{1}{s^2}\Big|_{s=j100} + \measuredangle \frac{10^6}{s+100}\Big|_{s=j100}$$

$$\cong -180° - 45°$$

Not Exactly $-180°$ so $KH \neq -1$ but

$$\cong -225°$$

$$G(s) = \frac{\frac{10^6}{s^2(s+100)}}{1 + \frac{10^6}{s^2(s+100)}} = \frac{10^6}{s^2(s+100) + 10^6}$$

has poles with pos real parts!

unstable

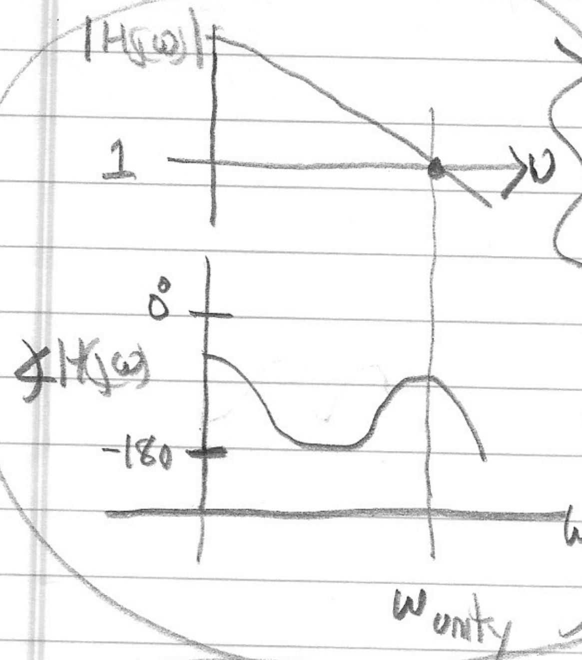
Gaurantee

If $90° \geq \angle H(j\omega) > -180°$

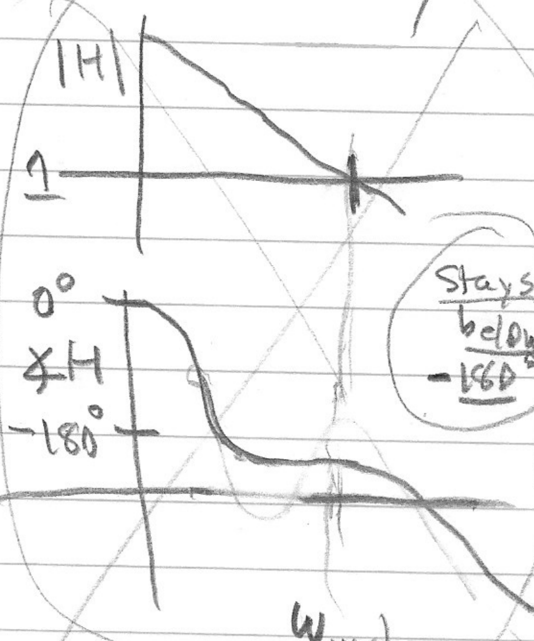
$\forall\ \omega$ s.t. $|H(j\omega)| \geq 1$

And $\angle H(j\omega_{unity}) > -180°$ by a margin

Then the bigger the margin the better the stability



Okay

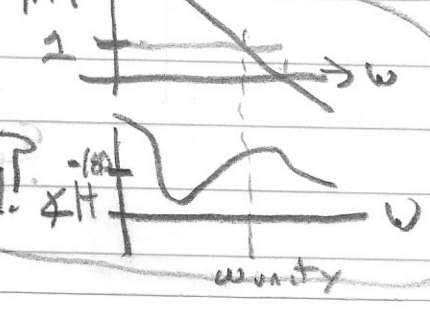


Stays below $-180°$

Not okay even though $H(j\omega_{unity}) \neq -1$

What about?

Dip below $-180°$ when $|H(j\omega)| \gg 1$?



Open-Loop ⇒ Closed-Loop Freq Response

$$G(s) = \frac{K(s)\,H(s)}{1 + K(s)\,H(s)}$$

$$G(j\omega) = \frac{K(j\omega)\,H(j\omega)}{1 + K(j\omega)\,H(j\omega)}$$

<u>If</u> $K(j\omega)\,H(j\omega) \gg 1$

<u>Then</u> $G(j\omega) \simeq 1$ $\quad |G(j\omega)| = 1$
$\measuredangle G(j\omega) = 0$

<u>If</u> $K(j\omega)\,H(j\omega) = 1$

<u>Then</u> $G(j\omega) = \frac{1}{2}$

<u>But if</u> $K(j\omega)\,H(j\omega) \simeq -1$

<u>Then</u> $G(j\omega) \simeq$ infinity

<u>Must Keep</u> $K(j\omega)\,H(j\omega)$ away from -1

or $\begin{cases} |K(j\omega)\,H(j\omega)| = 1 \\ \measuredangle K(j\omega)\,H(j\omega) = \pm 180^\circ \end{cases}$

6A

Phase Margin Definition

$$\text{Phase Margin} \equiv \measuredangle K(j\omega)H(j\omega)\Big|_{\omega=\omega_{unity}} + 180^\circ$$

$$\boxed{\omega_{unity}:\ \omega \text{ such that } |K(j\omega_{unity})H(j\omega_{unity})| = 1}$$

Gain Margin Definition

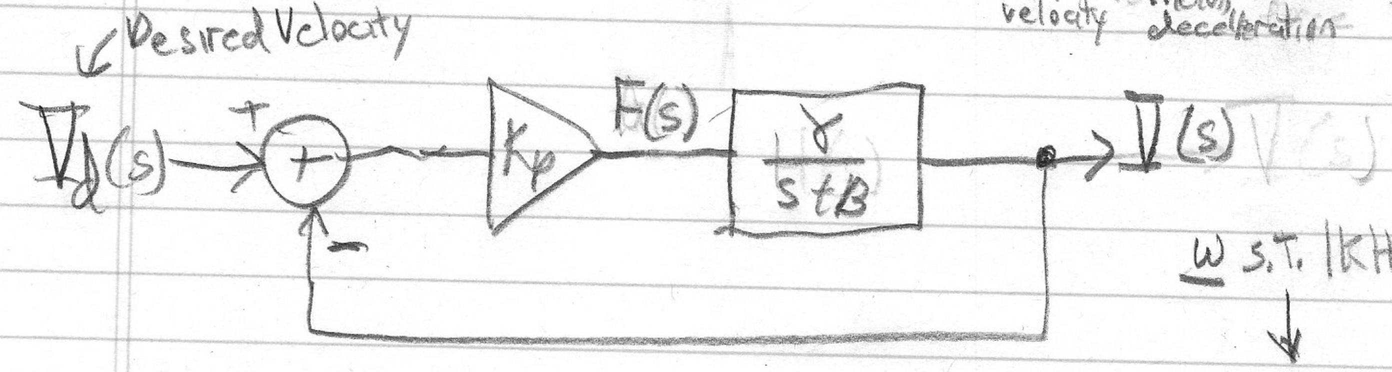
$$\text{Gain Margin} \quad -20\log|K(j\omega_{180})H(j\omega_{180})|$$

