1 $\boxed{3/30/20}$

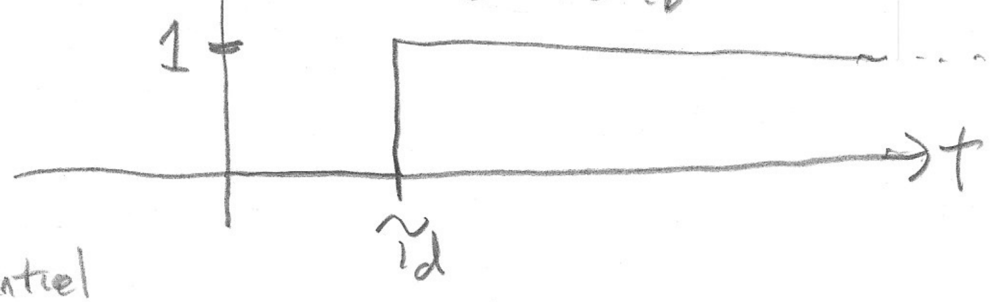
6.302 Three-Minute Transforms

(Okay, maybe > 3 minutes)

C.T. Signals

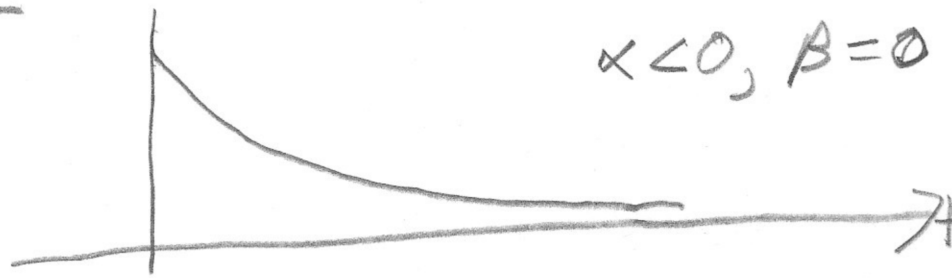
Delayed Unit Step

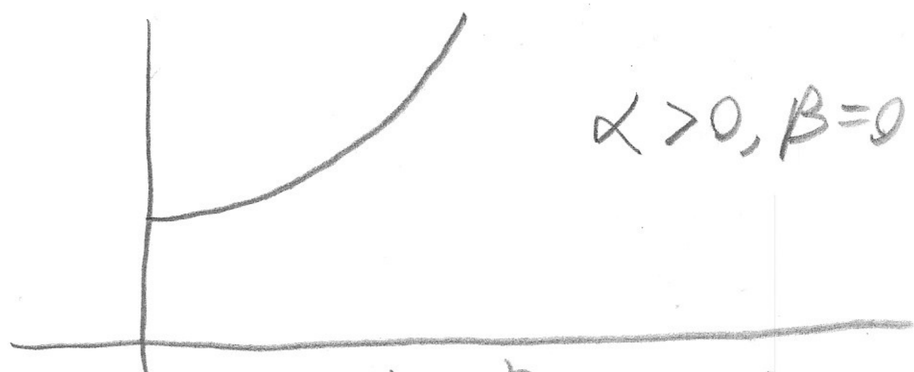
$u(t-\tau_d) = 0 \quad t < \tau_d$
$ = 1 \quad t \geq \tau_d$

