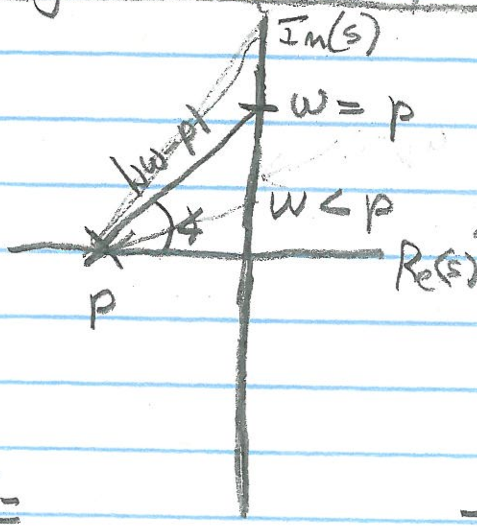
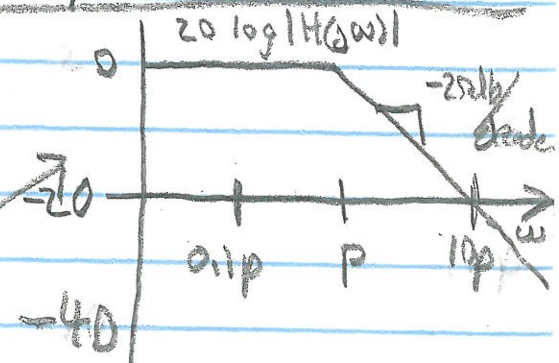
6.302 3/14/19 (Prelimary)

Lead and Lag Compensation

Sketching frequency response Reminder

Real Pole



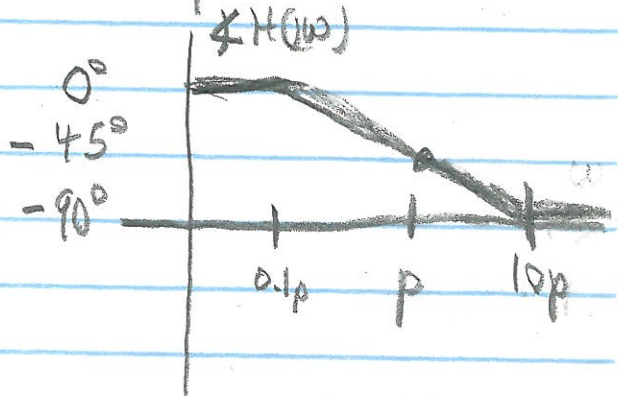


$H(s) = \frac{-p}{s-p}$

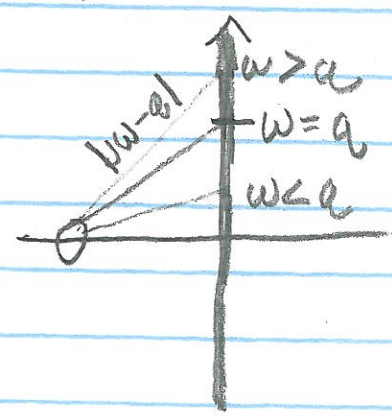
$20 \log |H(j\omega)| =$

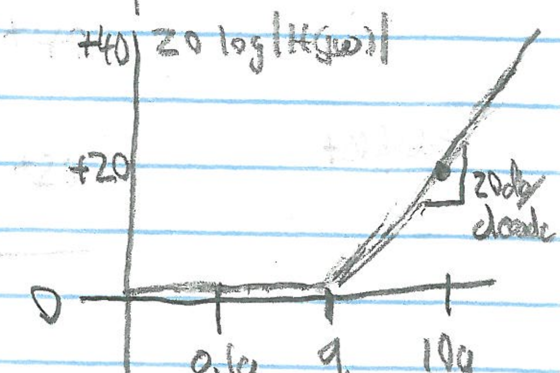
$20 \log |p| - 20 \log |j\omega - p|$

$\measuredangle H(j\omega) = -\arctan\left(\frac{\omega}{p}\right)$



Real Zero



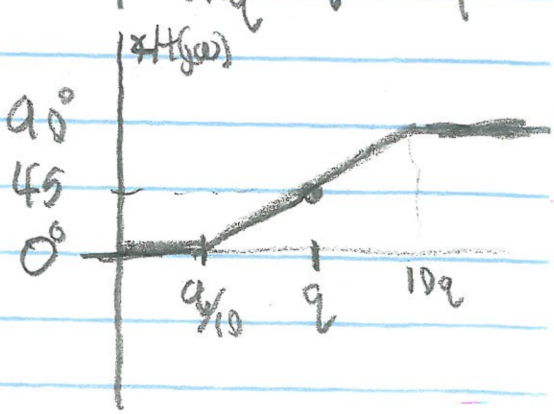


$H(s) = \frac{s-q}{-q}$

$20 \log |H(j\omega)| =$

$20 \log |j\omega - q| - 20 \log |q|$

$\measuredangle H(j\omega) = \arctan\left(\frac{\omega}{q}\right)$



1A

Straight Lines in Approximate Bode

A zero Example

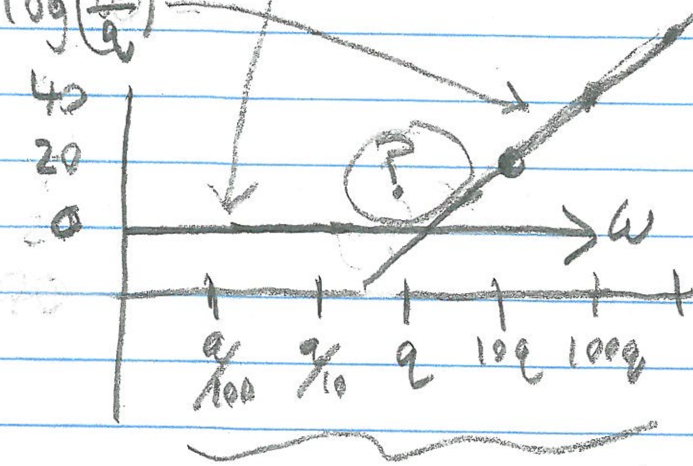
$$H(s)\Big|_{s=j\omega} = \frac{s-q}{-q}\Big|_{s=j\omega} = 1-\frac{s}{q}\Big|_{s=j\omega} = \underbrace{\left|1-j\frac{\omega}{q}\right|}_{=\sqrt{1+\left(\frac{\omega}{q}\right)^2}}$$

$$20\log|H(j\omega)| = 20\log\sqrt{1+\left(\frac{\omega}{q}\right)^2}$$

$$\cong 20\log 1 \quad \omega \ll q = 0$$

$$\simeq 20\log\left(\frac{\omega}{q}\right)$$

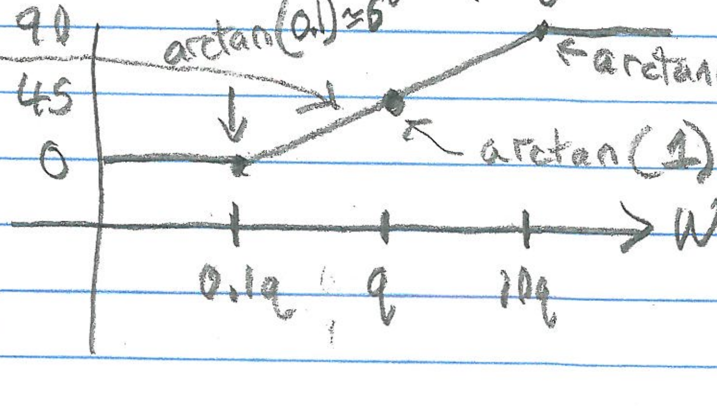
$20\log|H(j\omega)|$



In ? region: not so important

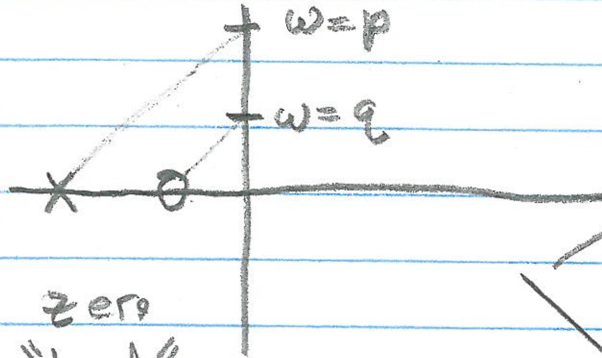
$$\measuredangle H(j\omega) = \arctan\left(\frac{\omega}{q}\right)$$ (positive since q (zero) is negative)