$$\boxed{\omega_{-180} \text{ is } \omega \text{ such that } \measuredangle K(j\omega_{-180})H(j\omega_{-180}) = -180^\circ}$$

Example for Phase Margin

Velocity plus friction: $\frac{d}{dt}V = -\beta V + \gamma F$

velocity — friction deceleration — input force

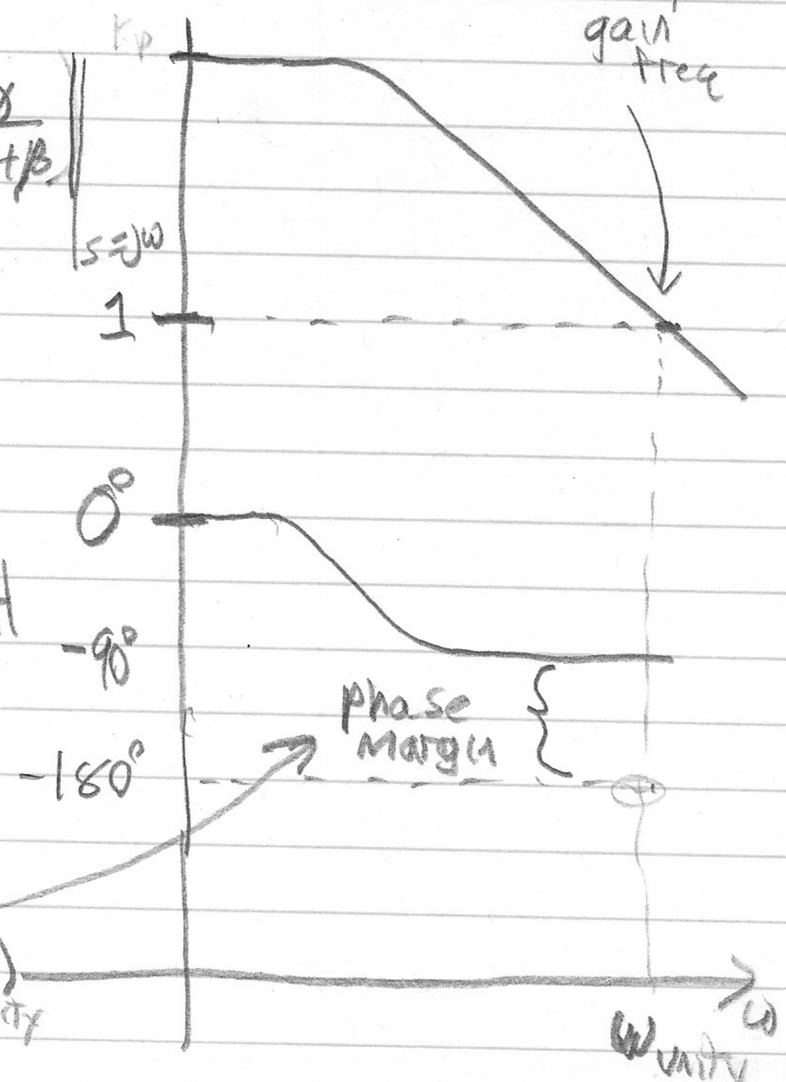


ω s.t. $|KH| = 1$

Unity gain freq

$$\left| K_p H(s) \right|\Big|_{s=j\omega} = \left| K_p \frac{\gamma}{s+\beta} \right|\Big|_{s=j\omega}$$