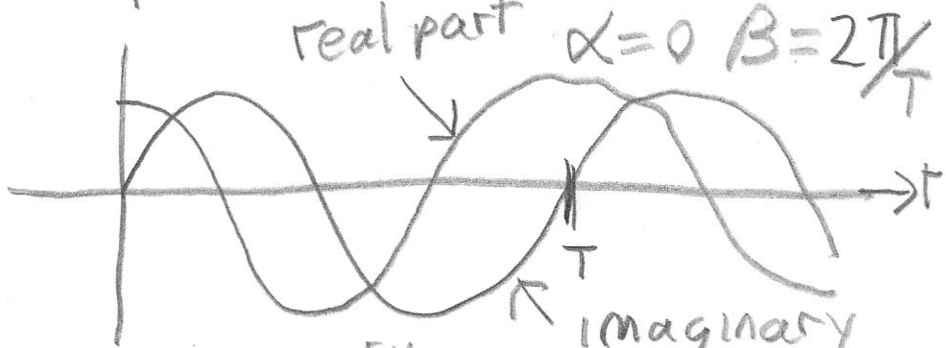


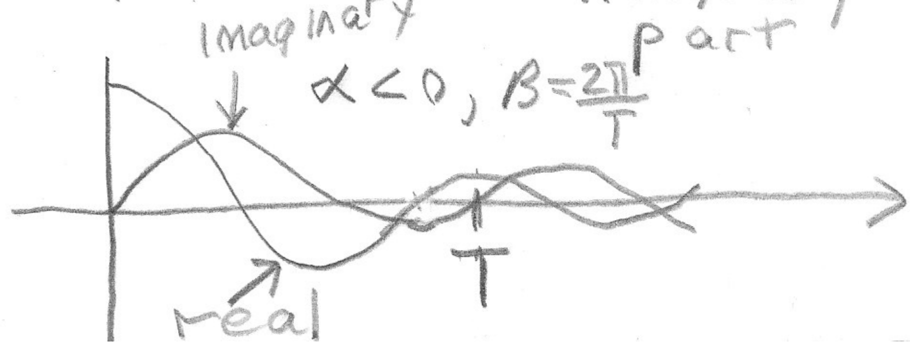
Complex Exponential

$e^{(\alpha+\beta j)t}u(t)$



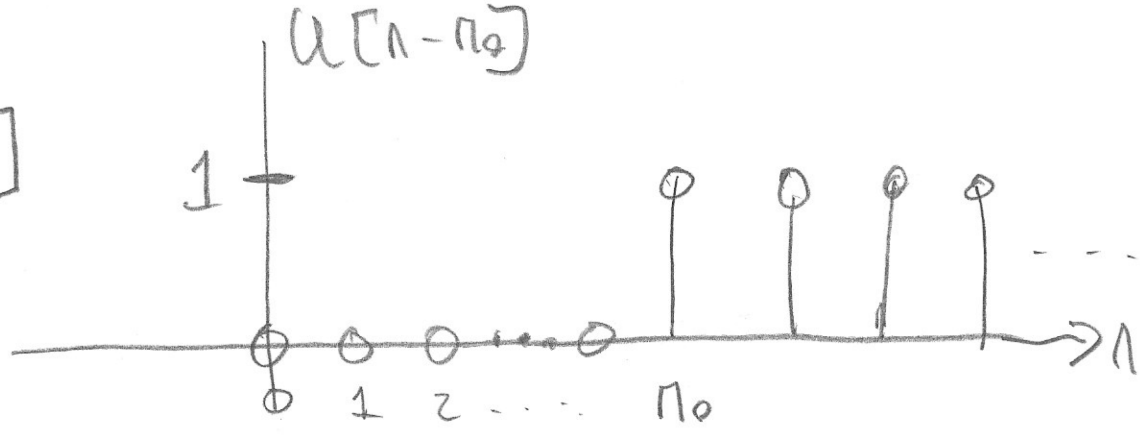






② # D. T. Signals

$u[n-n_0]$



$(\alpha+\beta j)^n u[n]$

or

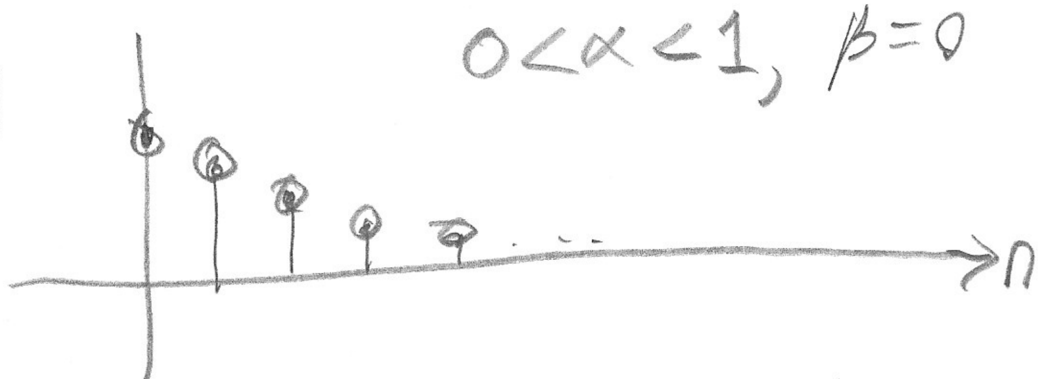
$Me^{j\theta n}u[n]$

$M=\sqrt{\alpha^2+\beta^2}$

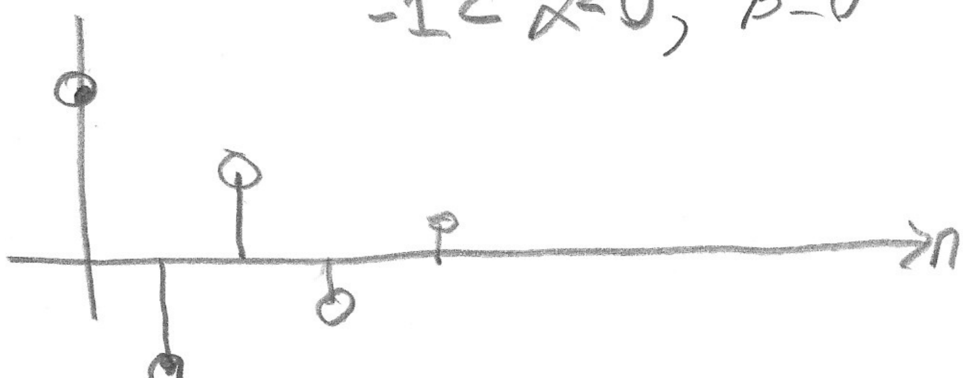
$\alpha = M\cos\theta$

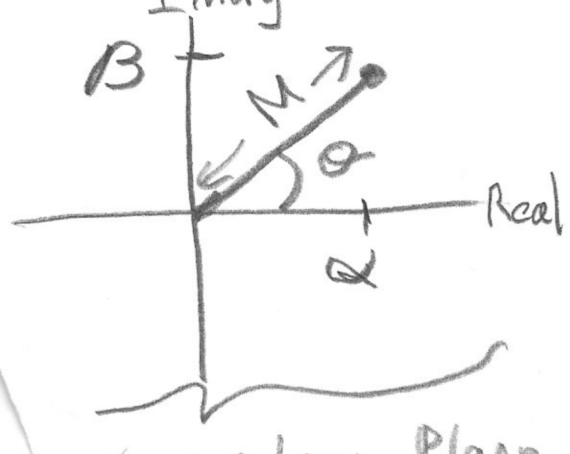
$\beta = M\sin\theta$

$0<\alpha<1,\ \beta=0$



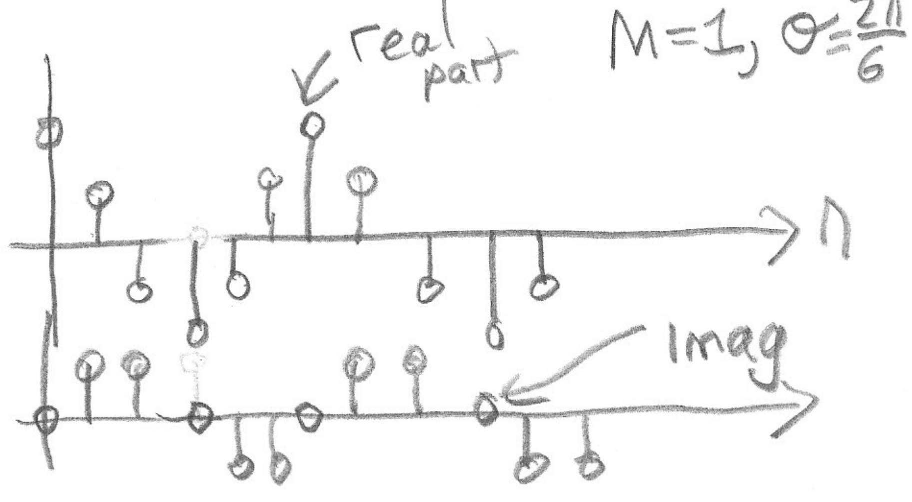
$-1<\alpha<0,\ \beta=0$





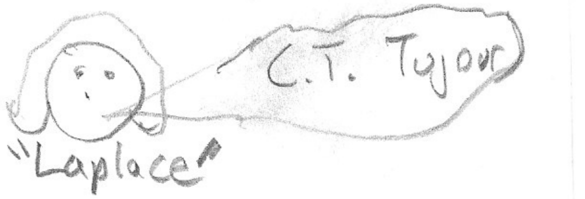
Complex Plane

$M=1,\ \theta=\frac{2\pi}{6}$



3

C.T. Laplace