Straight line good enough

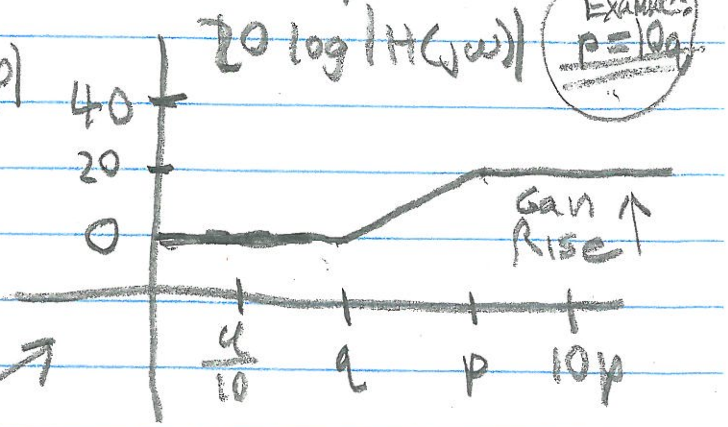


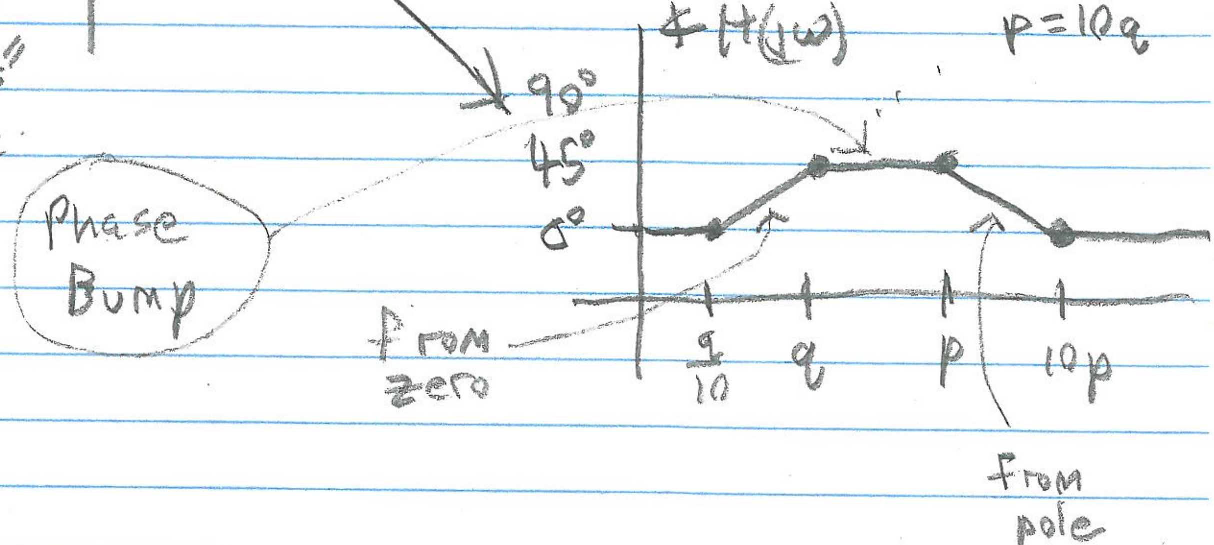
Lead Frequency Response

$$H(s) = \frac{p}{q}\left(\frac{s-q}{s-p}\right) \quad |q|<|p|$$



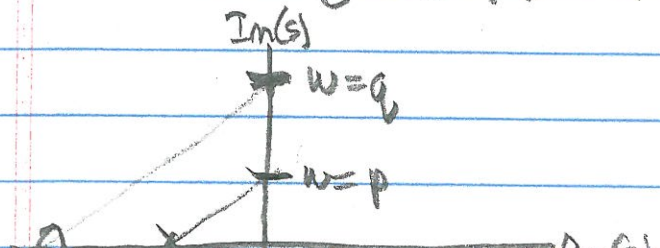
zero "leads" pole

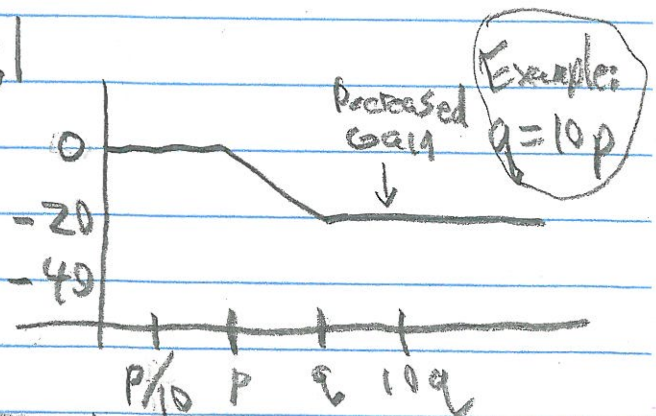




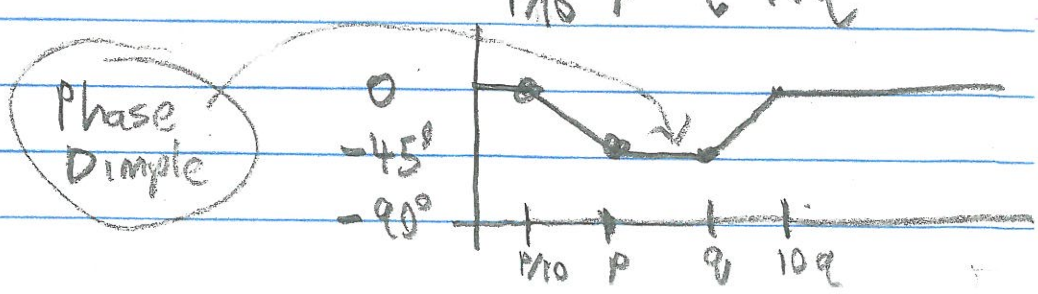
Lag Frequency Response

$$H(s) = \frac{p}{q}\left(\frac{s-q}{s-p}\right) \quad |p|<|q|$$

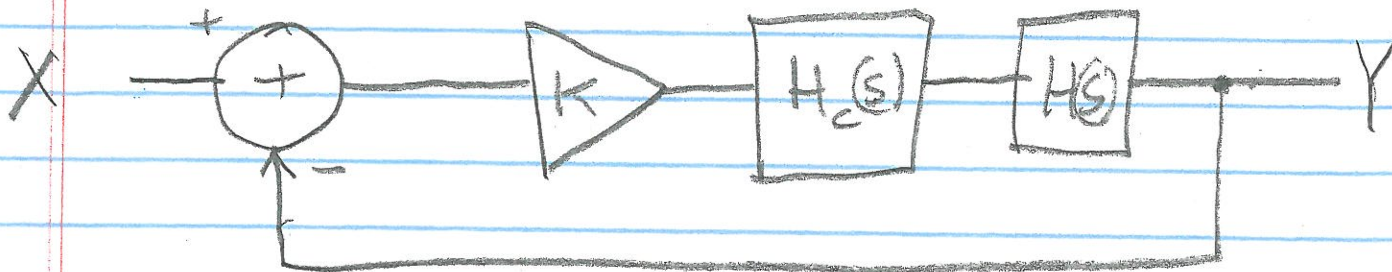




Zero "lags behind" pole



Feedback System



$Y(s) = \underbrace{G(s)}_{\text{closed loop}} X(s) \qquad G(s) = \dfrac{K H_c(s) H(s)}{1 + K H_c(s) H(s)}$

Want $|K H_c(j\omega) H(j\omega)|$ as large as possible for as wide a frequency range $[0, \omega_{max}]$ as possible.

Why

If $|K H_c(j\omega) H(j\omega)| \gg 1$

- Good tracking ($G(j\omega) \simeq 1$)
- Good disturbance rejection

$$\frac{1}{1 + K H_c(j\omega) H(j\omega)} \simeq 0$$

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<u>Also Want</u>

$$\text{At } \omega_{unity} \left(|K H_c(j\omega_{unity}) H(j\omega_{unity})| = 1 \right)$$

$$\measuredangle K H_c(j\omega_{unity}) H(j\omega_{unity}) > -180^\circ$$

<u>Why</u> (Heuristic)

If $K H_c(j\omega_{unity}) H(j\omega_{unity}) \Rightarrow -1$

$(\text{Magnitude} = 1,\ \measuredangle = -180^\circ)$

then

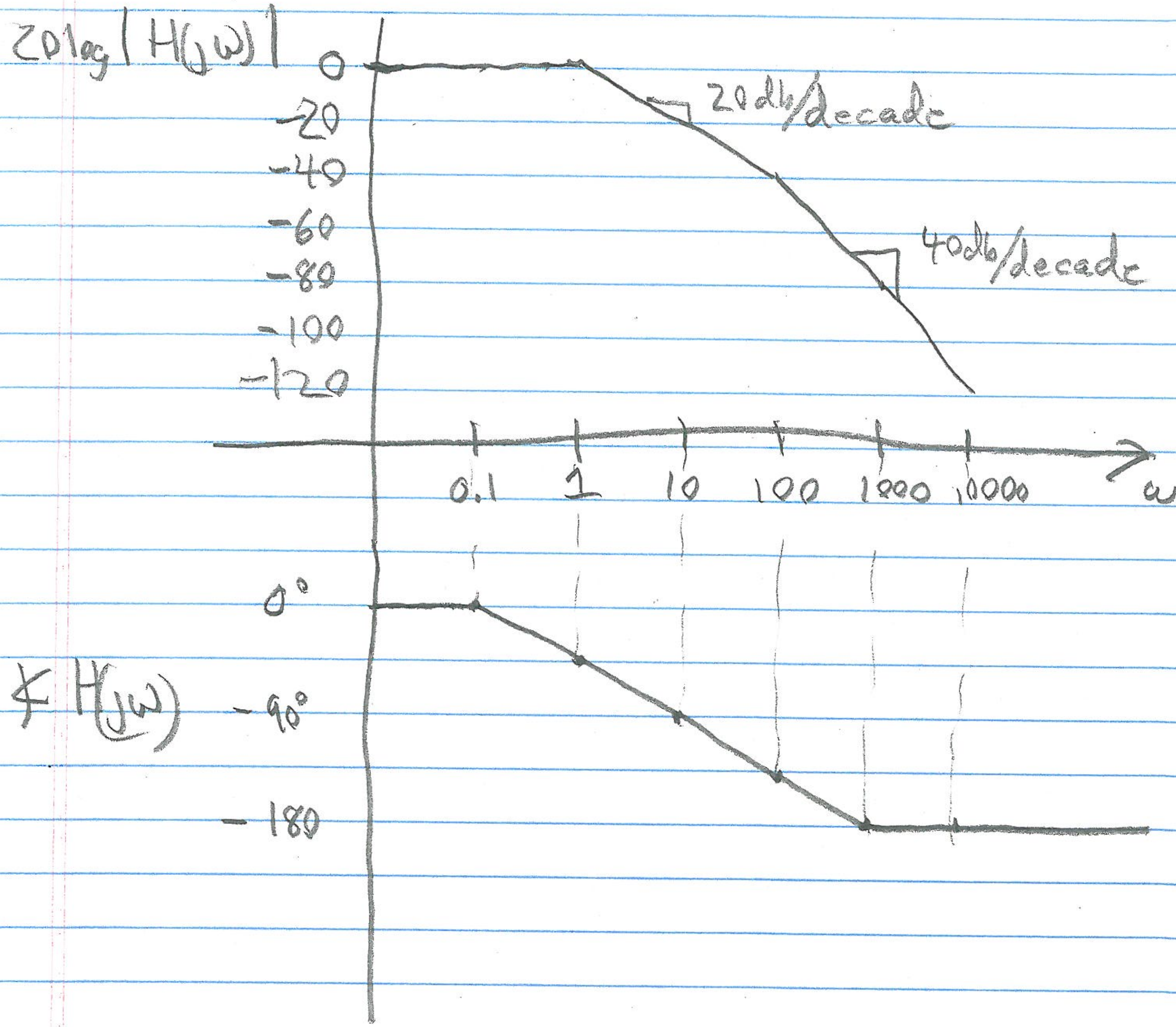
$$\frac{1}{1 + K H_c(j\omega_{unity}) H(j\omega_{unity})} \Rightarrow \infty$$

Instability!

Bad Disturbance Rejection!