$\measuredangle K_p H$



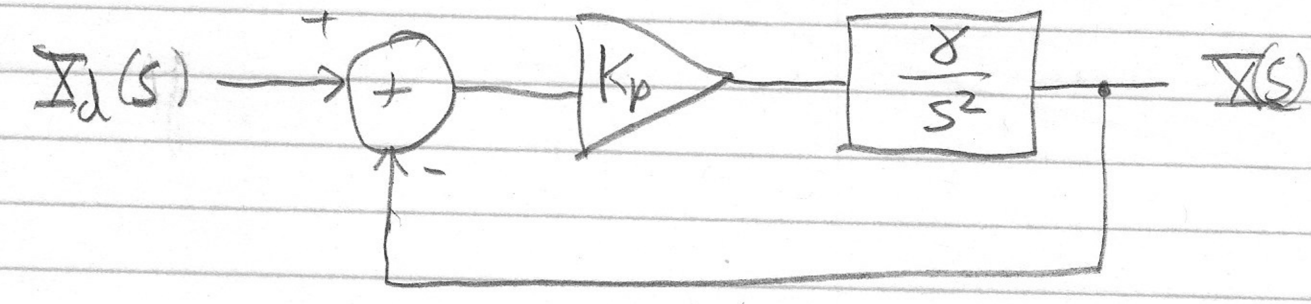
phase margin =

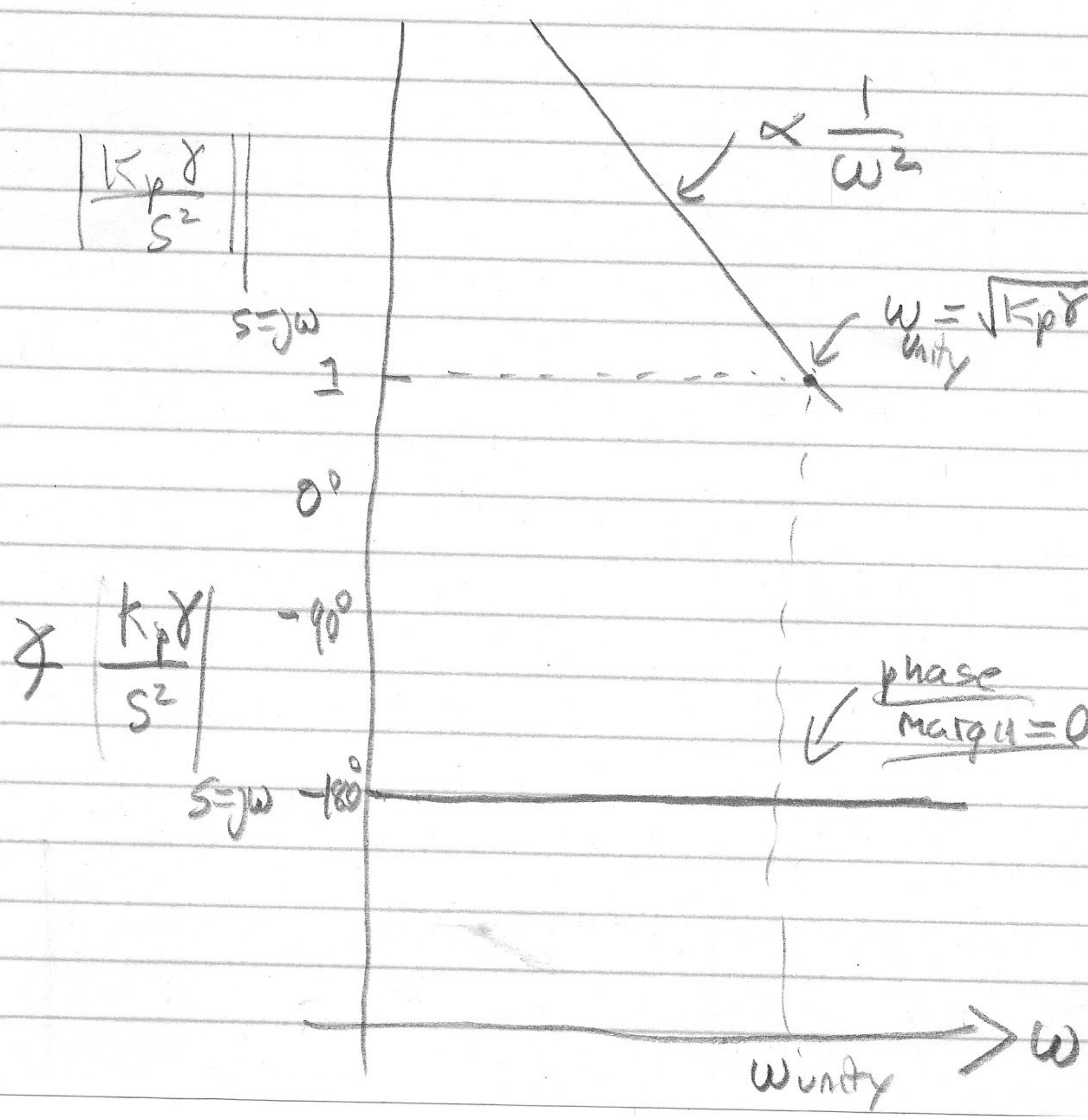
$180° + \measuredangle K_p(j\omega_{unity}) H(j\omega_{unity})$