$$\mathcal{L}(x(t)) = X(s) \equiv \underbrace{\int_0^\infty x(t)e^{-st}\,dt}_{\text{Unilateral Laplace}}$$

If $y(t) = \left(\frac{d}{dt}x\right)(t)$

$$\mathcal{L}(y) = Y(s) = s\,X(s)$$

$\boxed{\text{If } x(0) = 0}$

If $y(t) = \int_{-0}^{t} x(\tau)\,d\tau$

$$\mathcal{L}(y) = Y(s) = \frac{1}{s}X(s)$$

If $w(t) = A\,x(t) + B\,y(t)$

$$\mathcal{L}(w) = W(s) = A\,X(s) + B\,Y(s)$$

If $x(t) = Ae^{-at}u(t)$

$$\mathcal{L}(x) = \frac{A}{s+a}$$ $\boxed{(-a \text{ is a "pole"})}$

If $x(t) = (Ae^{-at} + Be^{-bt})u(t)$

$$\mathcal{L}(x) = \frac{A}{s+a} + \frac{B}{s+b} = \frac{(A+B)s + (Ab+Ba)}{s^2 + (a+b)s + ab}$$

4

D.T. Z Transforms

$$\mathcal{Z}(x) \equiv X(z) \equiv \underbrace{\sum_{n=0}^{\infty} x[n] z^{-n}}_{\text{unilateral "z" transform}}$$