Example

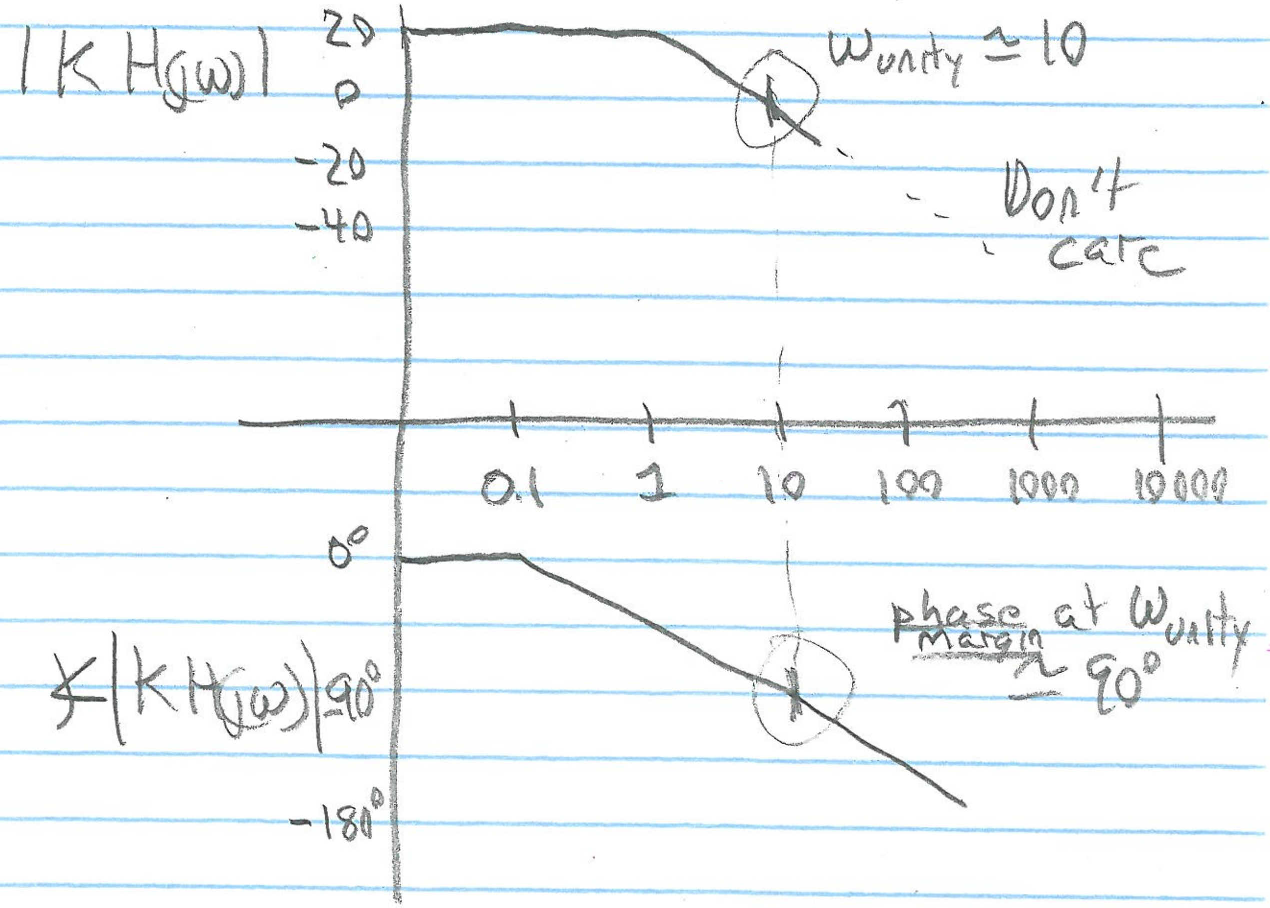
$$H(s) = \frac{100}{(s+1)(s+100)}$$



Design $H_c(s)$ for $K=10, 100$ & $10{,}000$

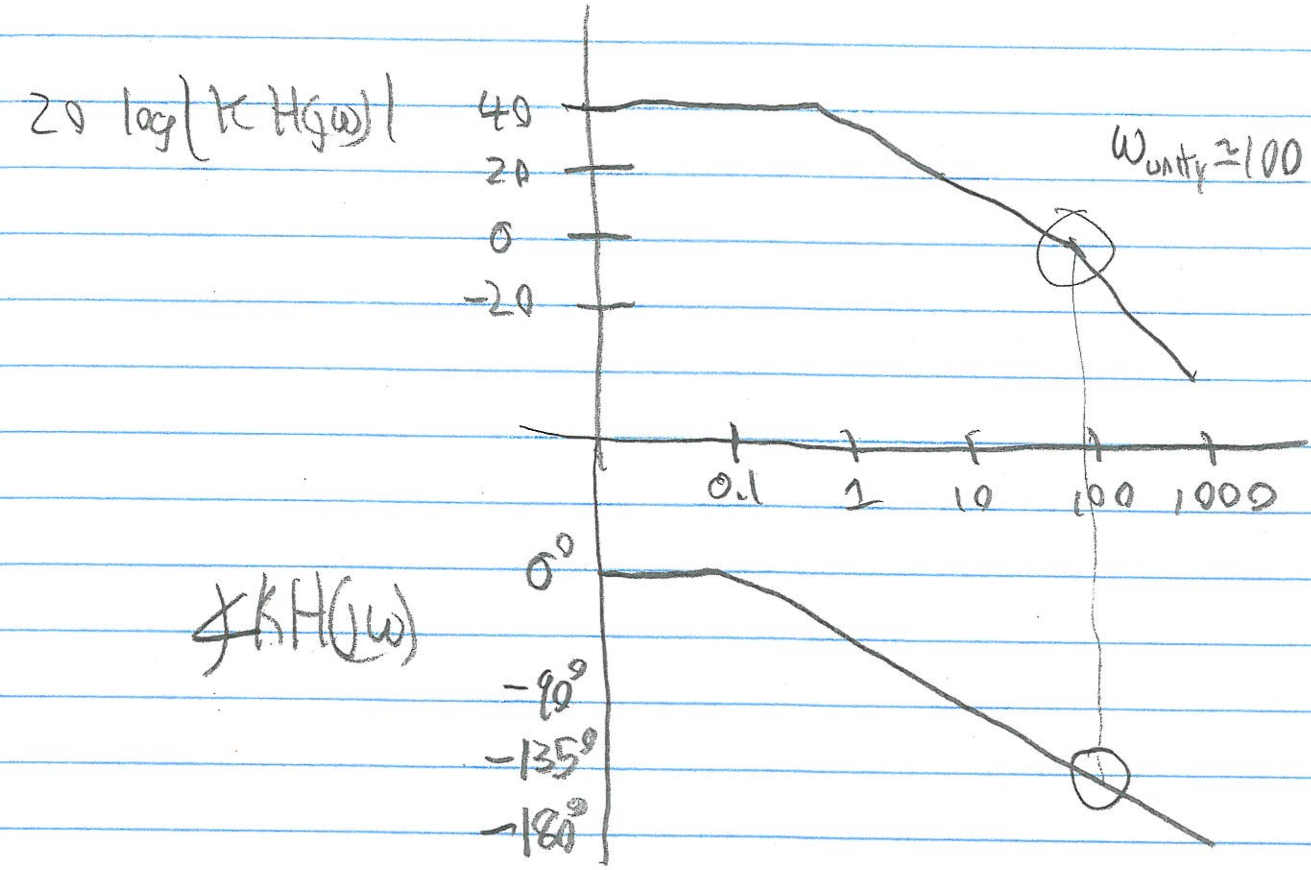
What is the phase margin?

$\underline{K = 10}$



$\underline{K=10}$, $H_c(s) = 1$, phase margin $\simeq 90°$

K = 100



For K = 100 phase margin $\simeq 45°$
okay but could be better. Lead? Lag?

Idea 1 Place a lead compensator with $q = -100$ $p = -1000$

Return to Later!

$$H_c(s) = \frac{p}{q}\left(\frac{s-q}{s-p}\right) = 10\,\frac{s+100}{s+1000}$$

$$K H_c(s) H(s) = 100 \cdot \frac{100}{(s+1)(s+100)} \cdot \frac{(s+100)}{s+1000} = \frac{10\,000}{(s+1)(s+1000)}$$

→ But Pole-Zero Cancellation Problematic!

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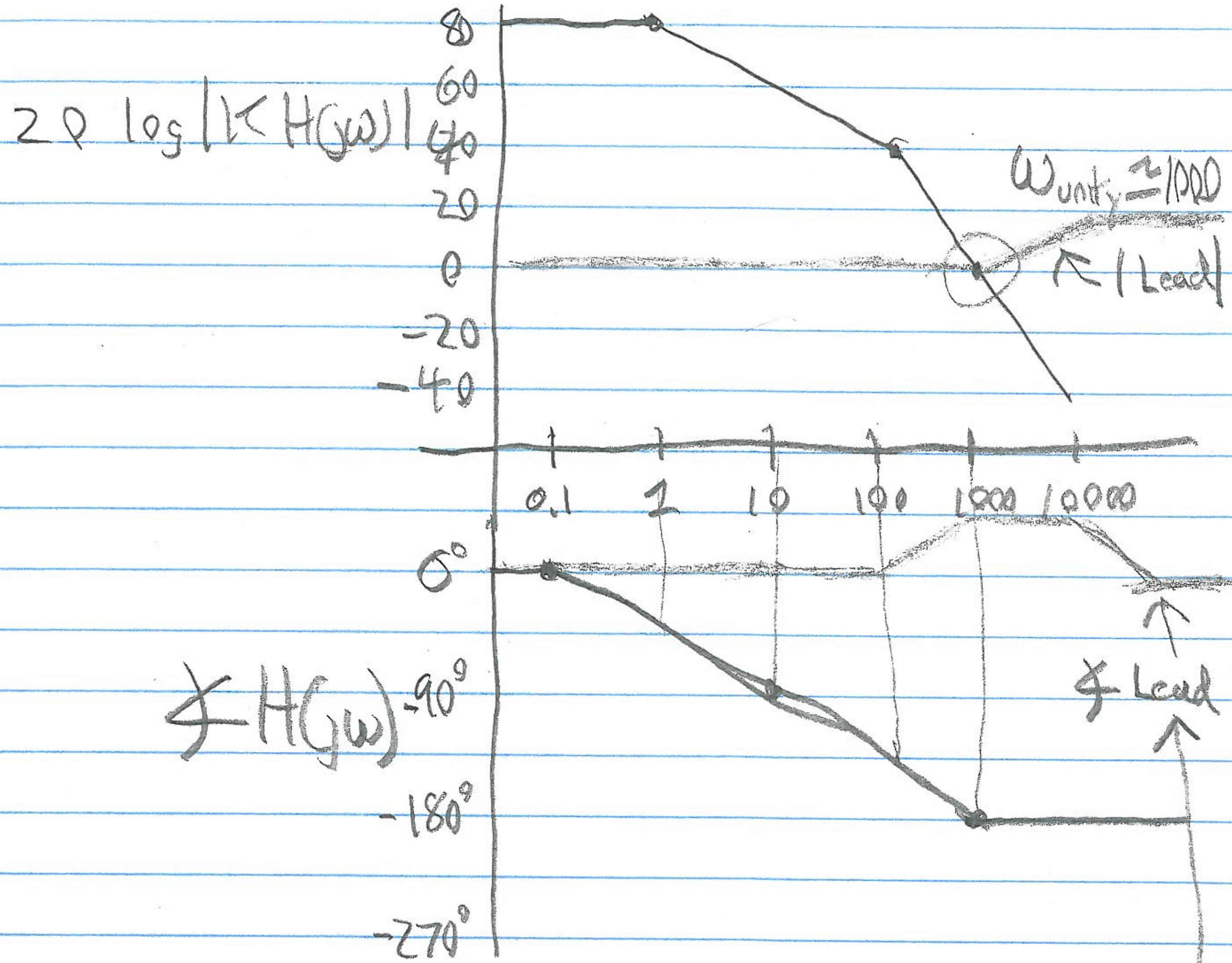
K = 10000