$= 90°$

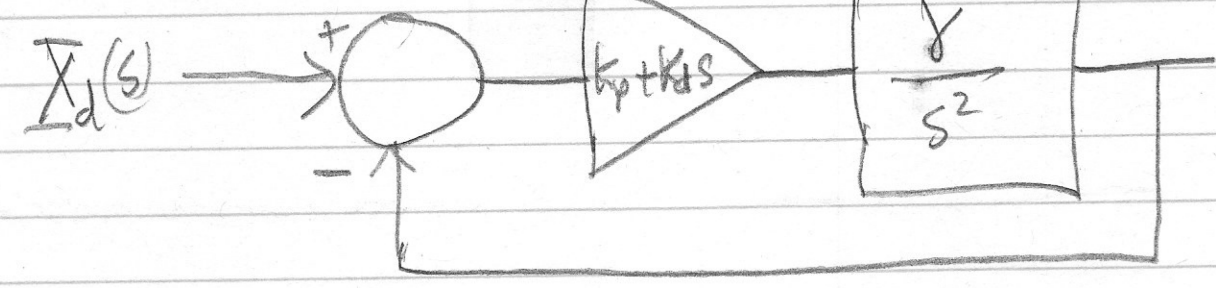
Position control using force

$$\frac{d^2}{dt^2} x(t) = \gamma f(t)$$

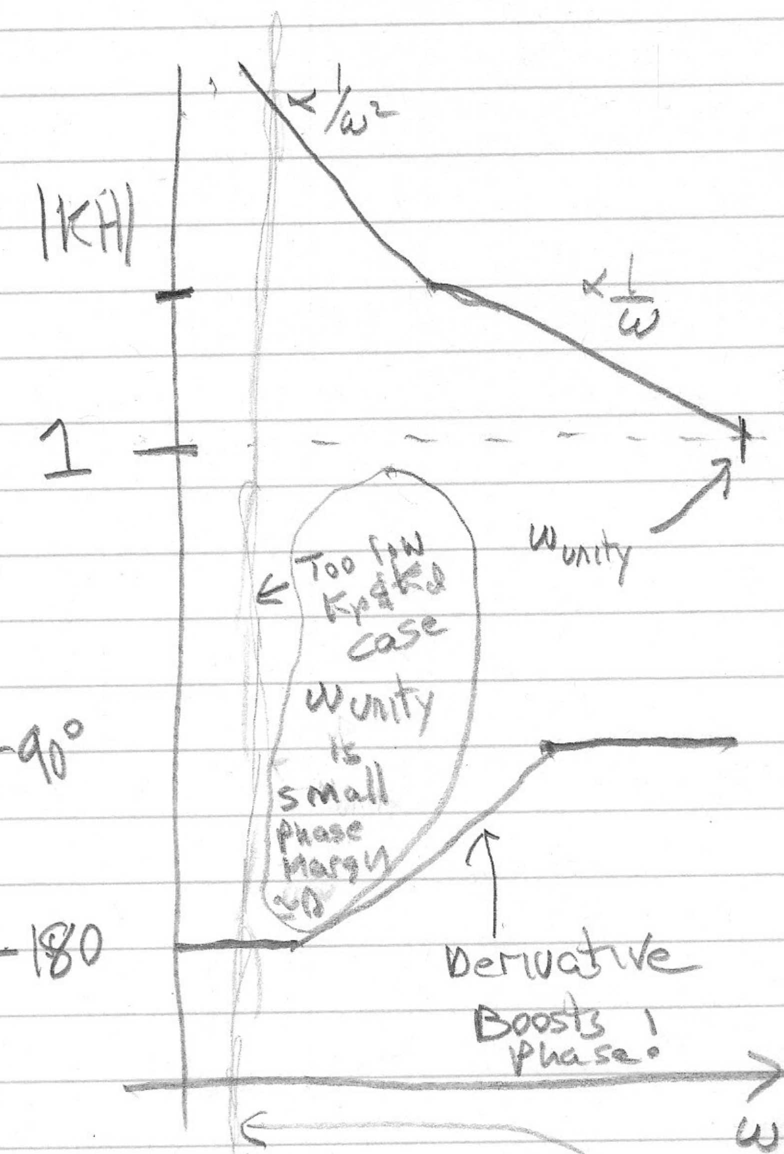




P. D. Compensated force-controlled Position



$$KH = \frac{\gamma(K_p + K_d s)}{s^2}$$



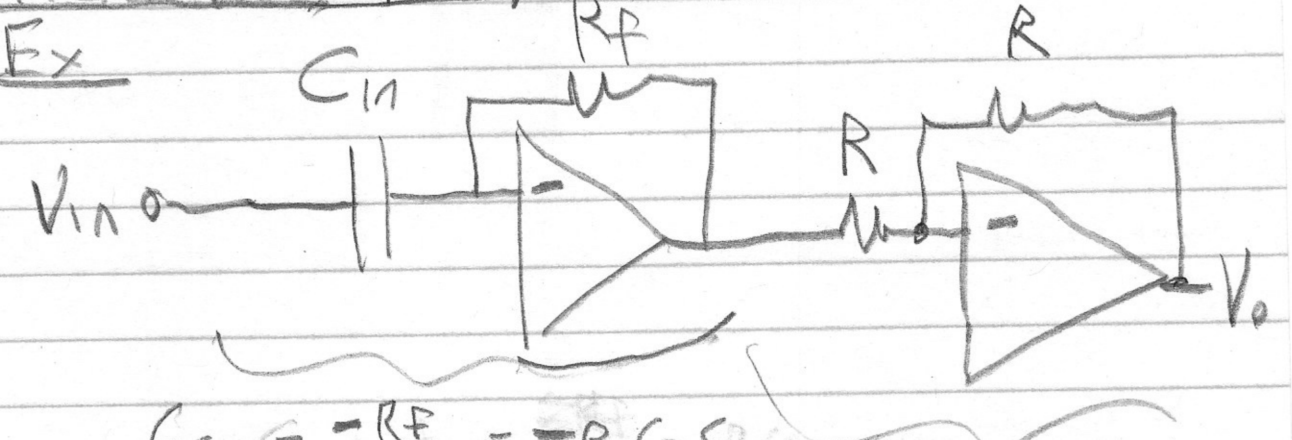
$$\measuredangle KH = \begin{cases} \lim_{\omega \to \infty} \cong \frac{\gamma K_d}{s} \to -90^\circ \\ \lim_{\omega \to 0} \cong \frac{\gamma K_p}{s^2} \to -180 \end{cases}$$

Note If K_d & K_p are both too low
very low ω_{unity}, might have $\measuredangle KH = -180^\circ$

Lead With a Op-Amp (Almost P.D.)

Differentaater Not practical

Ex



$$Gain = \frac{-R_f}{(1/C_{in}s)} = -R_f C_{in} s$$

~~Gain of -1~~

If Op-Amp Ideal!!

$$V_o(s) = \frac{R_f}{1/(C_{in}s)} V_{in}(s)$$

$$= R_f C_{in} s \, V_{in}(s)$$

$$V_o(t) \cong R_f C_{in} \frac{d}{dt} V_{in}(t)$$

$$H(j\omega) = jR_f C_{in} \omega$$

$$|H(j\omega)| = R_f C_{in} \omega$$

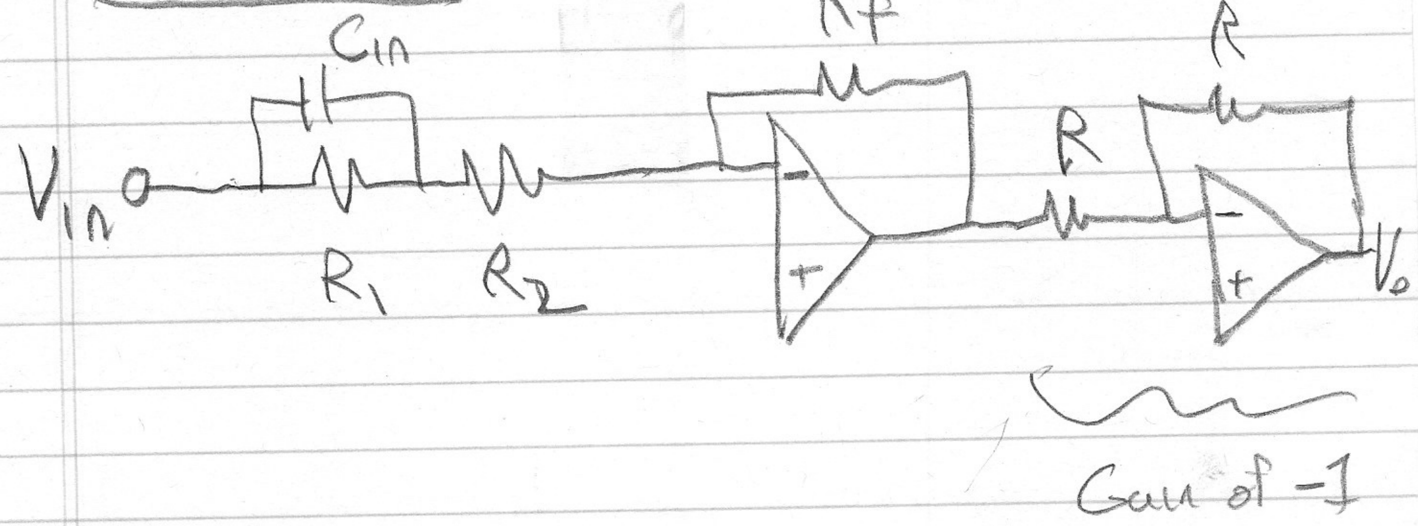
$$\angle |H(j\omega)| = +90^\circ$$

Requires infinite gain at infinite frequency

But deriv gives 90° phase boost

(11)