If $y[n] = x[n+1]$

$$\mathcal{Z}(y) = z X(z)$$

$$\boxed{x[n] = 0, n<0}$$

If $y[n] = x[n-1]$

$$\mathcal{Z}(y) = z^{-1} X(z)$$

If $w[n] = A x[n] + B y[n]$

$$\mathcal{Z}(w) = W(z) = A X(z) + B Y(z)$$

If $x[n] = A a^n u[n]$

$$\mathcal{Z}(x) = X(z) = \frac{A}{1-az^{-1}}$$

"pole"

If $x[n] = A a^n u[n] + B b^n u[n]$

$$\mathcal{Z}(x) = \frac{A}{1-az^{-1}} + \frac{B}{1-bz^{-1}} = \frac{(A+B)+(Ab+Ba)z^{-1}}{1-(a+b)z^{-1}+abz^{-2}}$$

5 C. T. Systems

Diff. Eqns.

$$\frac{d}{dt} y(t) = -\alpha y(t) + \beta v(t)$$

$$\frac{d}{dt} v(t) = -\gamma v(t) + \xi a(t)$$

$\frac{d}{dt}$

$$\Downarrow$$

$$\frac{d^2}{dt^2} y(t) + \alpha \frac{d}{dt} y(t) = \beta \frac{d}{dt} v(t)$$

<u>If</u> $\underline{\gamma = 0}$

$$\frac{d^2}{dt^2} y(t) + \alpha \frac{d}{dt} y(t) = \xi a(t)$$

<u>If</u> $\gamma \neq 0$?

$$\frac{d^2}{dt^2} y(t) + (\alpha + \gamma) \frac{d}{dt} y(t) + \alpha\gamma = \beta\xi a(t)$$

Laplace

$$Y(s) = \frac{\beta}{s+\alpha} V(s)$$

$$V(s) = \frac{\xi}{s+\gamma} A(s)$$

$$\Downarrow$$

$$Y(s) = \underbrace{\frac{\beta}{s+\alpha} \frac{\xi}{s+\gamma}}_{} A(s)$$

Transfer functions compose

$$Y(s) = \frac{\beta\xi \, A(s)}{s^2 + (\alpha+\gamma)s + \alpha\gamma}$$

$$Y(s) = H(s) A(s)$$

(6) D.T. Systems

$$y[n] = -\alpha y[n-1] + \beta v[n-1]$$

$$v[n] = -\gamma v[n-1] + \xi a[n-1]$$

$\Downarrow$

shifts, adds

see notes on difference equations

$$y[n] + (\alpha+\gamma) y[n-1] + \alpha\gamma y[n-2] = \beta \xi a[n-2]$$

Z-transforms

$$Y(z) = \frac{\beta z^{-1}}{1+\alpha z^{-1}} V(z)$$

$$V(z) = \frac{\xi z^{-1}}{1+\gamma z^{-1}} A(z)$$

$$Y(z) = \frac{\beta z^{-1} \cdot \xi z^{-1} A(z)}{(1+\alpha z^{-1})(1+\gamma z^{-1})}$$

Transfer functions compose

$$Y(z) = \frac{\beta \xi z^{-2} A(z)}{1 + (\alpha+\gamma) z^{-1} + \alpha\gamma z^{-2}}$$