For K = 10000 phase margin $\simeq 0°$
Bad, almost unstable

Idea 1 Place a lead with
$q = -1000$ $p = -10000$

$$K H_c(s) H(s) = 100{,}000 \left(\frac{s + 1000}{s + 10000}\right)\left(\frac{100}{(s+1)(s+100)}\right)$$

Actual Margin

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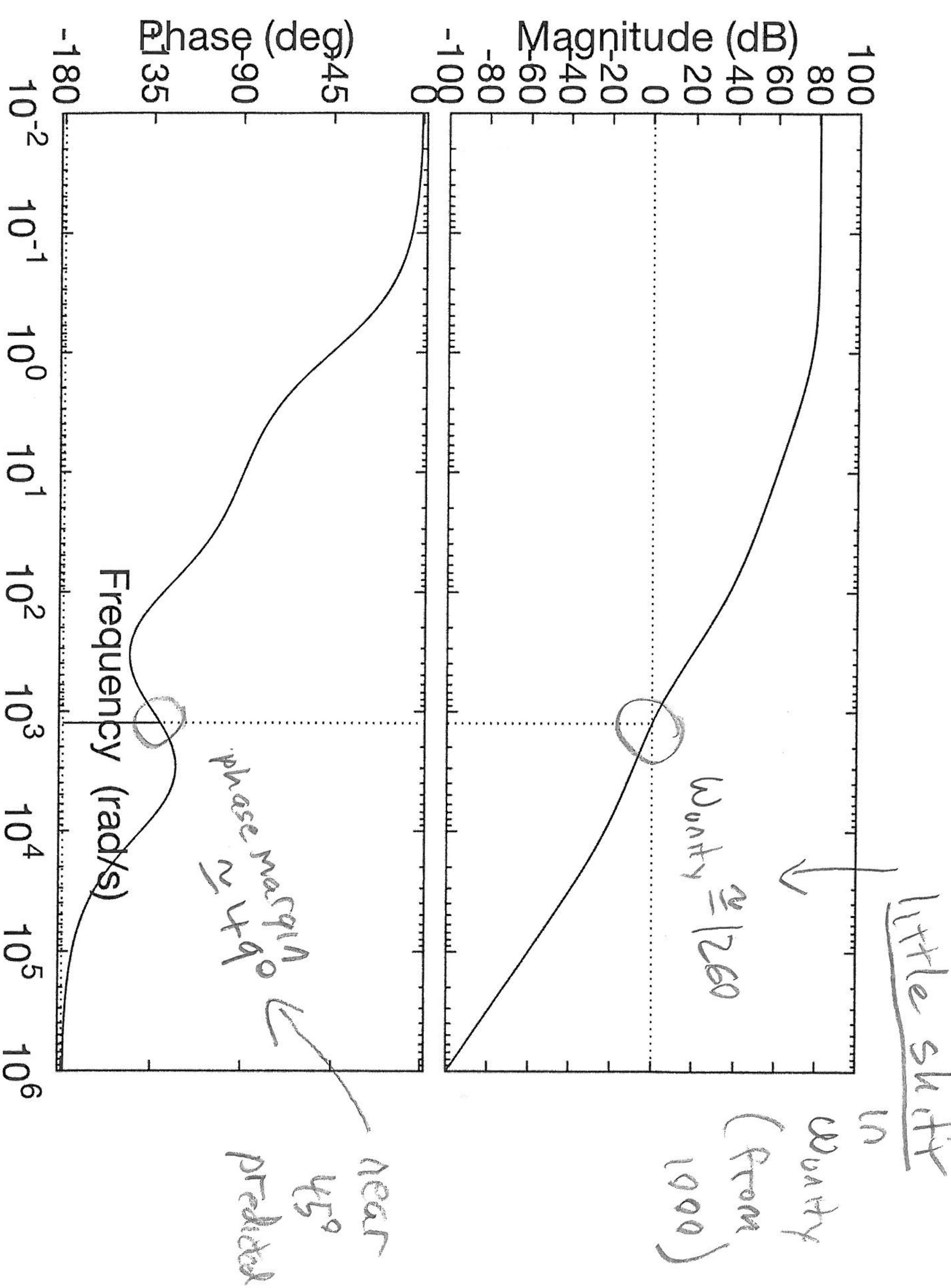