Lead



<u>At very low ω</u> $\left.\frac{1}{C_{in}s}\right|_{s=j\omega} \simeq \infty$ (Open)

$$V_o = \frac{R_f}{R_1 + R_2} V_{in}$$

<u>At very high ω</u> $\left.\frac{1}{C_{in}s}\right|_{s=j\omega} \simeq 0$ (Short)

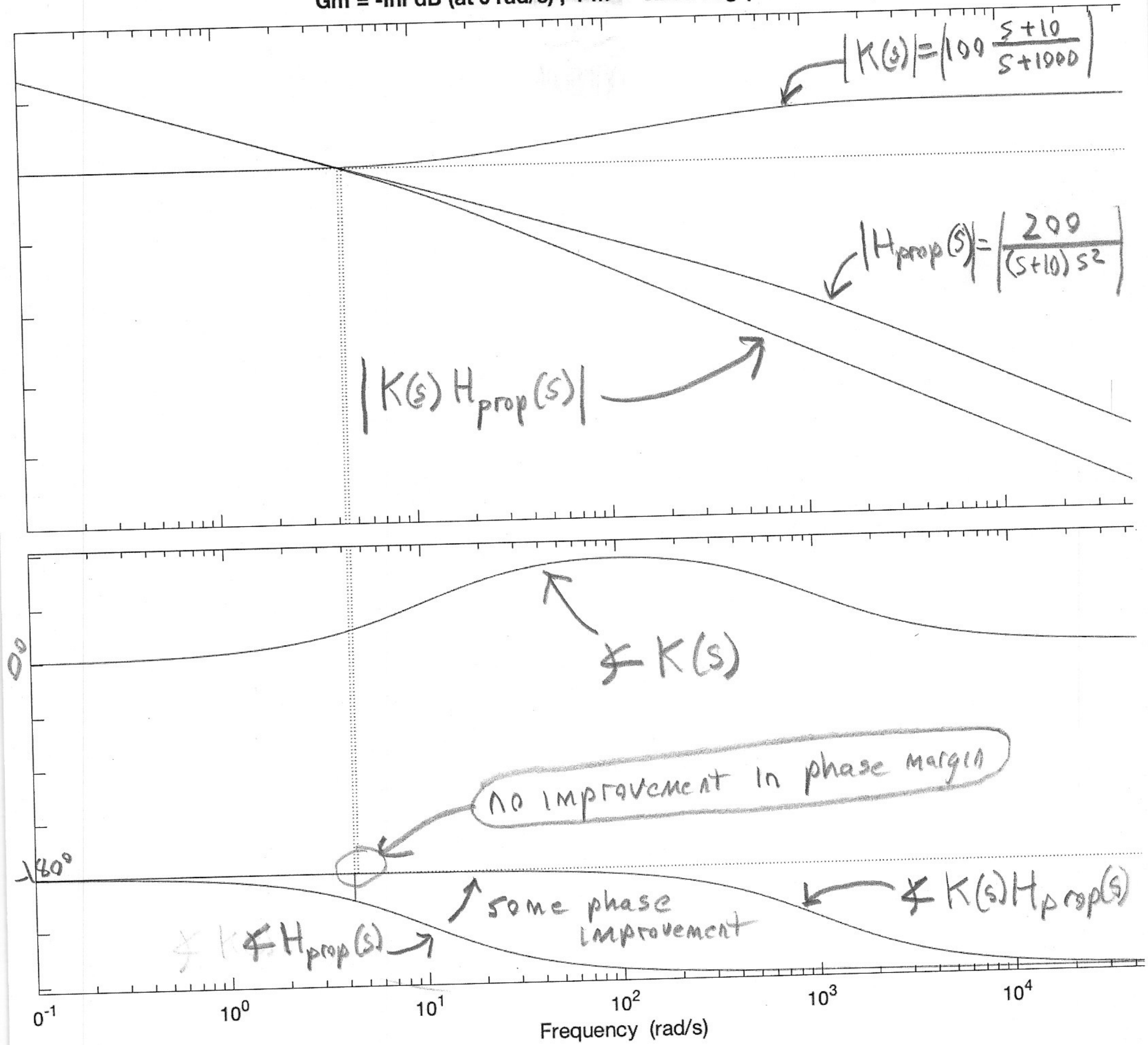
$$V_o = \frac{R_f}{R_2} V_{in}$$

<u>For what ω</u> is $\left.\frac{1}{C_{in}s}\right|_{s=j\omega} \simeq R_1$? $\left.\frac{1}{C_{in}s}\right| \simeq R_2$?

(12) $$H_{prop}(s) = \underset{\gamma}{\underset{\uparrow}{20}} \underbrace{\left(\frac{10}{s+10}\right)}_{\text{command to thrust pole}} \underbrace{\frac{1}{s}}_{\text{velocity to position pole}} \underbrace{\frac{1}{s}}_{\text{acceleration to velocity pole}} \qquad K(s) = \frac{(-p)(s-q)}{(-q)(s-p)}$$