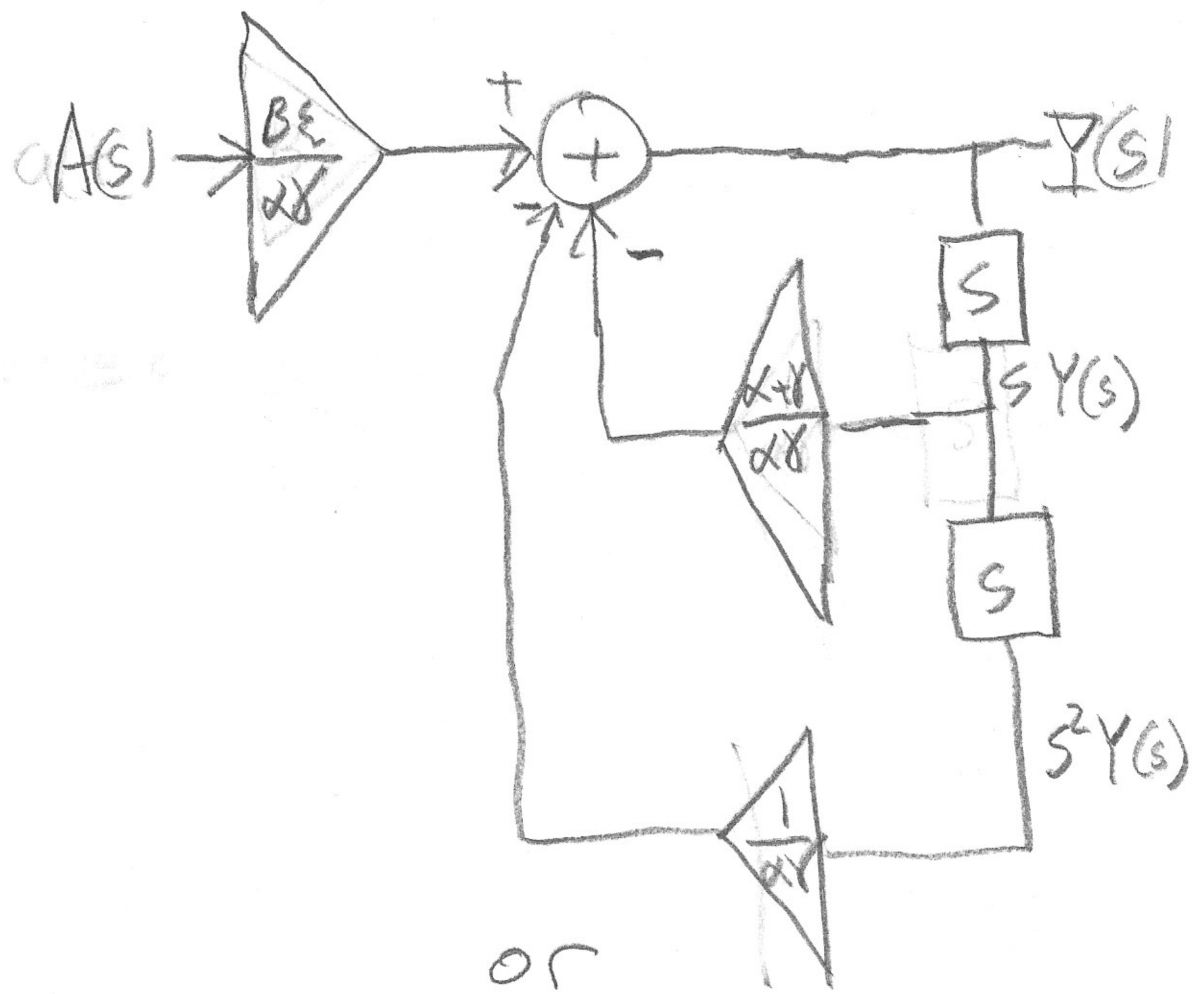
$$Y(z) = H(z) A(z)$$

7) C.T. Block Diagrams

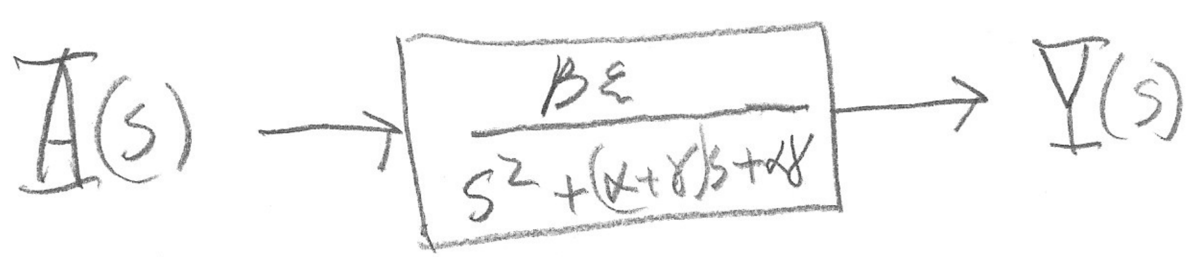
From a Differential Eqn

$$\frac{d^2}{dt^2}y(t) + (\alpha+\gamma)\frac{d}{dt}y + \alpha\gamma\, y = \beta\xi\, a(t)$$

$$s^2\, Y(s) + (\alpha+\gamma)\, s\, Y(s) + \alpha\gamma\, Y(s) = \beta\xi\, A(s)$$