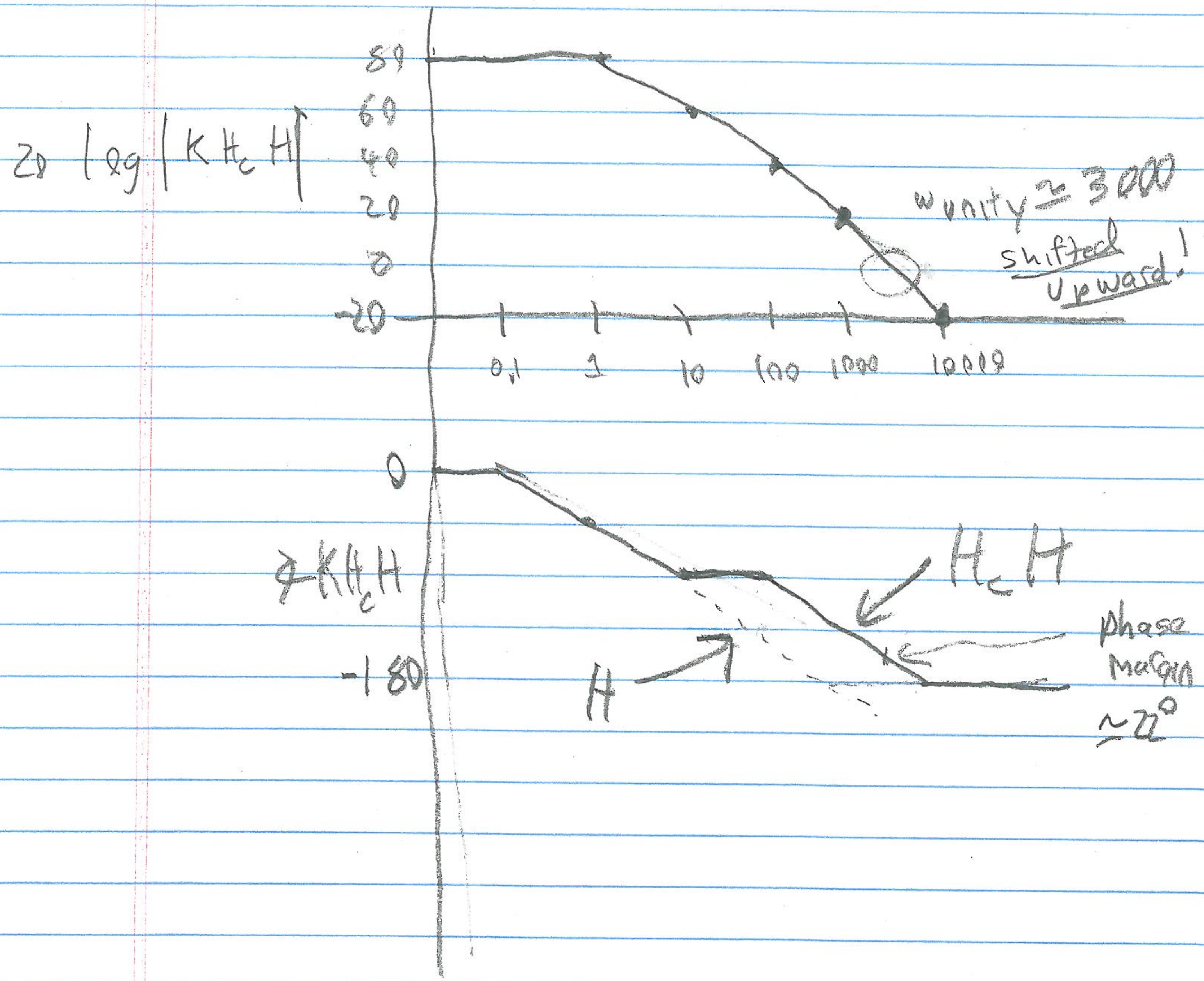
$K = 10,000$

Too Soon

10

$$H_c(s) = 10\left(\frac{s+100}{s+1000}\right)$$

Zero at too low a frequency



Actual Too Soon

$H_c(s) = 10 \frac{s+100}{s+1000}$

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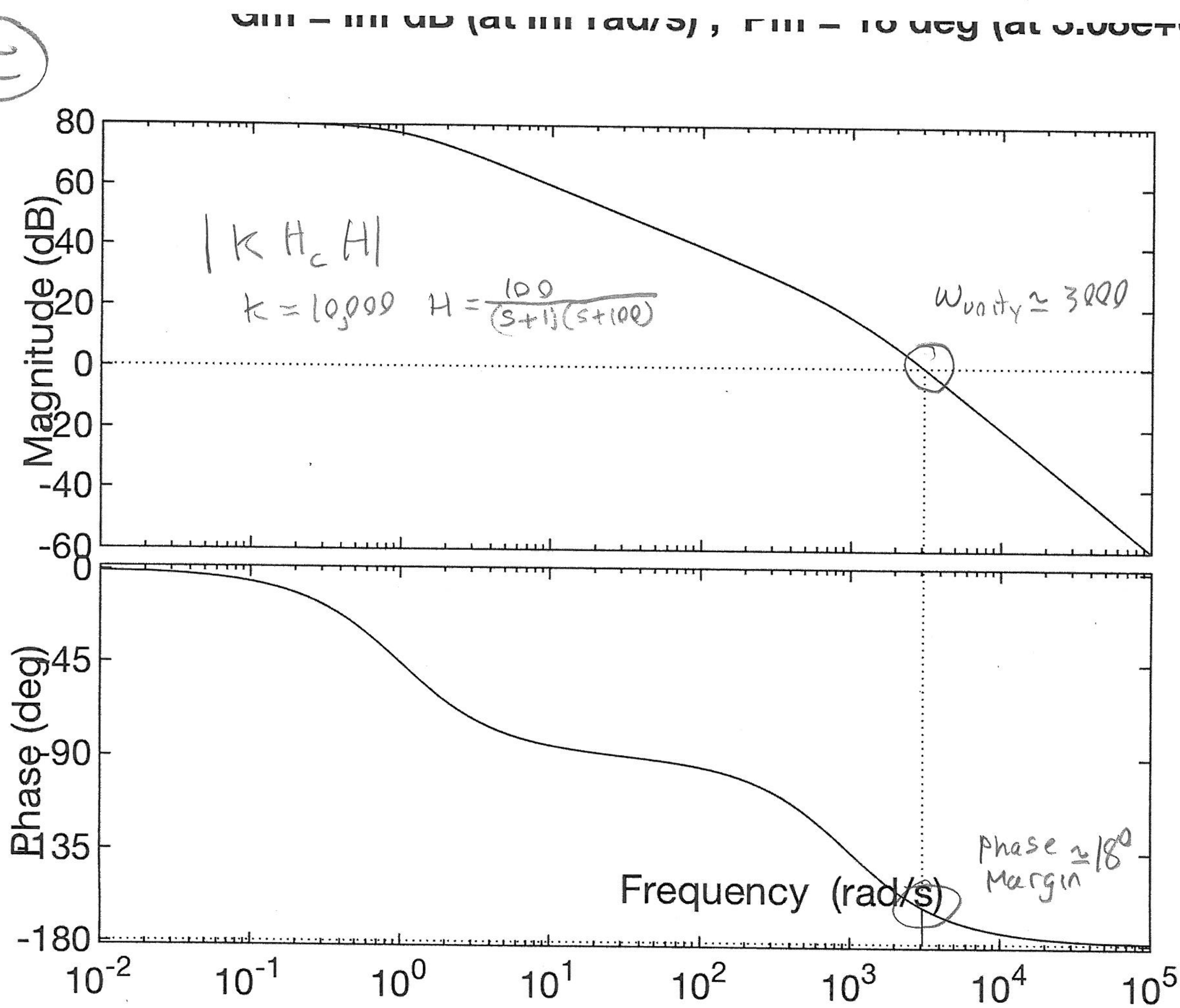
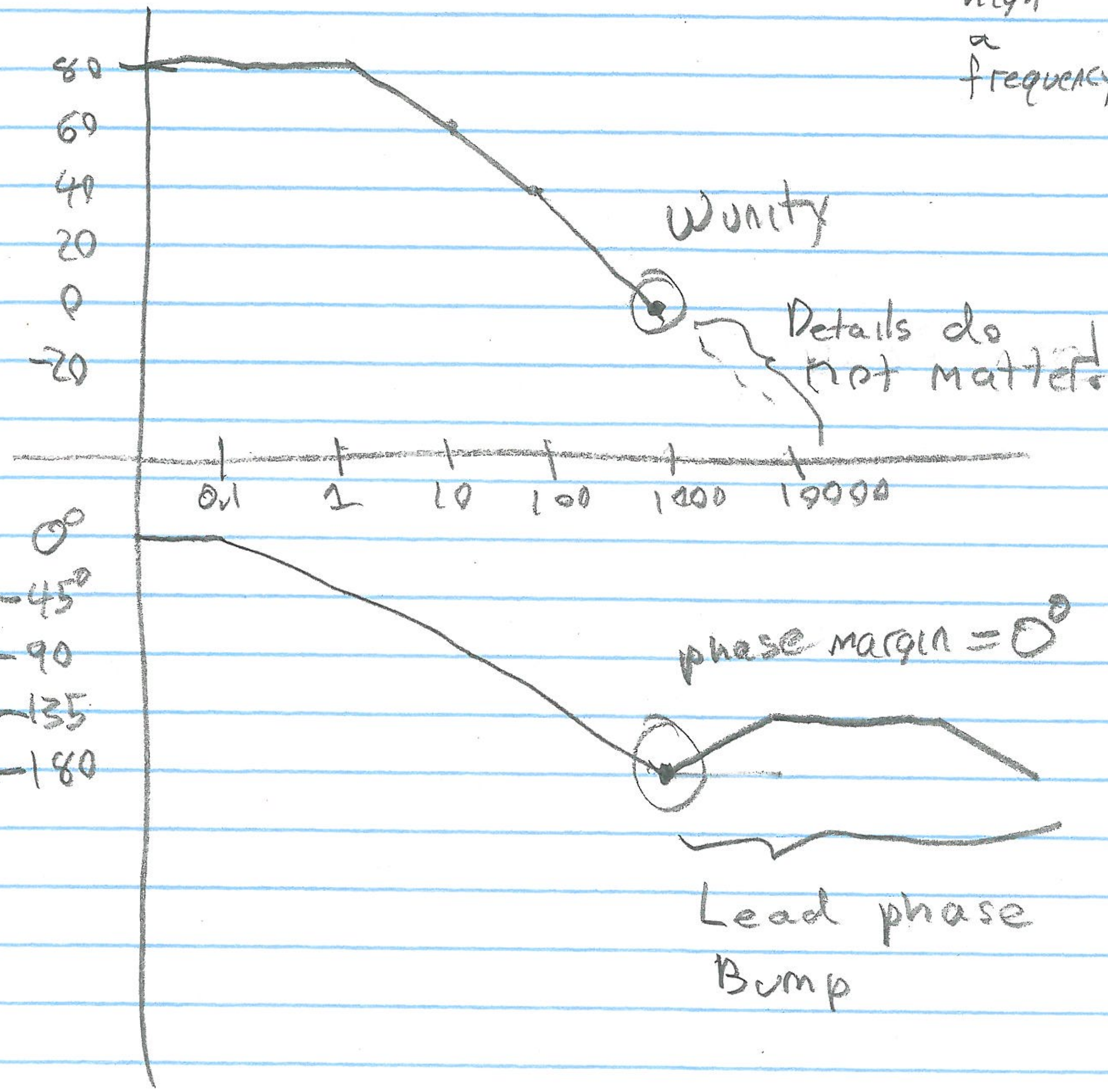


Too Late

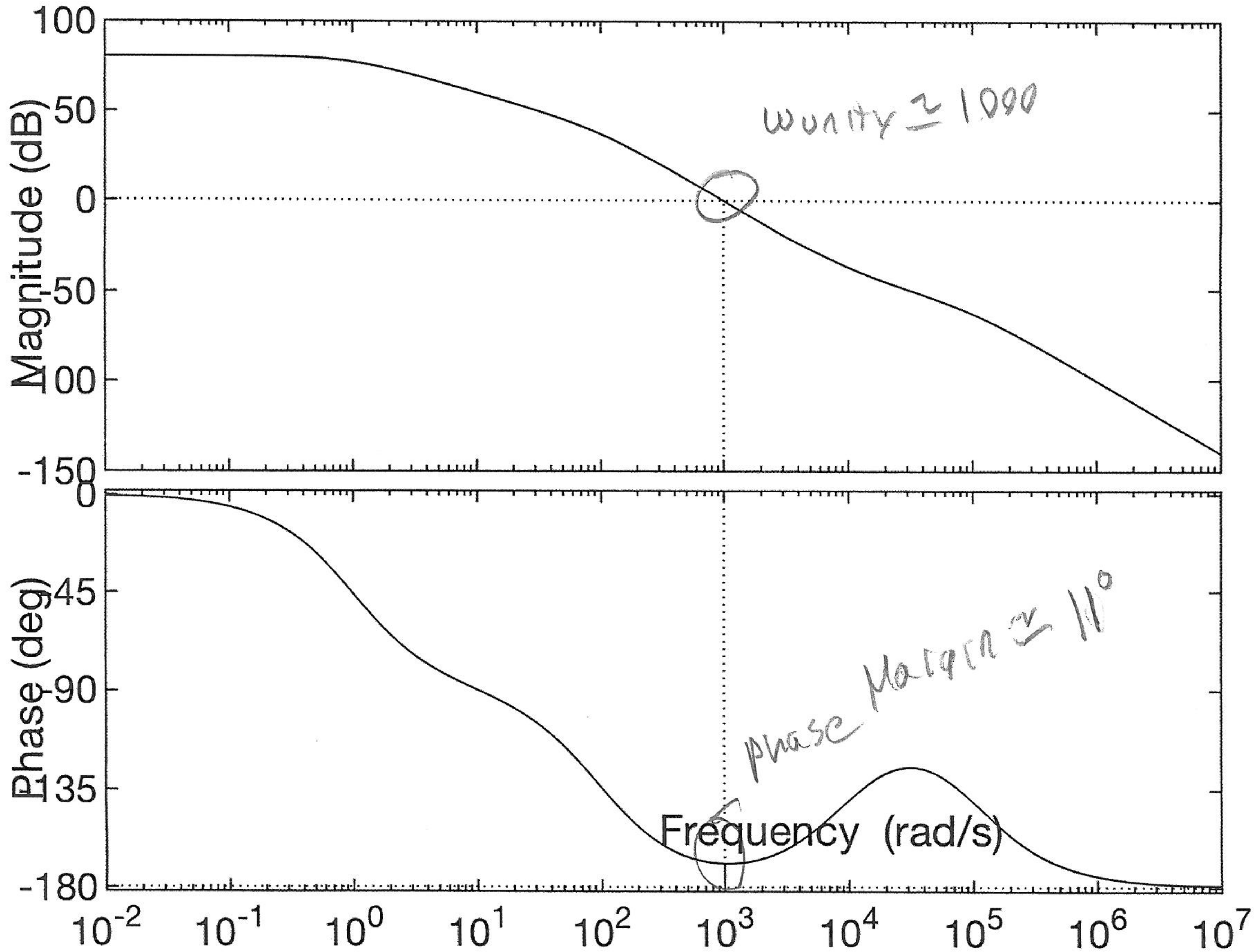
$$H_c(s) = 10\,\frac{s+10000}{s+100000}$$