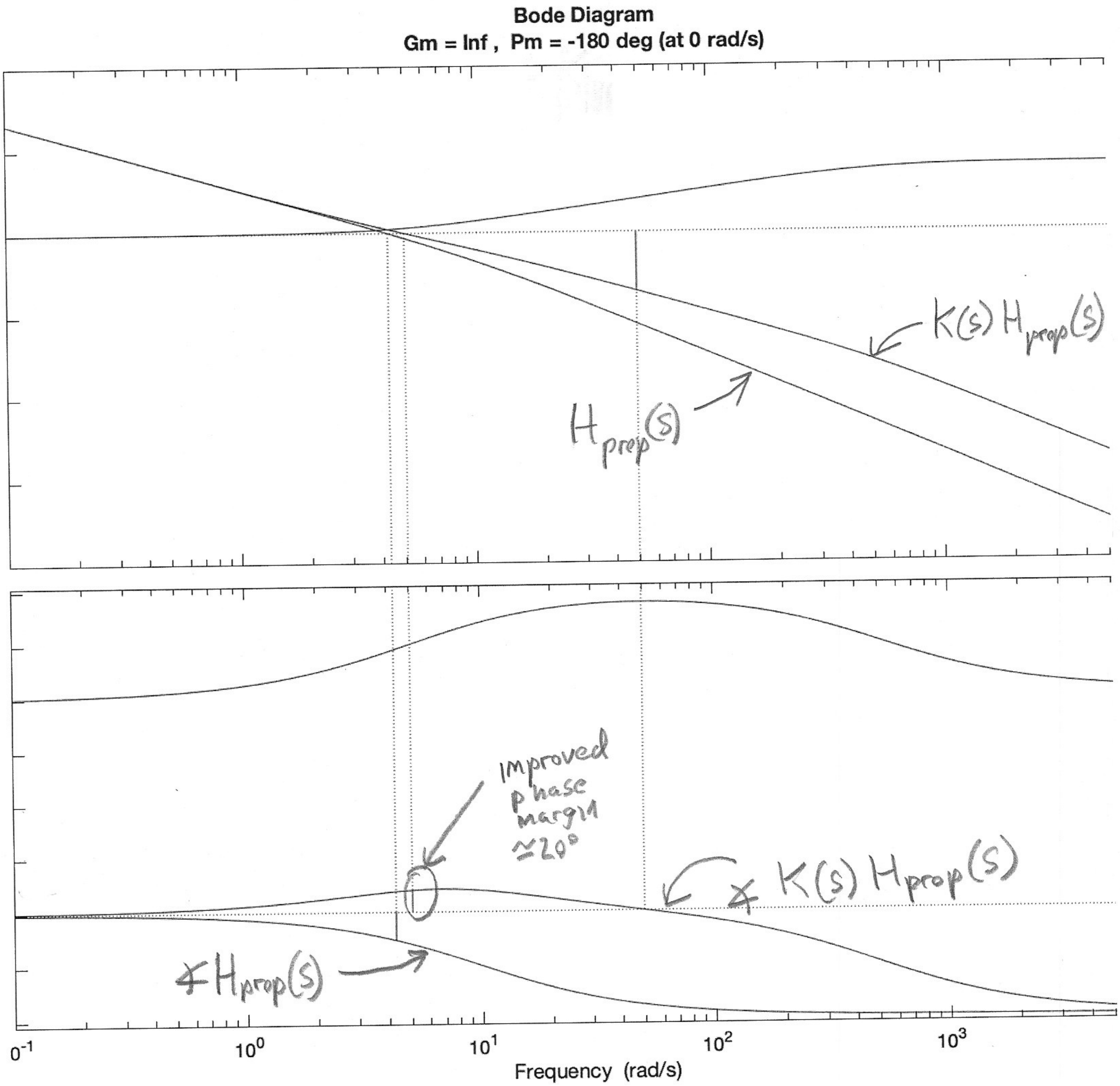
Bode Diagram

Gm = -Inf dB (at 0 rad/s) , Pm = -0.256 deg (at 4.47 rad/s)



Lead for copter arm

Better Lead: $K_{lead2}(s) = 100\left(\frac{s+5}{s+500}\right)$ B



step (feedback($\underbrace{((100*(s+5)/(s+500))}_{K(s)}*\underbrace{(200/((s+10)*s*s))}_{H(s)}$, 1))

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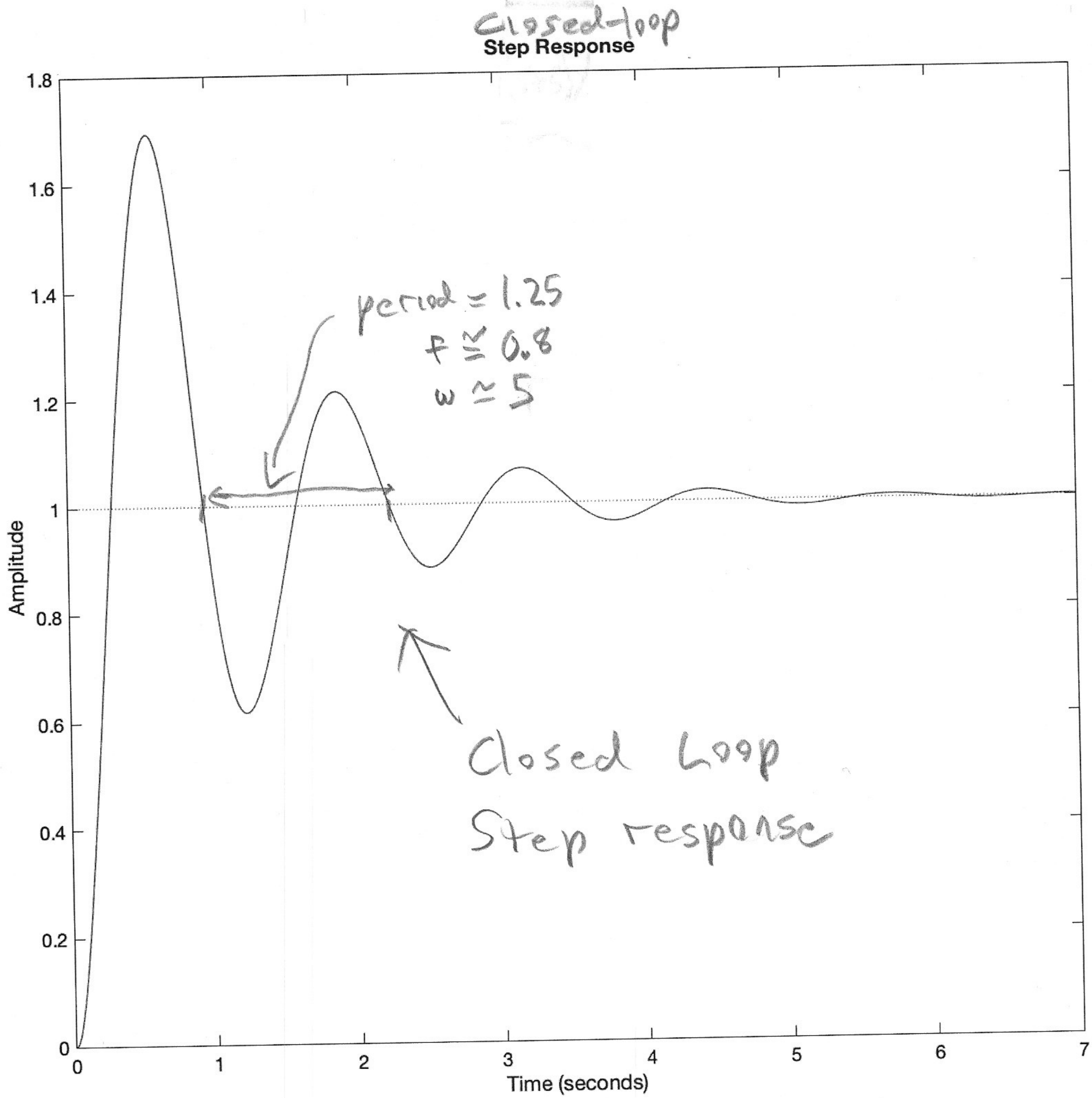

bode(feedback(Klead*Hprop, 1))