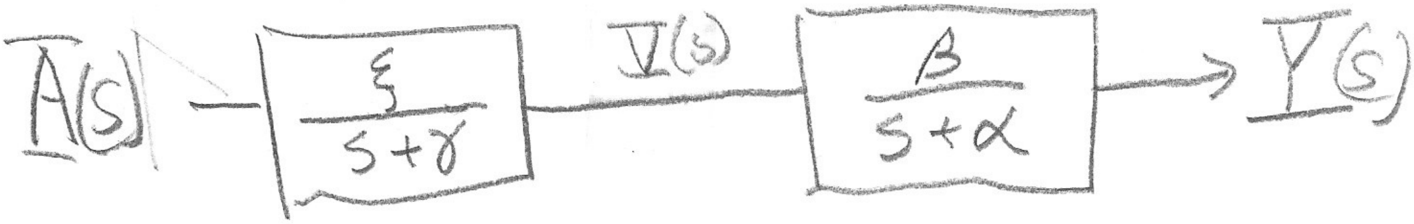


or

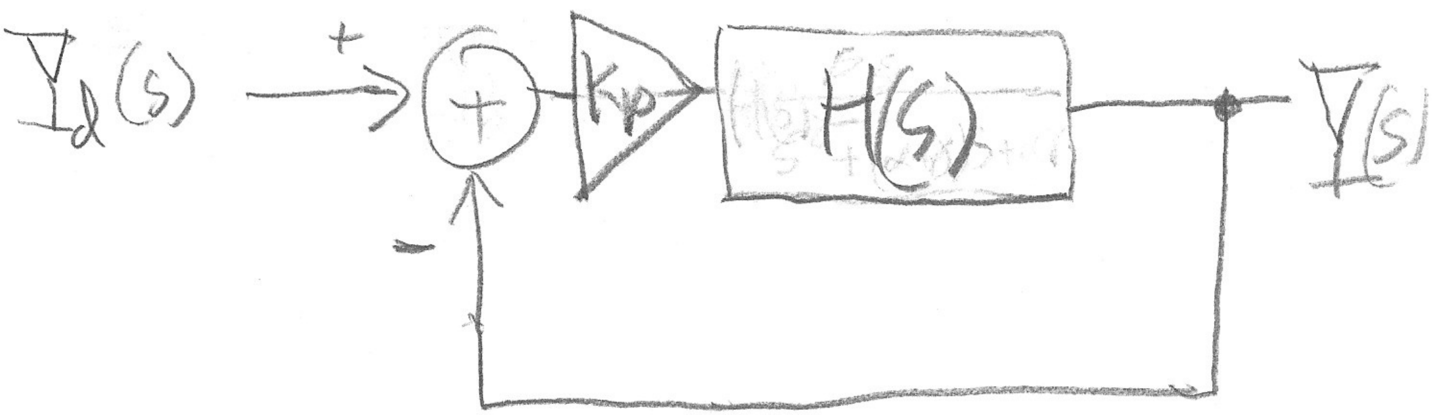


7A

Or



Suppose $\quad A(s) = K_p\left(Y_d(s) - Y(s)\right)$



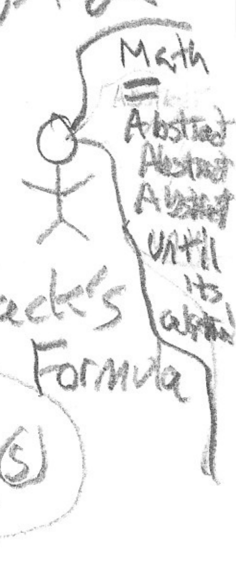
$$H(s) = \frac{\xi\beta}{s^2 + (\gamma+\alpha)s + \gamma\alpha}$$