Zero at too high a frequency



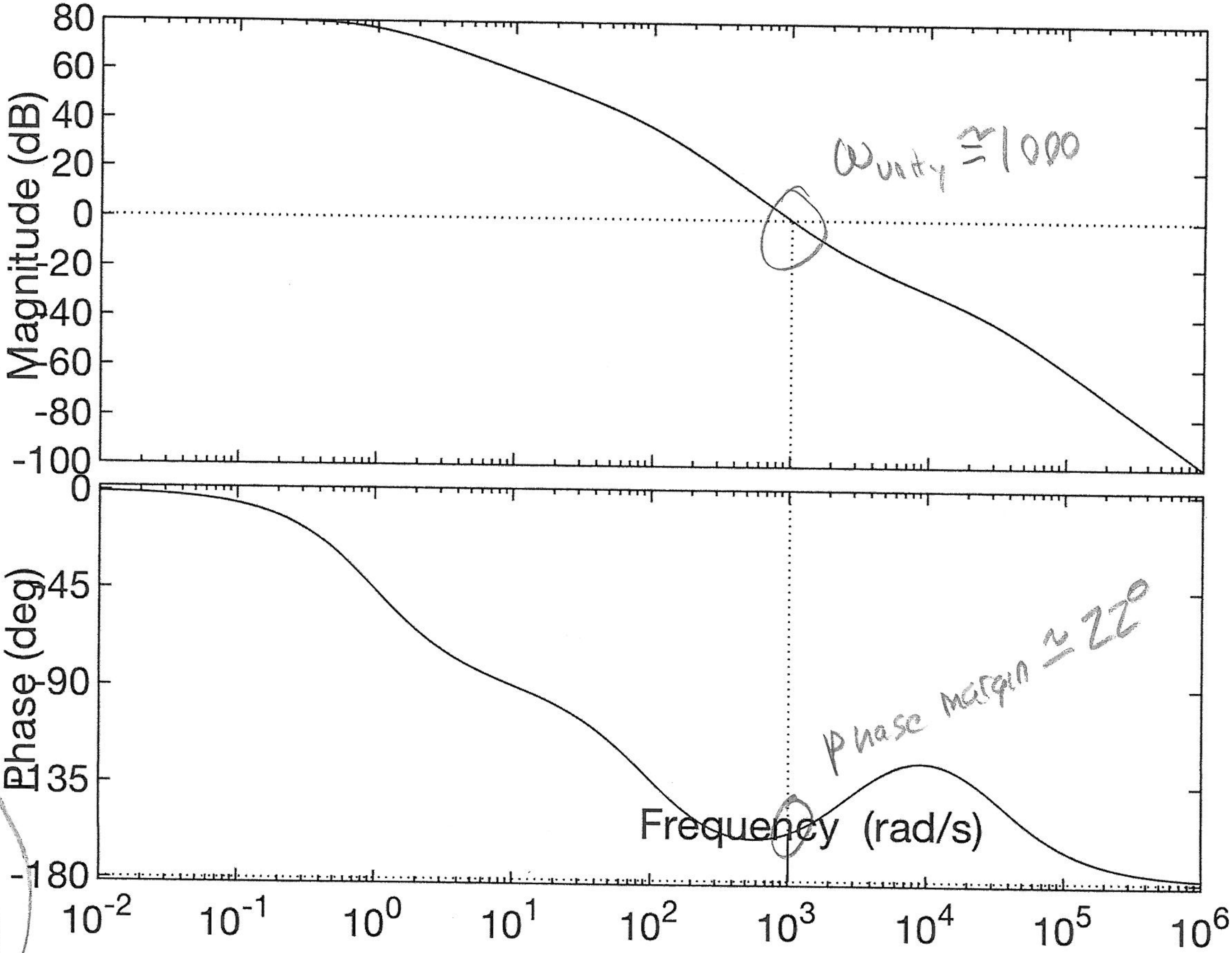
Actual Too Late $H_c(s) = 10\,\frac{s+10000}{s+100000}$ 13

Gm = Inf dB (at Inf rad/s) , Pm = 10.9 deg (at 1e+03



(14)

Lead a little late $\rightarrow H_c(s) = 10\left(\frac{s+3000}{s+30000}\right)$



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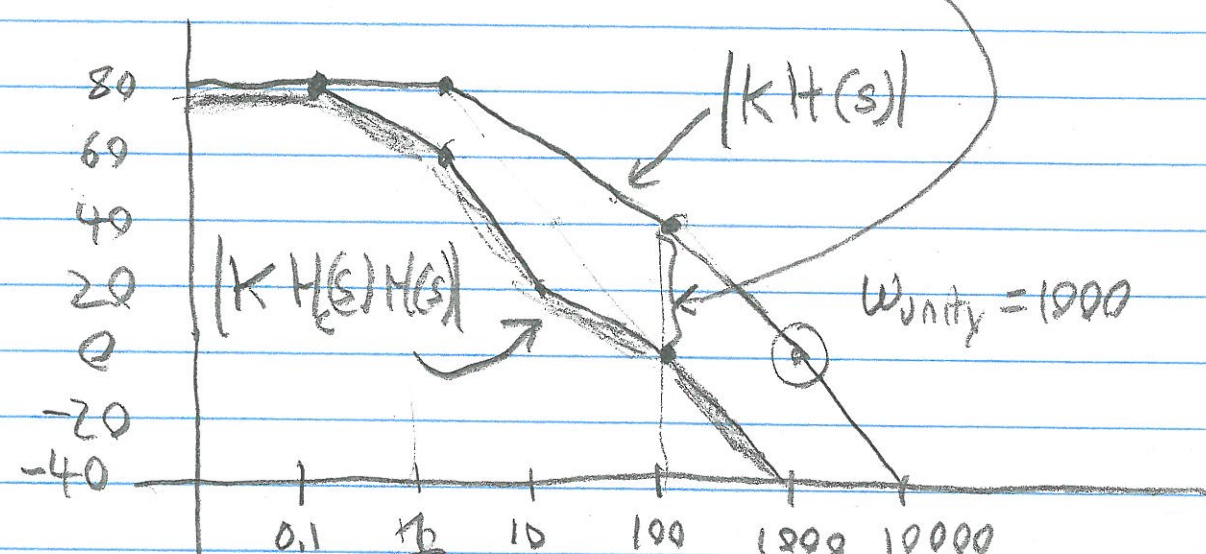
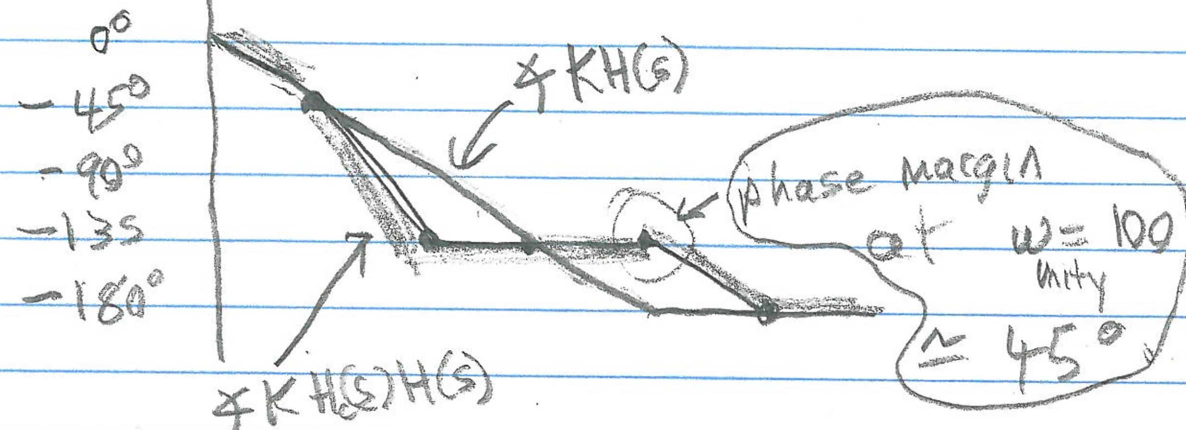
2nd Idea

$$H_c(s) = \frac{1}{100}\left(\frac{s+10}{s+0.1}\right)$$

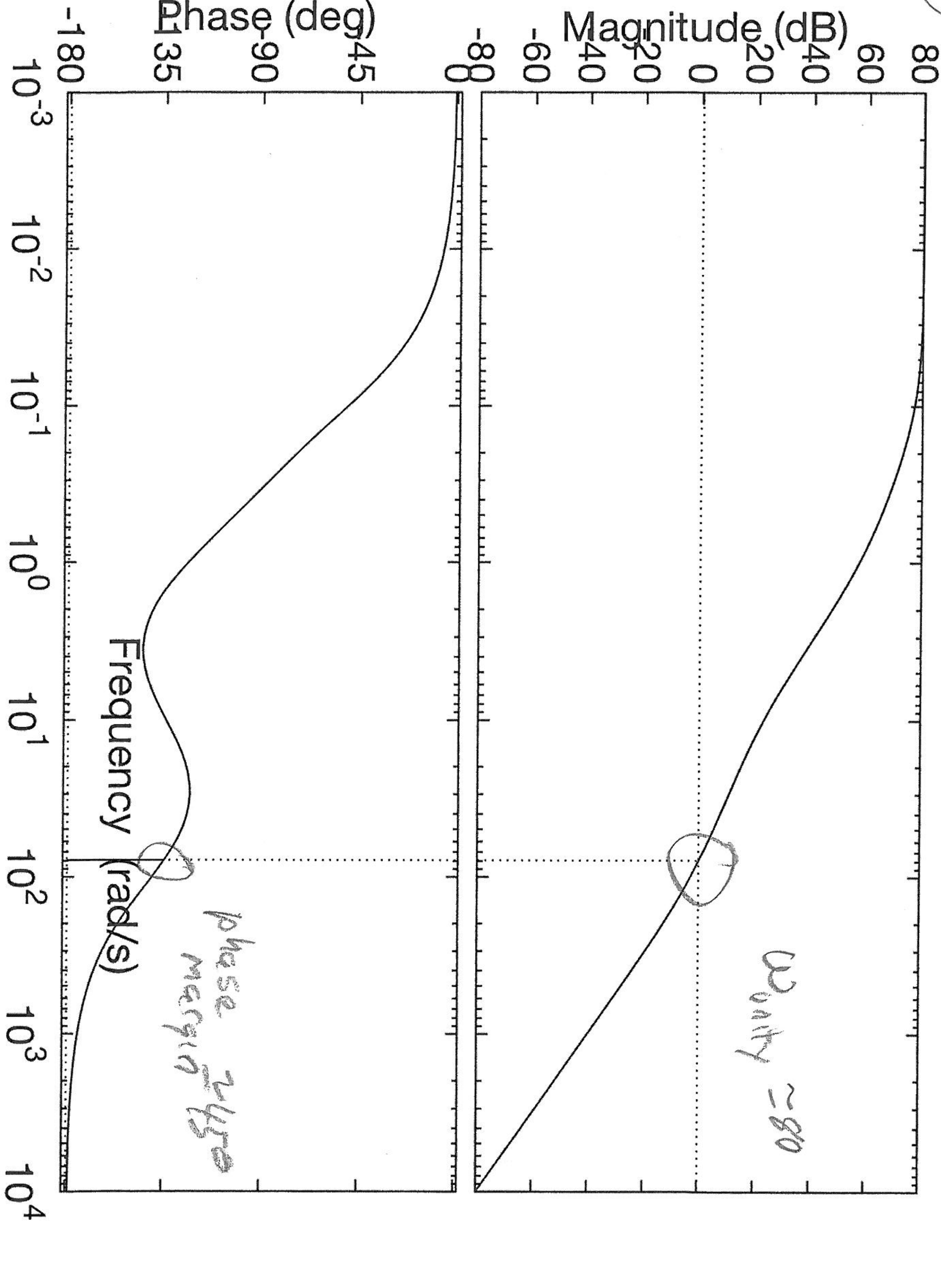
Lag to drop gain and reduce ω_{unity}. To reduce ω_{unity} to 100, reduce gain by 40dB!