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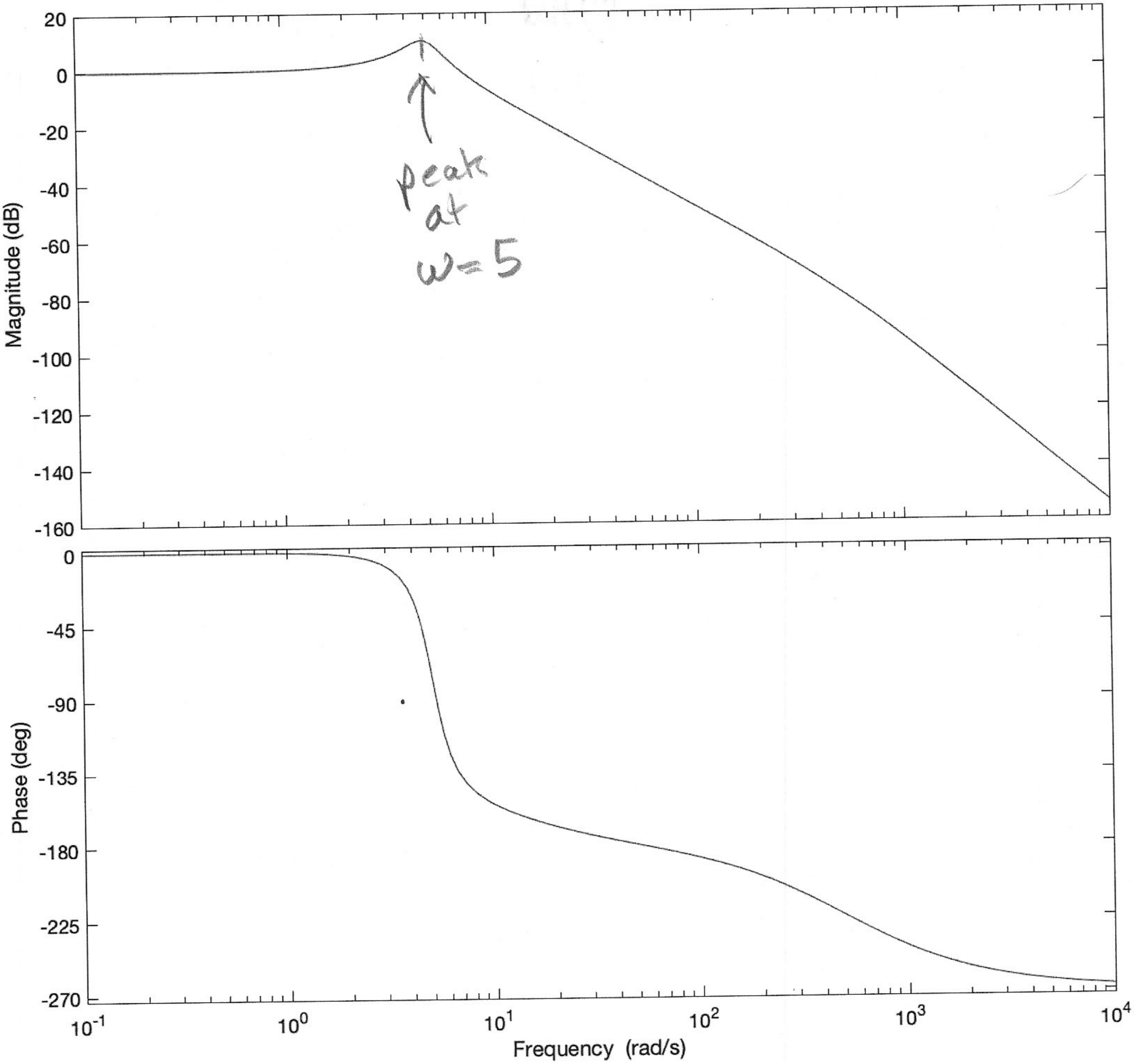
Closed-Loop