$$Y(s) = H(s)\,K_p\left(Y_d(s) - Y(s)\right)$$

$$\left(1 + K_p H(s)\right) Y(s) = K_p\, H(s)\, Y_d(s)$$

Black's Formula

$$Y(s) = \frac{K_p H(s)}{1 + K_p H(s)}\, Y_d(s)$$

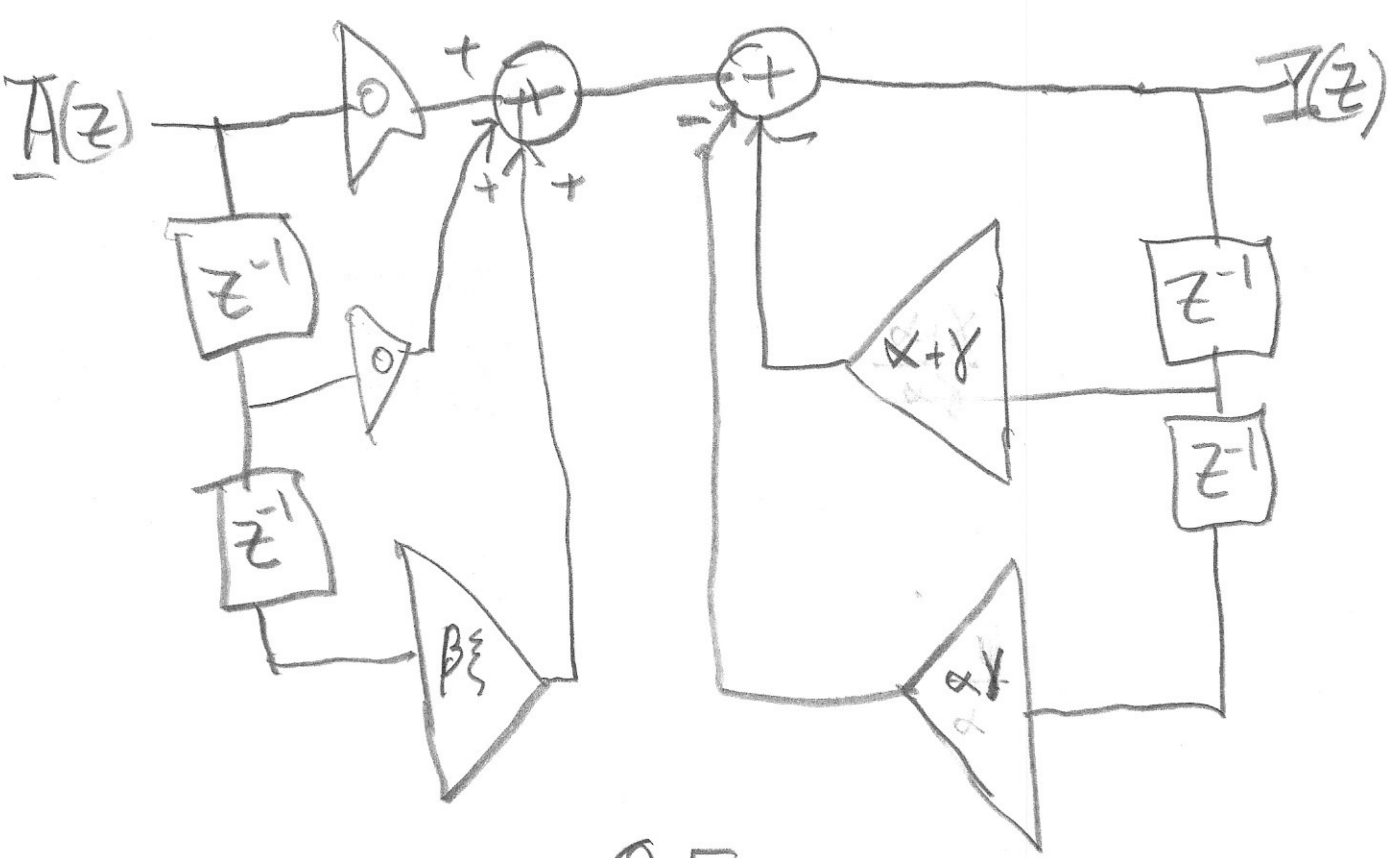


(8) D.T. Block Diagrams

From difference Equation

$$y[n] + (\alpha + \gamma)\, y[n-1] + \alpha\gamma\, y[n-2] = \beta\xi\, a[n-2]$$

$$Y(z) + (\alpha + \gamma)\, z^{-1}\, Y(z) + \alpha\gamma\, z^{-2}\, Y(z) = \beta\xi\, z^{-2} A(z)$$