$$K H_c(s)\, H(s) = 10100\,\frac{1}{100}\left(\frac{s+10}{s+0.1}\right)\left(\frac{100}{(s+1)(s+100)}\right)$$

Actual Lag $H_c(s) = \frac{1}{100}\left(\frac{s+10}{s+0.1}\right)$ (16)



Lag Too Soon $H_c(s) = \frac{1}{100}\left(\frac{s+1}{s+0.01}\right)$

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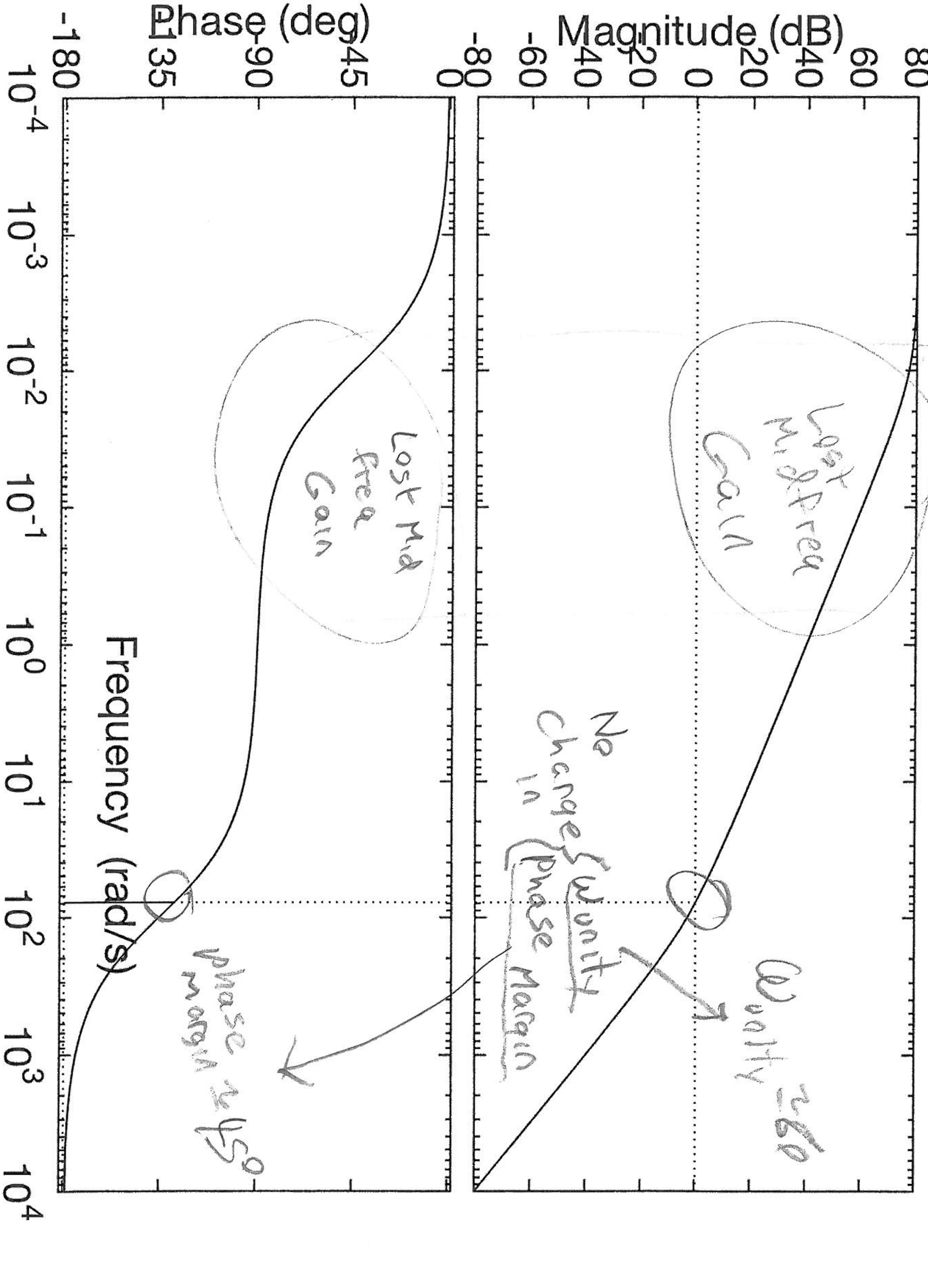
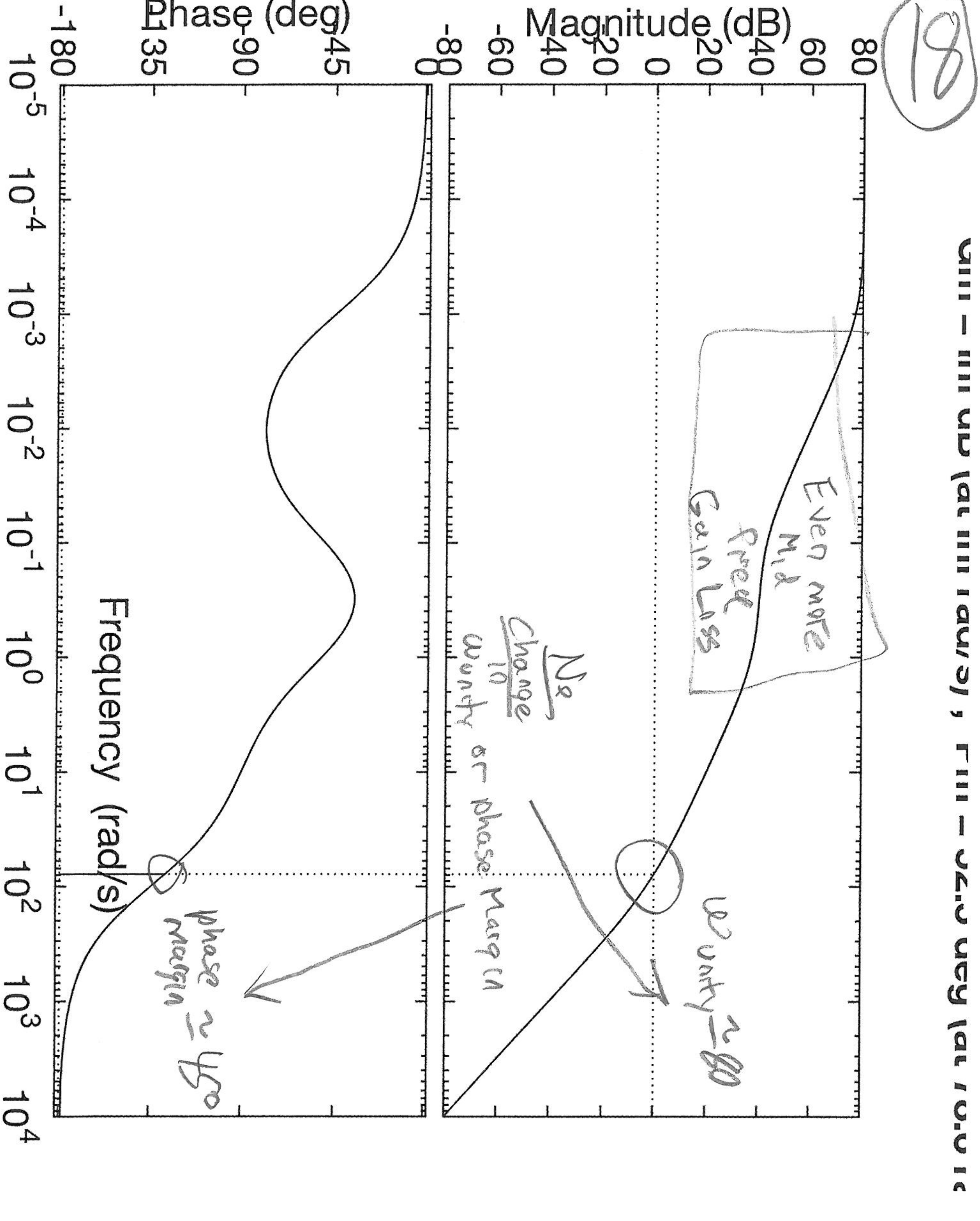


Lag Way Too Soon $H_c(s) = \frac{1}{100}\left(\frac{s+0.1}{s+0.001}\right)$

18



Observations

1) Lead compensators can be used to increase phase margin $\boxed{\text{without}}$ reduce gain at frequencies below ω_{unity}

$$|K H_{lead}(j\omega) H(j\omega)| \simeq |K H(j\omega)| \quad \omega < \omega_{unity}$$

2) Lead zero and pole must $\boxed{\text{surround}}$ ω_{unity} to be effective (Though ω_{unity} may not equal uncompensated ω_{unity})

3) If the phase margin is acceptable at frequencies below ω_{unity}, $\boxed{\text{low-frequency Lag}}$ can be used to reduce the ω_{unity}.

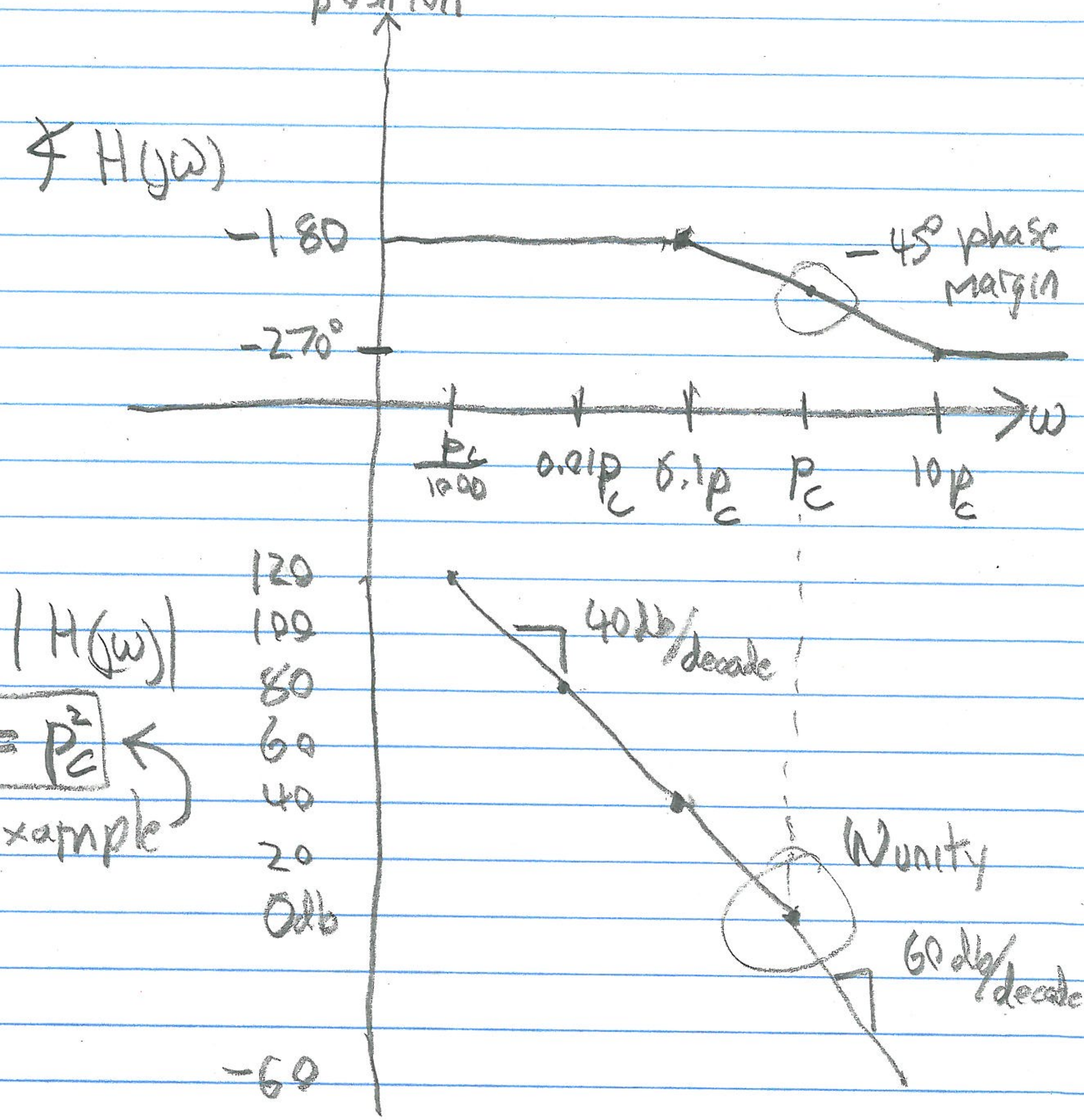
4) For many systems H, if a lag with pole p and zero q produces a given phase margin, then any $\alpha p, \alpha q$, $\alpha < 1$, will NOT reduce the phase margin.