Bode Diagram



Even Better Lead

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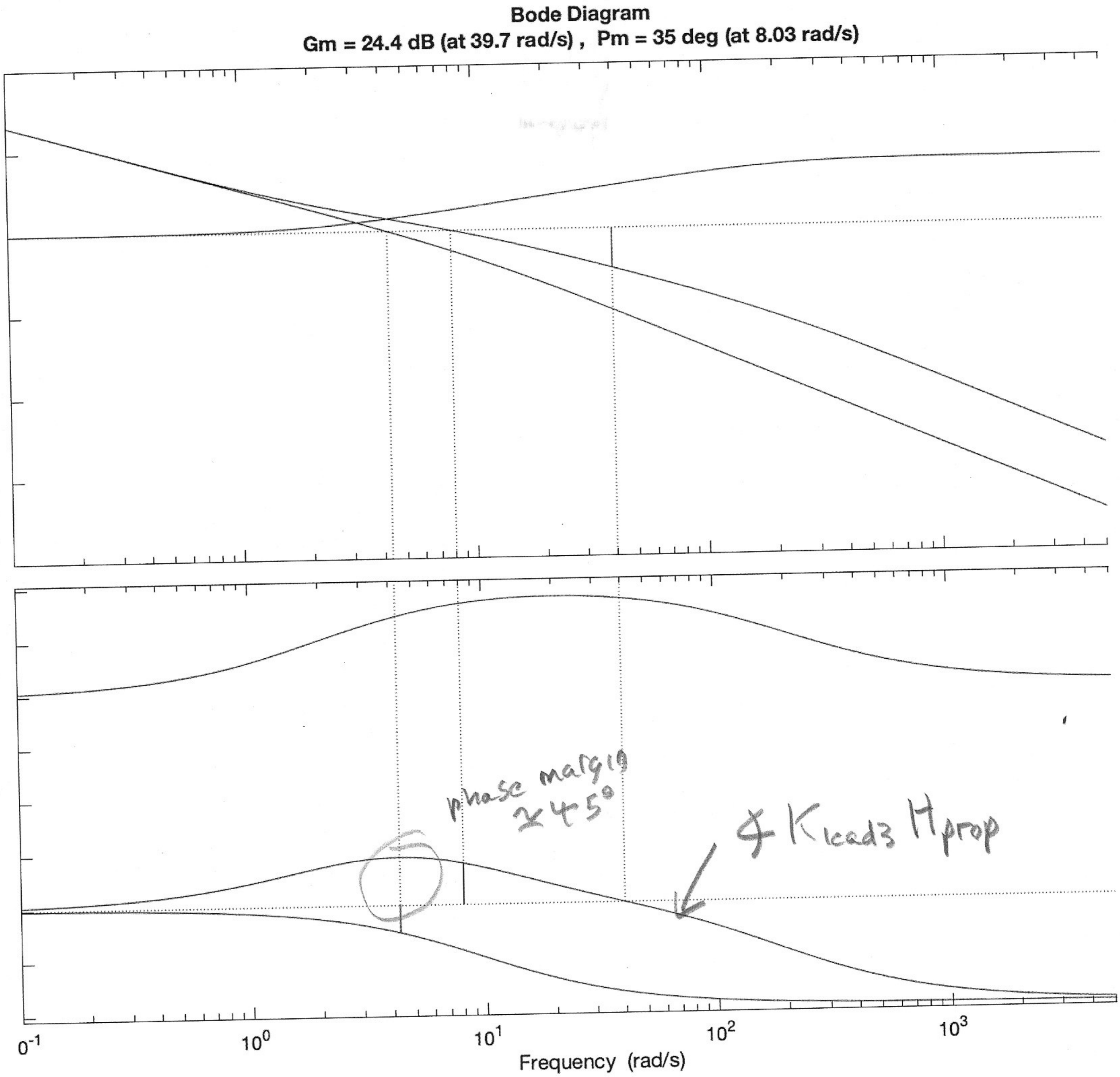
$$K_{lead3} = 100\left(\frac{s+2}{s+200}\right)$$



step(feedback(Klead3*Hprop, 1))

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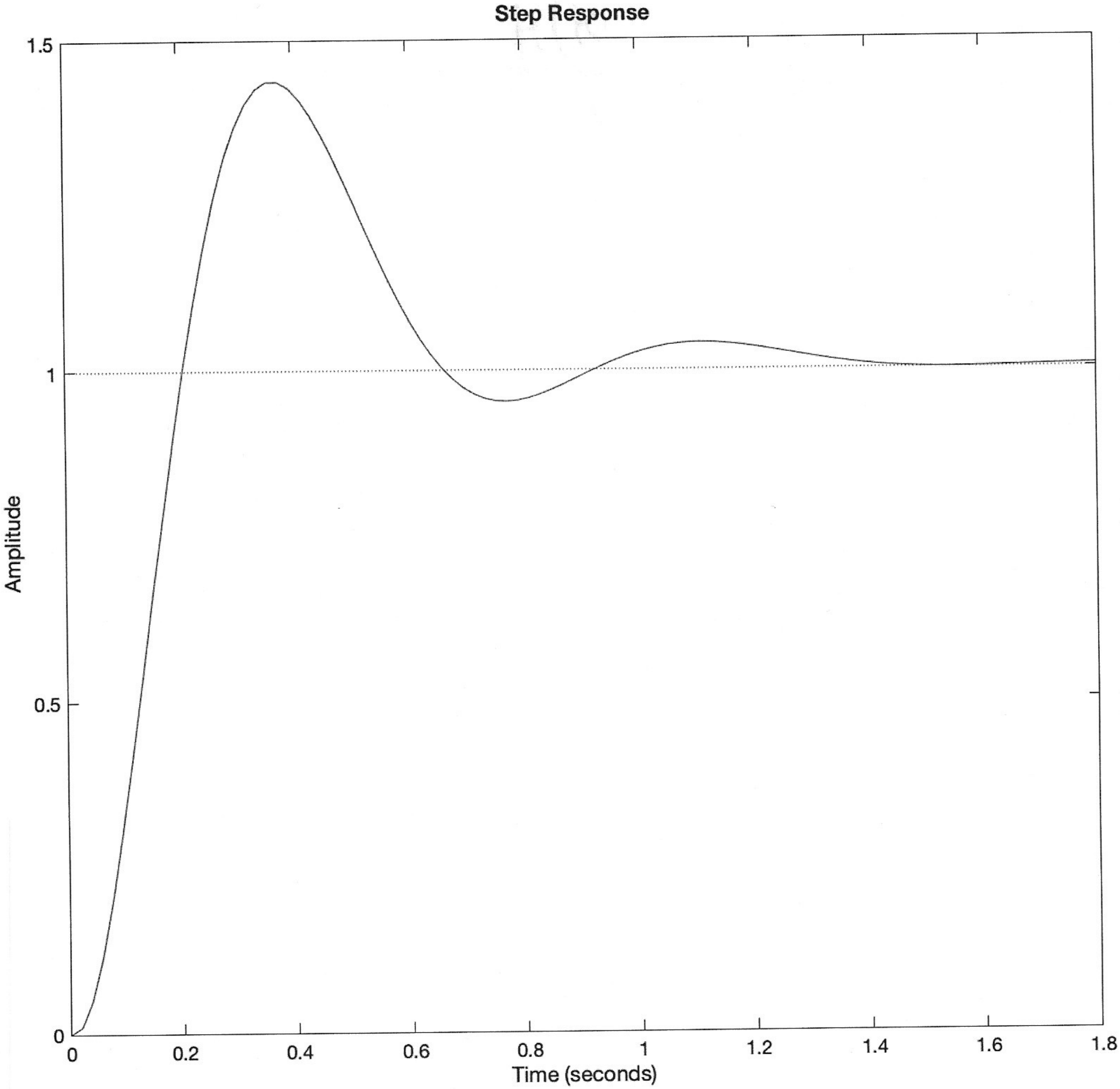
Closed-Loop



bode(feedback(Klead3*Hprop,1))

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Closed-loop