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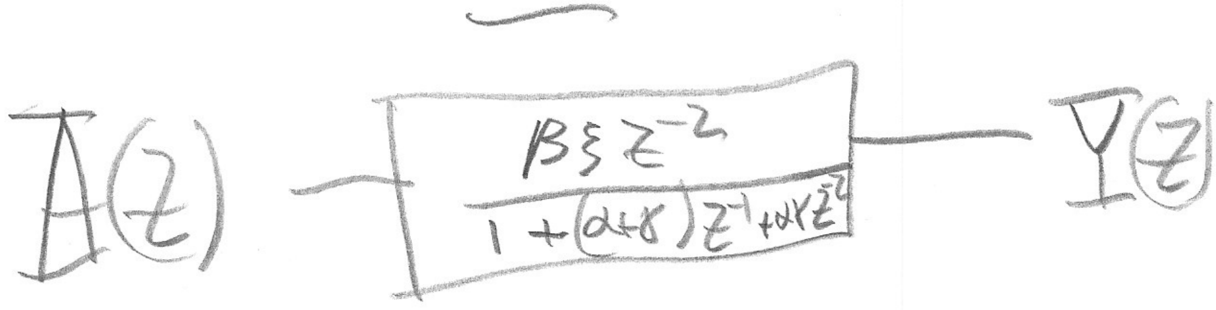
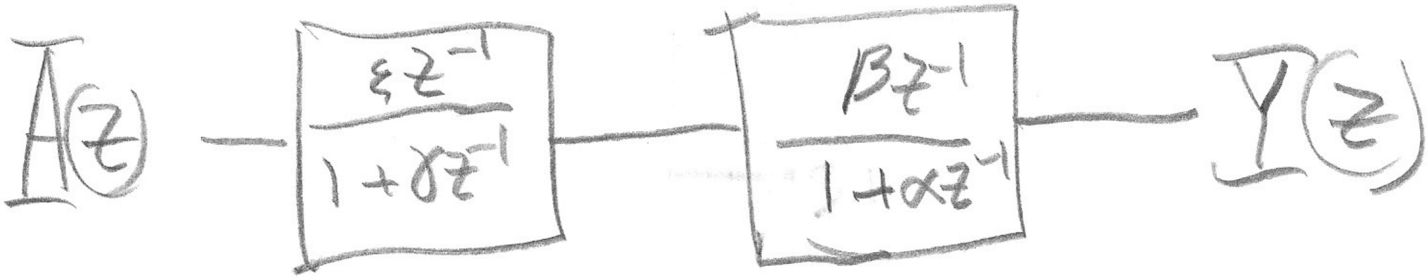
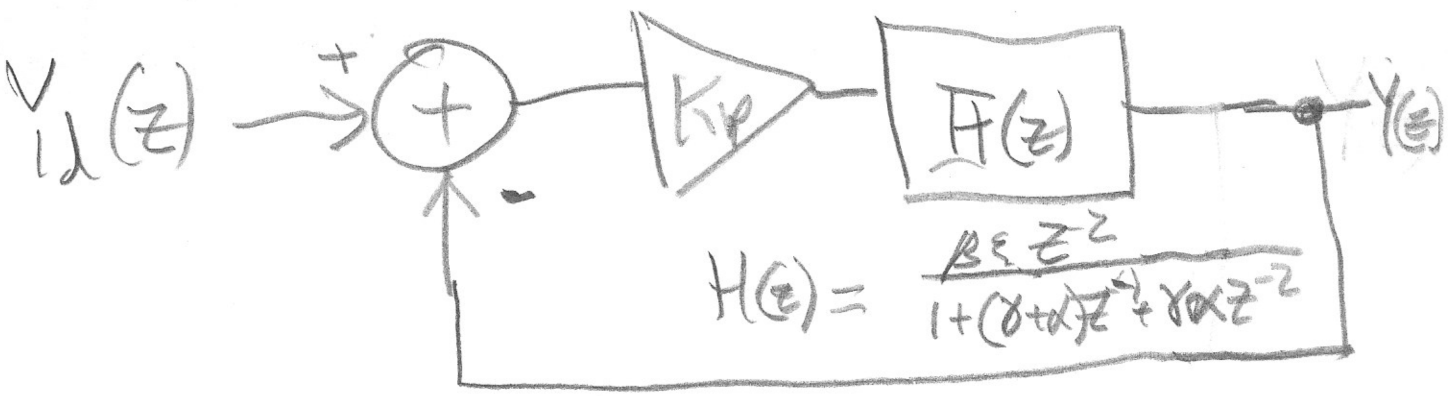


8A

<u>or</u>



<u>Suppose</u> $A(z) = K_p\left(Y_d(z) - Y(z)\right)$



$$Y(z) = H(z)\,K_p\left(Y_d(z) - Y(z)\right)$$

$$\left(1 + K_p H(z)\right) Y(z) = K_p H(z)\, Y_d(z)$$

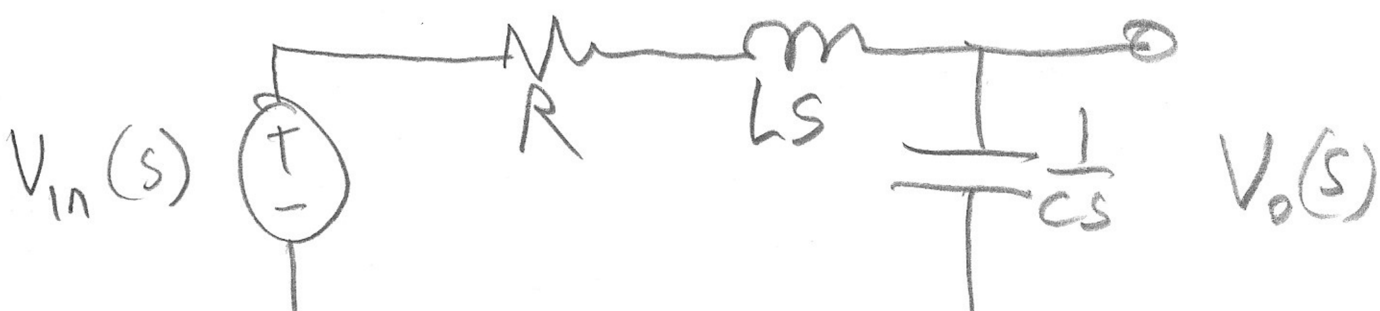
Black's Formula

$$Y(z) = \frac{K_p H(z)}{1 + K_p H(z)}\, Y_d(z)$$

9 Analyzing Circuits

$$V_o(s) = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_{in}(s) \quad \left(\text{Divider Formula}\right)$$

$$= \frac{1}{RCs + 1} V_{in}(s)$$



$$V_o(s) = \frac{\frac{1}{Cs}}{\underbrace{R + Ls + \frac{1}{Cs}}} V_{in}(s)$$

$$\frac{1}{LCs^2 + RCs + 1} = \frac{1}{LC} \cdot \left(\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \right)$$