Usual Mechanical Example

$$H(s) = \frac{\gamma(-p_c)}{s^2 (s-p_c)}$$

Force to position

command → force



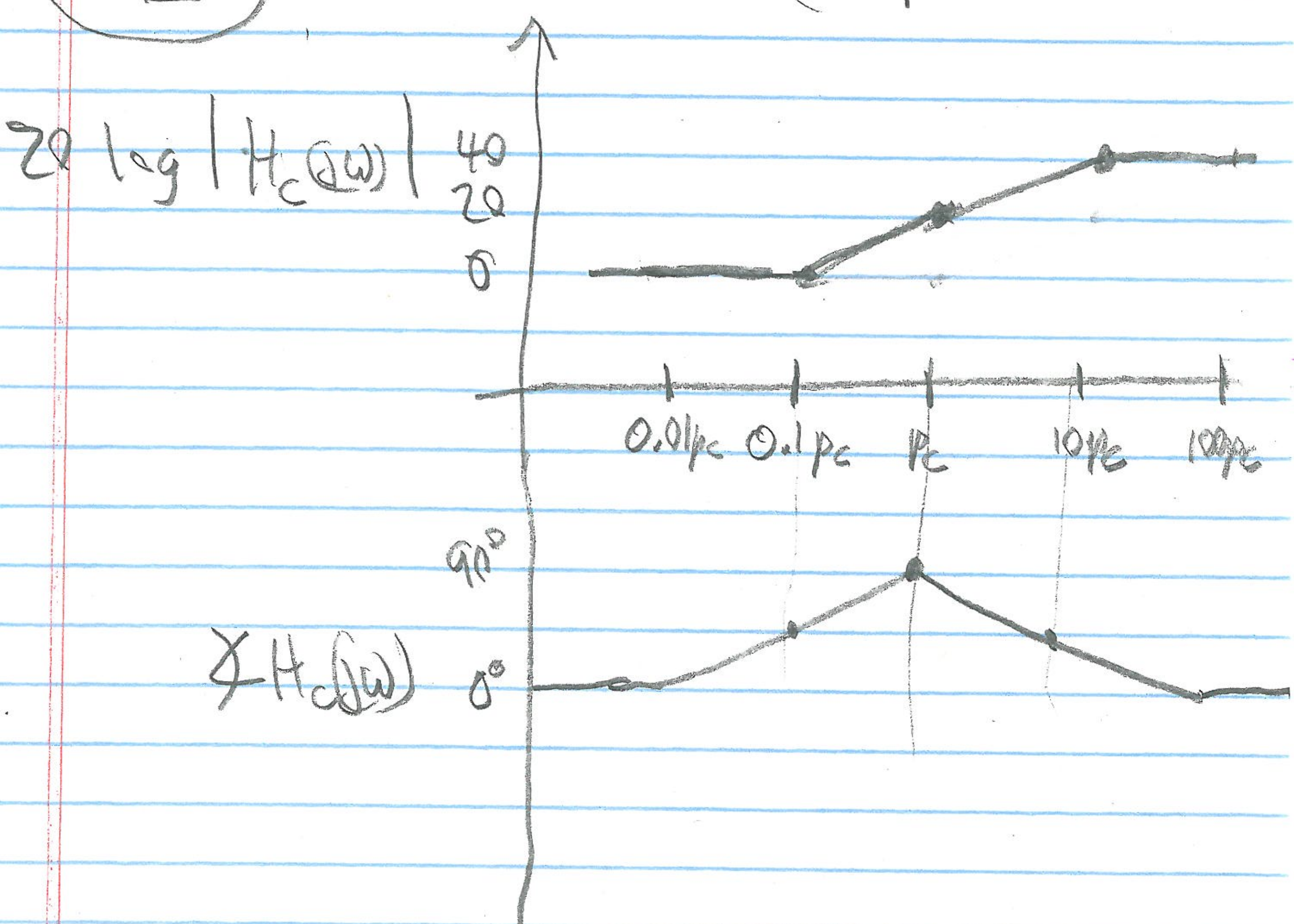
$\boxed{\gamma = p_c^2}$

Example

Lead Alone Could work

pole-zero seperation $= \underline{\underline{100}}$!

$$H_c(s) = 100\left(\frac{s + p_c/10}{s + 10p_c}\right)$$



Can you draw the diagrams for

$20\log|H_c(j\omega)\,H(j\omega)|$ and $\measuredangle H_c(j\omega)H(j\omega)$?

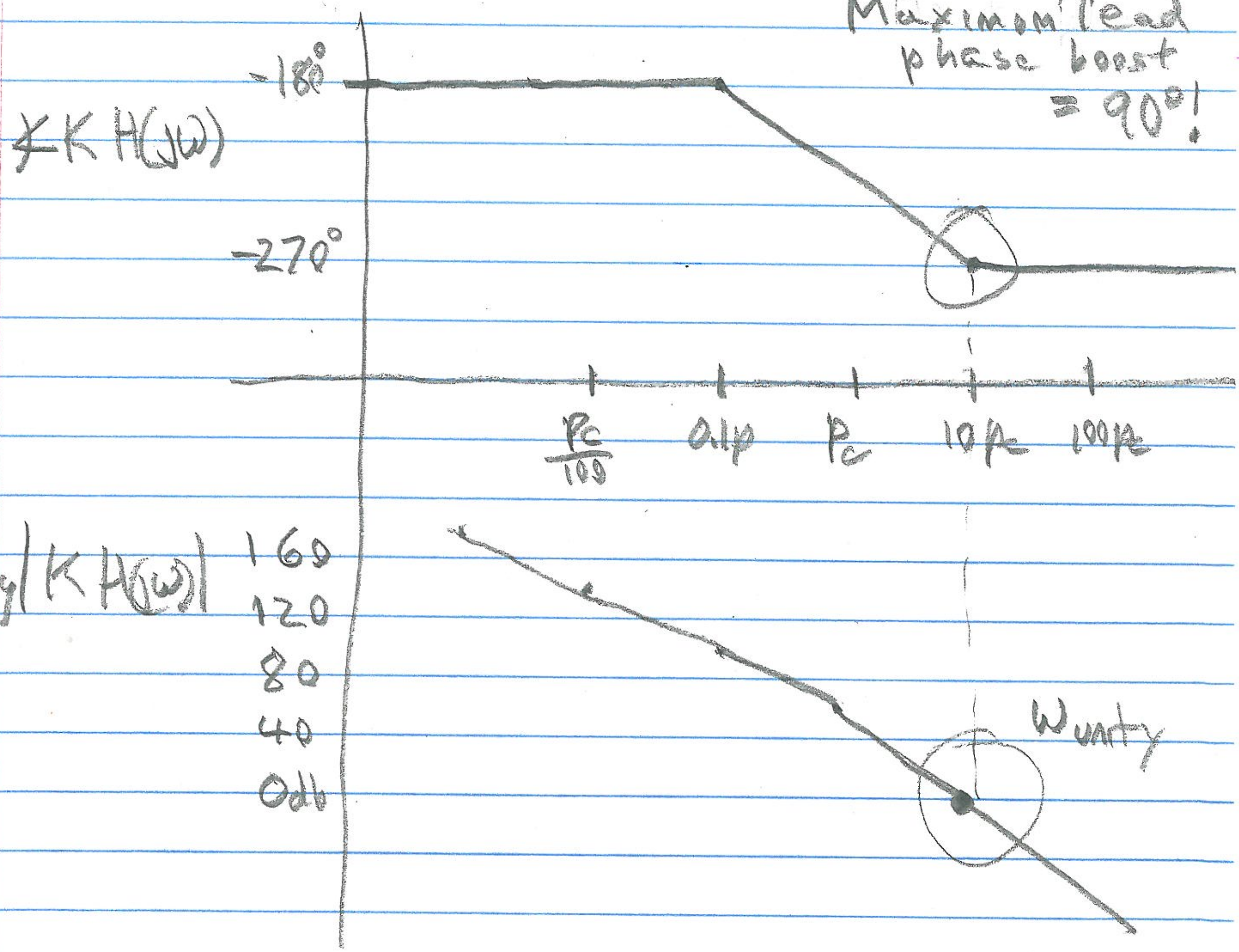
If $K = 1000$

with $K \cdot H_c(s) \cdot \frac{-p_c^3}{s^2(s-p_c)}$

Can $H_c(s)$ be a lead compensator and achieve a phase margin near 45°?

Answer: No! Phase margin at $\omega_{unity} \approx -90°$

Maximum lead phase boost $= 90°$!



For K=1000 Case

Combine Lead and Lag

Drops the gain to make $\omega_{unity} \cong P_c$

$$H_{lag}(s) = \frac{1}{1000}\left(\frac{s - P_c/100}{s - P_c/100{,}000}\right)$$

"Bumps" the phase up at ω_{unity}

$$H_{lead}(s) = 100\left(\frac{s - P_c/10}{s - 10P_c}\right)$$

Can you plot the diagrams?